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BIRECURRENT HYPERSURFACES OF A RIEMANNIAN MANIFOLD WITH CONSTANT CURVATURE

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0. Introduction

Let M be a hypersurface of dimension $n(\geqq 2)$ in an $(n+1)$-dimensional real space form $\overline{M}(c)$ with constant curvature c and H the second fundamental tensor of M. M is said to be *birecurrent* if there exists a covariant tensor field α of order 2 such that $\nabla^2 H = H \otimes \alpha$, where ∇ is the connection of M. Also, M is said to be *recurrent* if there exists a 1-form β such that $\nabla H = H \otimes \beta$. Matsuyama [2] recently proved that a recurrent hypersurface M in a real space form is locally symmetric and a complete irreducible birecurrent hypersurface M in a real space form is recurrent.

The main purpose of this paper is to characterize the birecurrent or recurrent hypersurface M of a Riemannian manifold with constant curvature c and to prove that M is classified as a cylinder, $M^n(c)$ or $M^r(c_1) \times M^{n-r}(c_2)$ where $1/c_1 + 1/c_2 = 1/c$.

1. Preliminaries

Let $\overline{M}(c)$ denote an $(n+1)$-dimensional connected Riemannian manifold of constant curvature c and let M denote an $n(\geqq 2)$-dimensional connected Riemannian manifold. We denote ϕ by a fixed isometric immersion of M into $\overline{M}(c)$. when the argument is local, M need not be distinguished from $\phi(M)$. Thus, for simplicity, a point x in M may be identified with the point $\phi(x)$ and a tangent vector X at x may also be identified with the tangent vector $d\phi(X)$ at $\phi(x)$ via the differential $d\phi$ of ϕ. We choose a local field $\{e_1, \ldots, e_n, e_{n-1}\}$ of orthonormal frames in $\overline{M}$ in such a way that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M and hence the other e_{n+1} is normal to M.

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Let $\{\bar{\omega}_1, \ldots, \bar{\omega}_n, \bar{\omega}_{n+1}\}$ be the field of dual frames associated with the above frame field. Throughout the present paper the following convention on the range of indices are used:

$$A, B, \ldots = 1, 2, \ldots, n, n+1,$$
$$i, j, \ldots = 1, 2, \ldots, n.$$

Then the structure equations of $\bar{M}$ are given by

(1. 1) $$d\bar{\omega}_A + \Sigma_B \bar{\omega}_{AB} \wedge \bar{\omega}_B = 0, \quad \bar{\omega}_{AB} + \bar{\omega}_{BA} = 0,$$

(1. 2) $$d\bar{\omega}_{AB} + \Sigma_C \bar{\omega}_{AC} \wedge \bar{\omega}_{CB} = c\bar{\omega}_A \wedge \bar{\omega}_B,$$

where $\bar{\omega}_{AB}$ denote the connection forms on $\bar{M}$. The restriction of these forms $\bar{\omega}_A$ and $\bar{\omega}_{AB}$ to M are simply denoted by ω_A and ω_{AB} without bar respectively. Hence we have $\omega_{n+1} = 0$. The metric on M induced from the Riemannian metric $\bar{g}$ in the ambient space $\bar{M}$ by the immersion ϕ is given by $g = 2\Sigma_i \omega_i \cdot \omega_i$. Then $\{e_1, \ldots, e_n\}$ becomes a field of orthonormal frames on M with respect to this metric g and $\omega_1, \ldots, \omega_n$ are the canonical forms on M. It is clear from $\omega_{n+1} = 0$ and the Cartan lemma that

(1. 3) $$\omega_{n+1i} = \Sigma_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

The quadratic form $\Sigma_{i,j} h_{ij} \omega_i \omega_j$ is called the *second fundamental form* on M. The (1, 1) tensor field A on M defined by

$$g(AX, Y) = \Sigma_{i,j} h_{ij} \omega_i(X) \omega_j(Y)$$

for any vector fields X and Y is called the *shape operator* of M. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the shape operator A_x at each point x in M are called the *principal curvatures* at x for the immersion. Furthermore, $\mathrm{Tr}\, A/n = \Sigma_i \lambda_i / n$ is called the *mean curvature* of M at x. In terms of the canonical forms ω_i and the connection forms ω_{ij}, the structure equations on the hypersurface M are given as follows:

(1. 4) $$d\omega_i + \Sigma_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0,$$

(1. 5) $$d\omega_{ij} + \Sigma_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij},$$

