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ON STABLE MINIMAL SURFACES IN THREE DIMENSIONAL MANIFOLDS OF NONNEGATIVE SCALAR CURVATURE

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1. Introduction

The following is the basic problem about the stability in Riemannian Geometry; given a Riemannian manifold N, find all stable complete minimal submanifolds of N. As answers of this problem, do Carmo-Peng [1] and Fischer-Colbrie and Schoen [3] showed that the stable minimal surfaces in $\boldsymbol{R}^3$ are planes and ·Schoen-Yau [5] and Fischer-Colbrie and Schoen [3] gave a solution for the case where the ambient space is a three dimensional manifold with nonnegative scalar curvature. In this paper we will remove the assumption of finite absolute total curvature in [3, Theorem 3].

2. The main result

We prove the following theorem.

THEOREM. *Let N be a complete oriented 3-manifold of nonnegative scalar curvature S. Let M be an oriented complete, stable minimal surface in N. If M is noncompact, then M is conformally equivalent to the complex plane $\boldsymbol{C}$ or the cylinder A. If the latter case occurs, then M is flat and totally geodesic and the scalar curvature of N is zero along M. Therefore if the scalar curvature of N is everywhere positive, then M cannot be a cylinder.*

If the Ricci curvature of N is nonnegative, then M is conformally equivalent to the complex plane $\boldsymbol{C}$ or M is a flat and totally geodesic cylinder.

In order to prove this theorem, we need following lemmas.

LEMMA 1. *If u is a positive function on M satisfying $\Delta u+(S-K+\frac{1}{2}\|B\|^2)u=0$ on M, where K is the Gaussian curvature of M and $\|B\|^2$*

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is the square of the length of the second fundamental form of M, then $\bar{d}s^2=u^2ds^2$ *is a complete metric on M with nonnegative Gaussian curvature where* ds^2 *is the original metric on M.*

Proof. Use the method of the proof in [2, Theorem 2.1].

LEMMA 2([4]). *Let M be a finitely connected, open Riemann surface on which a complete conformal metric* $e^{v(z)}|dz|$ *is defined and K a Gaussian curvature of M. Suppose that either* $\int_M K^+ dv<\infty$ *or* $\int_M K^- dv<\infty$ *where* $K^+(x)=max\{K(x),0\}$, $K^-(x)=-min\{K(x),0\}$ *and dv is the volume element of M. Then* $\int_M K\,dv\leqq 2\pi\chi(M)$ *where* $\chi(M)$ *denotes the Euler-Poincarè characteristic.*

REMARK 3. This is a result of S. Cohn-Vossen in the extended form.

The results of Theorem are proved in [3, Theorem 3] with the exception of the assertion that if M is conformally equivalent to the cylinder A, then M is flat and totally geodesic and the scalar curvature of N is zero along M. (This follows from [3, Theorem 3] only if M is assumed to have finite absolute total curvature).

Proof of Theorem. Let M be conformally equivalent to a cylinder. Since M is stable, there exists a positive solution u on M satisfying

$$\Delta u+(S-K+\frac{1}{2}\|B\|^2)u=0.$$

Then by Lemma 1, $\bar{d}s^2=u^2ds^2$ is a complete metric on M with non-negative Gaussian curvature $\bar{K}$. Since M is conformally equivalent to a cylinder, the Euler-Poincare characteristic $\chi(M)=0$. By Lemma 2, $\bar{K}\equiv 0$ on M. Hence $K-\frac{u\Delta u-|\nabla u|^2}{u^2}=0$. So $S+\frac{1}{2}\|B\|^2+\frac{|\nabla u|^2}{u^2}\equiv 0$ on M. Since $S\geqq 0$ and u satisfies $\Delta u+(S-K+\frac{1}{2}\|B\|^2)u=0$, we have $S\equiv 0$, $\|B\|^2\equiv 0$ and $K\equiv 0$ on M. Hence M is flat and totally geodesic and the scalar curvature of N is zero along M. This completes the proof.

REMARK 4. Schoen and Yau [6] proved that the cylinder is totally geodesic without assumption of finite absolute total curvature.

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