THE EXTENSION OF SOLUTIONS FOR THE CAUCHY PROBLEM IN THE COMPLEX DOMAIN

EUN GU LEE AND DOHAN KIM

Introduction

In [4], J. Leray introduced the notion of partial hyperbolicity to characterize the operators for which the non-characteristic Cauchy problem is solvable in the Gevrey class for any data which are holomorphic in a part of variables \( x'' = (x_2, ..., x_l) \) in the initial hyperplane \( x_1 = 0 \). A linear partial differential operator is called partially hyperbolic modulo the linear subvarieties \( S : x'' = \text{constant} \) if the equation \( P_m(x, \zeta_1, \zeta') = 0 \) for \( \zeta_1 \) has only real roots when \( \zeta' \) is real and \( \zeta'' = 0 \), where \( P_m \) is the principal symbol of \( P \).

Limiting to the case of operators with constant coefficients, A. Kaneko proposed a new sharper condition when \( S \) is a hyperplane [3].

In this paper, we generalize this condition to the case of general linear subvariety \( S \) and show that it is sufficient for the solvability of Cauchy problem for the hyperfunction Cauchy data which contains variables parallel to \( S \) as holomorphic parameters.

Let \( P(D) \) be an \( m \)-th order linear partial differential operator in \( \mathbb{R}^s \) with constant coefficients, and let \( P_m(D) \) be its principal part. Assume that \( x_1 = 0 \) is non-characteristic with respect to \( P \). We use the following notation for the separation of the independent variables; \( x = (x_1, x') = (x_1, x'', x''') \) with \( x'' = (x_2, ..., x_l) \) and similar notation for the complexification \( z = x + \sqrt{-1} y \) for the dual variables \( \zeta = \xi + \sqrt{-1} \eta \).

We let \( \Gamma \) be a convex open cone in \( \mathbb{R}^{s-l} \) and \( \Delta \) be a convex open cone such that \( \Delta \subset \subset \Gamma \), i.e., \( \bar{\Delta} \) is a compact subset of \( \Gamma \).

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Lemma. Consider the holomorphic Cauchy problem

\[(2.1) \quad \begin{cases} P(D)F(x) = 0 \\ \left( \frac{\partial}{\partial z_1} \right)^j F(x) \bigg|_{z_1=0} = F_j(x'), \ j = 0, \ldots, m-1 \end{cases} \]

The holomorphic data \(F_j(x')\) are given on a domain of the form \(\{x' \in \mathbb{C}^{n-1} | |x'| < A, \ y''' \in \Gamma, |y'| < B}\). Also, for some constant \(b, c > 0\), we have

\[(2.2) \quad -\text{Im} \ \zeta' \leq b \ |\text{Im} \ \zeta'''| + c |\zeta'''|\]

if \(\text{Re} \ \zeta''' \in \Delta^c\).

Then the solution can be continued onto the domain

\[W = \{z \in \mathbb{C}^n | 0 < x_1 < \delta, |x| < A', \lambda |y_1| < \text{dis}(y''', \partial \Gamma'), |y'| < B'\}\]

where \(A', B', \lambda\) and \(\delta\) are suitable positive constants.

Proof. First note that by the Cauchy–Kowalevsky theorem, the solution exists on a domain

\[\tilde{W} = \{z \in \mathbb{C}^n | |z_1| < k \ \text{dis}(y'''', \partial \Gamma), |x'| < A/2, |y''| < B', y''''' \in \Gamma, |y'''| < B/2\}\]

where \(k\) is a positive constant. Starting from this open set, we may use the method of Bony–Schapira [1]. Choose

\[z_0 = (t + \sqrt{-1}s, 0, \ldots, 0, \sqrt{-1}y_0)\]

where \(t > 0, y_0 \in \Gamma, |y_0| < \varepsilon\). If every real characteristic hyperplane passing through this point intersects \(\tilde{W}\), then the solution \(F(z)\) can be continued up to the interior of \(\text{ch}^{-1}\{z_0 \cup \tilde{W}\}\).

A characteristic hyperplane passing through \(z_0\) is expressed by the following equation

\[(2.3) \quad -\text{Re} \ <z - z_0, \sqrt{-1}\zeta> = x \cdot \eta + y \cdot \xi - t \eta_1 - s \xi_1 - y_0 \cdot \xi''' = 0\]

where \(\zeta = \xi + \sqrt{-1}\eta\) satisfies \(P_m(\zeta) = 0\). The fact that \(P_m(\zeta) = 0\) and the non-characteristic assumption imply that there exists \(M > 0\) such that

\[|\xi_1| \leq M |\zeta'|\].