(1. 6) $$\Omega_{ij} = -\Sigma_{k,l} (R_{ijkl}/2) \omega_k \wedge \omega_l,$$

where Ω_{ij} (resp. R_{ijkl}) denotes the curvature form (resp. the curvature tensor) on M. By means of the above structure equations of M and $\bar{M}$, the Gauss equation of the hypersurface is obtained as

(1. 7) $$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + h_{il}h_{jk} - h_{ik}h_{jl}.$$

Now, the components h_{ijk} of the covariant derivatives of the second fundamental form are defined by

(1. 8) $$\Sigma_k h_{ijk} \omega_k = dh_{ij} - \Sigma_k h_{kj} \omega_{ki} - \Sigma_k h_{ik} \omega_{kj}.$$

Since the ambient space $\bar{M}$ is of constant curvature, we get the Codazzi equation

(1.9) $$h_{ijk}-h_{ikj}=0.$$

The covariant derivative h_{ijkl} of h_{ijk} can be defined as follows:

(1.10) $$\Sigma_l h_{ijkl}\omega_l=dh_{ijk}-\Sigma_l(h_{ljk}\omega_{li}+h_{ilk}\omega_{lj}+h_{ijl}\omega_{lk}).$$

2. Birecurrent hypersurfaces

Let M be a birecurrent hypersurface of a Riemannian manifold $M^{n+1}(c)$ with constant curvature c, that is, the second fundamental form h_{ij} satisfies

(2.1) $$h_{ijkl}=\alpha_{lk}h_{ij}$$

for some tensor field α_{lk}. In particular, M is said to be *proper* if h_{ijkl} does not vanish identically.

From (2.1) and the Ricci formula for h_{ij}, it follows that

$$(\alpha_{lk}-\alpha_{kl})h_{ij}=-\Sigma_r(R_{lkir}h_{rj}+R_{lkjr}h_{ir}),$$

which together with the Gauss equation (1.7) implies that

(2.2) $$(\alpha_{kl}-\alpha_{lk})h_{ij}=c(h_{il}\delta_{jk}-h_{ik}\delta_{jl}+h_{jl}\delta_{ik}-h_{jk}\delta_{il}) +h_{il}{}^2h_{jk}-h_{ik}{}^2h_{jl}+h_{jl}{}^2h_{ik}-h_{jk}{}^2h_{il}$$

where we have put $h_{ij}{}^2=\Sigma_r h_{ir}h_{rj}$. Since the second fundamental form h_{ij} is diagonalizable, an orthonormal basis $\{e_i\}$ at a point x on M can be chosen in such a way that $h_{ij}=\lambda_i\delta_{ij}$, namely, $\lambda_1, \ldots, \lambda_n$ are the principal curvatures at the point x. Then the relationship (2.2) tells us that

(2.3) $$(\alpha_{kl}-\alpha_{lk})\lambda_i\delta_{ij}=(c+\lambda_l\lambda_k)(\lambda_j-\lambda_i)(\delta_{lj}\delta_{ki}-\delta_{li}\delta_{kj})$$

at each point of M. When $l=j$, $k=i$ in (2.3), it follows that

(2.4) $$(\alpha_{ij}-\alpha_{ji})\lambda_i\delta_{ij}=(c+\lambda_i\lambda_j)(\lambda_j-\lambda_i)(1-\delta_{ij})$$

for any indices i and j.

For the shape operator A and for a point x of M, we denote by $n_A(x)$ the number of distinct eigenvalues of A_x. It is a simple algebraic fact that (2.4) implies M has at most two distinct principal curvatures and satisfies

$$c+\lambda_i\lambda_j=0$$

for any distinct i and j, that is, we have $n_A(x)\leqq 2$. Say, λ and μ be the distinct principal curvatures and their multiplicities be denoted by r and s respectively. We then have

(2.5) $$c+\lambda\mu=0,$$

and if $c=0$, then the non-zero principal curvature, say λ, is simple, i.e. $r=1$.

Denote by h the trace of the shape operator A, Then the Codazzi equation and (2. 1) give rise to

(2. 6) $$(h-\lambda_i)\alpha_{ji}=0$$

for any indices i and j.

Let M_A be the set which consists of points in M at which $n_A(x)=2$. Suppose that M_A is not empty. Then M_A is clearly an open subset in M. In each connected component of M_A, λ and μ are well-defined and are distinct smooth functions everywhere so that the distributions $D_\lambda=\{X : AX=\lambda X\}$ and $D_\mu=\{X : AX=\mu X\}$ can be defined. They are mutually orthogonal smooth distributions in each connected component of M_A.

