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ON THE BOUNDARY BEHAVIOR OF ANALYTIC FUNCTIONS

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Let $w=f(z)$ be a meromorphic function in the unit disc D. Let $\zeta_0=e^{i\theta}$ be a fixed point on $\Gamma: |z|=1$ and A an open arc of Γ containing ζ_0. We suppose that E is a set of linear measure zero containing ζ_0 and contained in A. We associate with every $e^{i\theta}$ in $A-E$ an arbitrary curve Λ_θ in D terminating at $e^{i\theta}$ and the cluster set $C_{\Lambda_\theta}(f, e^{i\theta})$ of $f(z)$ at $e^{i\theta}$ along Λ_θ. We define the boundary cluster set $C^*_{\Gamma-E}(f,\zeta_0)$ of $f(z)$ at ζ_0 as

$$C^*_{\Gamma-E}(f,\zeta_0)=\bigcap_{r>0} M_r$$

where M_r denotes the closure of the union $\cup C_{\Lambda_\theta}(f, e^{i\theta})$ for all $e^{i\theta}$ in the intersection of $A-E$ with $|z-\zeta_0|<r$.

Using this definition, K. Noshiro ([2]) established the following theorem for meromorphic functions.

Theorem (K. Noshiro). *If $\alpha\in C_D(f,\zeta_0)-C^*_{\Gamma-E}(f,\zeta_0)$ is an exceptional value of $f(z)$ in a neighborhood of ζ_0, then either α is an asymptotic value of $f(z)$ at ζ_0, or there exists a sequence of points $z_n\in \partial D$ converging to ζ_0 such that α is an asymptotic value of $f(z)$ at each z_n.*

In this note we define a new boundary cluster set and extend Noshiro's theorem to an arbitrary simply connected domain.

For our proof we will need the following theorem.

Theorem (W. Seidel, [5]). *Let $f(z)$ be a non-constant function of class (U). If $f(z)\neq\alpha$ in the unit disc, then there exists at least one θ such that*

$$\lim_{r\to 1} f(re^{i\theta})=\alpha.$$

Let $f(z)$ be a meromorphic function in a simply connected domain

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D, $\tilde{E}$ a D-conformal null set of prime ends of D such that E the union of impressions of prime ends in $\tilde{E}$ contains ζ_0. We associate with every accessible boundary point a with $P(a)$ in $\tilde{D}-\tilde{E}$ ($\tilde{D}$ is the set of all prime ends of D) an arc Λ at $P(a)$ in D terminating at $z(a)$ and the cluster set $C_\Lambda(f, z(a))$ of $f(z)$ at $z(a)$ along Λ. We define a new boundary cluster set $C^*_{\tilde{D}-\tilde{E},\{\Lambda\}}(f,\zeta_0)$ of $f(z)$ as follows

$$C^*_{\tilde{D}-\tilde{E},\{\Lambda\}}(f,\zeta_0)=\bigcap_{r>0} M_r$$

where M_r is the closure of the union $\cup C_\Lambda(f, z(a))$ for all accessible points a with $P(a)\in\tilde{D}-\tilde{E}$ and $z(a)$ in $\{z: |z-\zeta_0|<r\}$. Then it is easy to see that

$$C^*_{\tilde{D}-\tilde{E},\{\Lambda\}}(f,\zeta_0)\subset C_\Gamma(f,\zeta_0)\subset C_D(f,\zeta_0).$$

We prove the following analogue of Noshiro's Theorem.

THEOREM. *Let D be a simply connected domain in the z-plane, which is not the whole plane, and let ζ_0 be a boundary point of D, contained in the union of impressions of prime ends in $\tilde{E}$, a D-conformal null set. Let $f(z)$ be single-valued and meromorphic in D. If $\alpha\in C_D(f,\zeta_0)-C^*_{\tilde{D}-\tilde{E}}(f,\zeta_0)$ is an exceptional value of $f(z)$ in a neighborhood of ζ_0, then either α is an asymptotic value of $f(z)$ at ζ_0, or there exists a sequence of points z_n in the boundary of D converging to ζ_0, such that α is an asymptotic value of $f(z)$ at each z_n.*

Proof. Let $\Omega=C_D(f,\zeta_0)-C^*_{\tilde{D}-\tilde{E}}(f,\zeta_0)$. We choose a positive number r, sufficiently small, such that $f(z)\neq\alpha$ in $D\cap D_r$, where $D_r=\{z: |z-z_0|<r\}$, and α lies outside M_r. Let d_1 be the distance between α and M_r.