We consider the following two cases.

(1) Case \(|\xi'| \leq |\eta'|\).

The point

\[x_1 = 0, \ x' = \frac{t \eta_1 + s \xi_1}{|\eta'|^2} \eta', \ y_1 = \eta' = 0, \ y''' = y_0\]

satisfies (2.3). Since we have

\[|\xi_1| \leq |\zeta_1| \leq M |\zeta'| \leq \sqrt{2} M |\eta'|\]

and similarly, \(|\eta_1| \leq \sqrt{2} M |\eta'|\), this point is contained in \(\tilde{W}\) provided that
The extension of solutions for the Cauchy problem in the complex domain

\[ |x'| \leq \frac{t|\eta_1| + |s||\xi_1|}{|\eta'|} \leq \sqrt{2} M(t + |s|) < A/2 \]

and \( \varepsilon < B/2 \).

(2) Case \( |\xi'| \geq |\eta'| \).

First suppose that \( |\xi''| \leq |\xi'''| \).

The point

\[ x = 0, \quad y_1 = 0, \quad y'' = t\eta_1 + s\xi_1 \]

satisfies (2.3). Since we have

\[ |\xi_1| \leq M |\xi'| \leq \sqrt{2} M |\xi'| \leq 2M |\xi'''| \]

and \( |\eta_1| \leq 2M |\xi'''| \), this point is contained in \( \tilde{W} \) provided that

\[ |y''| \leq \frac{t|\eta_1| + |s||\xi_1|}{|\xi'''|} \leq 2M(t + |s|) < B/2 \]

and \( \varepsilon < B/2 \).

Next consider \( \zeta \in C^n \) such that \( |\xi'''| \geq |\xi''| \) and \( \text{Re} \xi''' \in \Delta^0 \). Then there exists \( \gamma \in \Gamma \) with \( |\gamma| = 1 \) such that

\[ \xi''' \cdot \gamma < -C |\xi'''| \]

where \( C \) is a constant (independent of \( \xi''' \)). The point

\[ x = 0, \quad y_1 = y'' = 0, \quad y''' = y_0 + \frac{t\eta_1 + s\xi_1}{\xi''' \cdot \gamma} \gamma \]

satisfies (2.3). Without loss of generality, we can assume that \( t\eta_1 + s\xi_1 \leq 0 \). (If this is not the case, then we can replace \( \zeta \) by \( -\zeta \).)

Since we have

\[ |\xi_1| \leq M |\xi'| \leq 2M |\xi'''| \]

and, similarly, \( |\eta_1| \leq 2M |\xi'''| \), this point is contained in \( \tilde{W} \) provided that

\[ |y'''| \leq |y_0| + \frac{t|\eta_1| + |s||\xi_1|}{|\xi''' \cdot \gamma|} \leq \varepsilon + \frac{2M(t + |s|)}{C} < B/2. \]

Finally, consider \( \zeta \in C^n \) such that \( |\xi'''| \geq |\xi''| \) and \( \text{Re} \xi''' \in \Delta^0 \). If \( \eta_1 > 0 \), then we have

\[ s\xi_1 \leq t\eta_1 + s\xi_1 \leq 0. \]

Therefore, the point

\[ x = 0, \quad y_1 = \frac{t\eta_1 + s\xi_1}{\xi_1}, \quad y'' = 0, \quad y''' = y_0 \]

satisfies (2.3). This point is contained in \( \tilde{W} \) provided that

\[ |z_1| \leq |s| < k \text{dis}(y_0, \partial \Gamma) \]
and $\varepsilon < B/2$.

If $\eta_1 \leq 0$, by hypothesis, we have a decomposition of form

$$\eta_1 = \alpha + \beta + \gamma$$

where $|\alpha| \leq b|\eta''|$, $|\beta| \leq c|\zeta''|$, $|\gamma| \leq c|\eta''|$. The point

$$x_1 = 0, \quad x'' = \frac{t\gamma}{|\eta''|^2} \eta'', \quad x''' = \frac{t\alpha}{|\eta''|^2} \zeta''',$$

$$y_1 = s, \quad y'' = \frac{t\beta}{|\zeta''|^2} \zeta'', \quad y''' = \gamma_0$$

satisfies (2.3). If $|x'| \leq t(b^2 + c^2)^{1/2} < A/2$, $|y'| \leq tC < B/2$ and $|z_1| = |s| \leq k \text{dis}(\gamma_0, \partial I')$, $\varepsilon < B/4$, then this point is contained in $\hat{W}$.