In the following, we only consider that the hypersurface M is proper. At first we prove

LEMMA 1. *Let* $V=\{x\in M_A : \Sigma_{r,s}\ \alpha_{rs}\alpha_{rs}(x)=0\}$. *If* M *is proper, then* $M-V$ *is loally cylindrical.*

Proof. Since M is proper, M_A-V is non void and open and hence so is $M-V$, which means that there exists a point x in $M-V$ such that $\alpha_{ij}(x)\neq 0$ for some indices i and j. Thus we see, using (2. 6), that $h=\lambda_j$ on $M-V$ and hence, no loss of generality, we can put $\lambda_j=\lambda$. Since λ and μ have multiplicities r and s respectively, it follows that

$$(r-1)\lambda+s\mu=0,$$

which together with (2. 5) implies that we have $r=1$, $\mu=0$ and $c=0$ on M_A-V. In fact, unless otherwise, λ and μ are constant, which together with (2. 1) yields $h=0$ and hence $\lambda=\mu=0$ on M_A-V. This is a contradiction. Thus the Gauss equation says that M_A-V is flat and consequently a locally cylinder. This completes the proof of the lemma.

LEMMA 2. *Suppose that the interior* V_0 *of a connected component of* V *is not empty. Then* $\mu=0$, $c=0$ *or* λ *and* μ *are non-zero constant on* V_0.

Proof. Since λ and μ are smooth functions on V_0, it follows from (2. 5) that

$$\lambda_{,i}\mu+\lambda\mu_{,i}=0,$$

where $d\lambda=\Sigma_i\lambda_{,i}\omega_i$. This shows that

$$\lambda_{,ij}\mu+\lambda\mu_{,ij}+\lambda_{,i}\mu_{,j}+\lambda_{,j}\mu_{,i}=0.$$

However, we have $h_{ijkl}=0$ on V_0 and hence $\lambda_{,ij}=\mu_{,ij}=0$. Thus the last relationship is reduced to

$$\lambda_{,i}\mu_{,j}+\lambda_{,j}\mu_{,i}=0$$

for any indices i and j. Accordingly we have $\lambda_{,j}\mu_{,j}=0$ for any j, which means $\lambda_{,j}{}^2\mu_{,i}=0$ for any i and j.

Let V_λ be the set which consists of points in V_0 at which grad $\lambda \neq 0$. We then see that $\mu_{,i}=0$ on V_λ for arbitrary index i and hence μ is constant on V_λ. Thus (2.5) tells us that $\mu=0$ and $c=0$ on V_λ, which completes the proof.

Let M be a birecurrent hypersurface of $M^{n+1}(c)$. Then M is totally umbilic provided that $M_A=\phi$ because the number of distinct principal curvatures are less then or equal to 2.

The case where $M_A\neq\phi$ is considered. Suppose that V is void. As in the proof of Lemma 1, it follows that $\mu=0$, $c=0$ and $r\leqq 1$ on M_A. Thus, by continuity of principal curvatures, it is seen that $\mu=0$, $c=0$ and $r\leqq 1$ on M. The same conclusions are obtained in the case where the interior of V is empty. Therefore, we only consider the case where the interior of V does not vanish. But, Lemma 2 tells us that $\mu=0$, $c=0$ or λ and μ are non-zero constant on M, the latter case arising when $V=M$ by means of the continuity of principal curvatures.

Summing up we have

THEOREM 3. *Let M be a birecurrent and complete hypersurface of a Riemannian space form $M^{n+1}(c)$ with constant curvature c.* (1) *If M is proper, then it is a cylinder and $c=0$.* (2) *If M is not proper, then M is $M^n(c_0)$ or $M^r(c_1)\times M^{n-r}(c_2)$, where $1/c_1+1/c_2=1/c$.*

Let M be a recurrent hypersurface of $M^{n+1}(c)$, namely,

$$h_{ijk}=\alpha_k h_{ij}$$

for some function α, where $d\alpha=\Sigma_i\alpha_i\omega_i$. Then we have

$$h_{ijkl}=\alpha_{kl}h_{ij}+\alpha_k\alpha_l h_{ij},$$

where we have used $\Sigma_l\alpha_{kl}\omega_l=d\alpha_k-\Sigma_l\alpha_l\omega_{kl}$. Thus, by the direct consequence of Theorem 3, we have

COROLLARY 4. *Let M be a recurrent and complete hypersurface of a Riemannian space form $M^{n-1}(c)$ with constant curvature c. Then M is a cylinder, $M^n(c_0)$ or $M^r(c_1)\times M^{n-r}(c_2)$, where $1/c_1+1/c_2=1/c$.*

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