Let G be any component of $D\cap D_r$ We choose two accessible boundary points a_1 and a_2 such that $z(a_1)\neq z(a_2)$, $z(a_1)$ and $z(a_2)$ belong to $\partial G\cap\partial D$, and, $P(a_1)$, $P(a_2)\notin\tilde{E}$. We take a suitable last part Λ'_{a_1} of Λ_{a_1} and a suitable last part Λ'_{a_2} of Λ_{a_2} lying in G where Λ_{a_1} and Λ_{a_2} are arcs used in defining $C^*_{\tilde{D}-\tilde{E}}(f,\zeta_0)$. We can choose a_1 and a_2 so that it is possible to construct a cross-cut L of G by connecting two end points, lying in G, of Λ'_{a_1} and Λ'_{a_2} by a Jordan arc such that there is a positive number d_2 such that $|f(z)-\alpha|\geqq d_2>0$ on L.

Let d be a positive number less than $\min(d_1, d_2)$. Let G_0 be the domain surrounded by a part of ∂G near ζ_0 and L. Next we take the neighborhood $D_\alpha=\{w : |w-\alpha|<d\}$ so that $D_\alpha \subset \Omega$.

Let Δ be any component of the inverse image of D_α in the domain D_0. The fact that α is a cluster value of $f(z)$ at ζ_0 guarantees the existence of such Δ for some G. We note that Δ is not necessarily simply connected. The component Δ is not compact in G_0 (otherwise, we have $D=f(\Delta)$, which gives a contradiction).

Let $\tilde{\Delta}$ be the universal covering surface of Δ. Then $\tilde{\Delta}$ is conformally equivalent to the unit disc. We map the universal covering surface $\tilde{\Delta}$ of Δ conformally upon the unit disc $|t|<1$ in a one-to-one manner. Denote by $z=z(\zeta)$ the mapping function.

We consider the two bounded analytic functions $z=z(t)$ and $w=w(t)=f(z(t))$ in $|t|<1$. Since the functions $z=z(t)$ and $w=f(z(t))$ are bounded and analytic, there exists a set E on $|t|=1$ such that both functions have radial limits at every point of E and the complement of E is of linear measure zero.

We denote by E_t the set of $t=e^{i\varphi}$ such that both the radial limits $z(e^{i\varphi})$ and $w(e^{i\varphi})$ exist and such that $|w(e^{i\varphi})-\alpha|<d$.

If there exists a point $e^{i\varphi}$ in E_t such that $z(e^{i\varphi})$ coincides with some $P(a)\in\tilde{D}-\tilde{E}$, then there exists an asymptotic path Λ_a terminating at $z(a)$ along which $f(z)$ converges to the asymptotic value $\beta=w(e^{i\varphi})$. If $\beta=\alpha$, our statement is true. If $\beta\neq\alpha$, applying Noshiro's theorem, we see that there is an asymptotic path in G ending at $z(a)$ along which $f(z)\to\alpha$.

Suppose that there is no point $e^{i\varphi}$ in E_t such that $z(e^{i\varphi})$ coincides with some a with $P(a)\in\tilde{D}-\tilde{E}$. Then since $\tilde{E}$ is a conformal null set, E_t must be of linear measure zero. Hence $(w(t)-\alpha)/d$ is a function of class (U). Since α is an exceptional value of $w(t)$, α must be a radial limit of $w(t)$ by Seidel's theorem. Accordingly, α is an asymptotic value of $f(z)$ at an accessible point a, $P(a)$ belonging to $\tilde{E}$. This completes the proof of the theorem.

As an immediate consequence of this theorem we have

Corollary. *Let D be a simply connected domain which is not the whole plane, $\tilde{E}$ a D-conformal null set. Let $f(z)$ be an analytic function in D. Suppose that at every prime end P of D that does not belong to $\tilde{E}$ there is a curve γ_p such that*

$$\limsup_{\substack{z \to p \\ z \in \gamma_p}} |f(z)| \leq m$$

and such that $f(z)$ does not have the asymptotic value ∞ at any prime end of D. Then $f(z)$ is bounded in D.

Proof. Assume the contrary. Then there exists a point $\zeta_0 \in \partial D$ such that $\infty \in C_D(f, \zeta_0)$. Since ∞ is not an asymptotic value of $f(z)$ at any prime end of D.

$$\infty \notin C^*_{\bar{D}-\bar{E}, \{\gamma_p\}}(f, \zeta_0).$$

Thus we have

$$\infty \in C_D(f, \zeta_0) - C^*_{\bar{D}-\bar{E}, \{\gamma_p\}}(f, \zeta_0).$$

Since $f(z)$ is analytic in D, ∞ is an exceptional value of $f(z)$ at ζ_0. Therefore, by the Theorem, ∞ is an asymptotic value of $f(z)$ at some boundary point of D. But then ∞ is an asymptotic value of $f(z)$ at some prime end of D, which is contrary to hypothesis.

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