Now by (1) and (2), if we choose $K, s, \varepsilon > 0$ such that $t + |s| < K$, $|s| \leq k \text{dis}(\gamma_0, \partial I')$, $\varepsilon < B/4$ where

$$K = \min \left\{ \frac{B}{4M}, \frac{BC}{8M}, \frac{A}{2 \sqrt{2} M'}, \frac{A}{2(b^2 + c^2)^{1/2}}, \frac{B}{2C} \right\},$$

then the solution $F(z)$ can be continued up to $\text{ch}[\{z_0 \cup \hat{W}\}]$ for every $t > 0$, $\varepsilon > 0$. When we let them vary under these conditions and make the unions of these convex domains, we clearly obtain a domain of the form $W$.

**Theorem.** Assume that for some constant $b, c > 0$.

$$-\text{Im} \zeta_1 \leq b \text{Im} \zeta'' | + c |\zeta''|,$$

if $\text{Pm}(\zeta) = 0$ and $\text{Re} \zeta'' \in \Delta$. Assume that the hyperfunction data $u_j(x')$, $j = 0, \ldots, m-1$, can be expressed as the boundary values of functions $F_j(z')$ holomorphic in $[\mathbb{R}^{n-1} \times i(\mathbb{R}^{l-1} \times I')] \cap \{|z'| < \delta\}$. Then on a neighborhood of the origin we can solve the following boundary value problem

$$\begin{cases}
P(D)u = 0 \\
\left(\frac{\partial}{\partial x_1}\right)^j u \big|_{x_1 = 0} = u_j(x'), \quad j = 0, \ldots, m-1.
\end{cases}$$

**Proof.** With the initial data $F_j(z')$, we are going to solve the holomorphic Cauchy problem (2.1). Put $A = B = \delta / \sqrt{2}$. Then by Lemma, the holomorphic solution $F(z)$ can be continued to the domain

$$W = \{z \in \mathbb{C}^n | 0 \leq x_1 < A', |x'| \leq A', \lambda |y_1| < \text{dis}(\zeta''', \partial I'),$$

$$|y'| < B', |y''| < B'\}$$

which is a wedge with its edge tangent to the real axis. Thus $F(z)$ continued there defines a hyperfunction solution $u(x)$ of $P(D)u = 0$ on
$x_1 > 0$ locally on a neighborhood of the origin. Moreover, by [3, Lemma 2.6], the boundary values of $u$ agree with the given data $u_j$. Therefore the proof is complete.

Similarly, for the boundary value problem to $x_1 < 0$, we can prove the sufficient condition

$$\text{Im } \zeta_1 \leq b |\text{Im } \zeta'''| + c |\zeta'''| \quad \text{if } P_m(\zeta) = 0, \quad \text{Re } \zeta''' \in \Delta^0$$

for some constant $b, c > 0$.

**Corollary 1.** Assume that for some constant $b, c > 0$,

$$|\text{Im } \zeta_1| \leq b |\text{Im } \zeta'''| + c |\zeta'''|$$

if $P_m(\zeta) = 0$ and $\text{Re } \zeta''' \in \Delta^0$. Assume that the hyperfunction data $u_j(x')$ can be expressed as the boundary values of functions $F_j(x')$ holomorphic in $\{R_n \times i(R_l \times l') \} \cup \{|z'| < \delta\}$. Then the Cauchy problem

$$
\begin{align*}
P(D)u &= 0 \\
\left( \frac{\partial}{\partial x_1} \right)^j u \big|_{x_1 = 0} &= u_j(x'), \quad j = 0, \ldots, m - 1
\end{align*}
$$

admits a hyperfunction solution which contains the same holomorphic parameters.

**Corollary 2.** Let $\Delta \subset S^{n-1}$ be an open subset. Assume that the data $u_j(x')$ contain $x''$ as holomorphic parameters and satisfy

$$S. S. u_j \subset \subset R_n \times \Delta dx'''.
$$

Assume that for any compact subset $L$ of $\Delta$, there exists $b, c > 0$ such that

$$-\text{Im } \zeta_1 \leq b |\text{Im } \zeta'''| + c |\zeta'''|$$

if $P_m(\zeta) = 0$ and $\text{Re } \zeta''' / |\text{Re } \zeta'''| \in L$. Then the Cauchy problem

$$
\begin{align*}
P(D)u &= 0 \\
\left( \frac{\partial}{\partial x_1} \right)^j u \big|_{x_1 = -0} &= u_j(x'), \quad j = 0, \ldots, m - 1
\end{align*}
$$

admits a hyperfunction solution which contains the same holomorphic parameters.

**References**


Seoul National University
Seoul 151-742, Korea