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LIPSCHITZ CONTINUOUS METRIC PROJECTIONS AND SELECTIONS

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1. Introduction

Let X be a normed linear space and M be a subspace of X. The metric projection onto M is the (generally set-valued) mapping $P_M : X \longrightarrow 2^M$ defined by

$$P_M(x) : = \{y \in M : \|x-y\| = d(x, M)\},$$

where $d(x, M) = \inf \{\|x-y\| : y \in M\}$, i. e., $P_M(x)$ is the set of all "best approximations" or "nearest points" to x from M. M is called a *proximinal* subspace if $P_M(x)$ is nonempty for each x. M is called a *Chebyshev* subspace if $P_M(x)$ is a singleton for each x. A metric selection for M is any selection s for P_M, that is, $s : X \longrightarrow M$ and $s(x) \in P_M(x)$ for each $x \in X$. The *kernel* of P_M is the set

$$\text{Ker } P_M : = \{x \in X : 0 \in P_M(x)\}.$$

A metric selection s is homogeneous if $s(\alpha x) = \alpha s(x)$ for each $x \in X$ and $\alpha \in \boldsymbol{R}$. A metric selection s is additive modulo M if $s(x+m) = s(x) + m$ for each $m \in M$ and $x \in X$.

In this article we are interested in characterizing when metric projection is Lipschitz continuous and determining when metric selections which are also Lipschitz continuous exist.

2. Lipschitz continuous metric projections

Through this section, unless otherwise specified, M will denote a proximinal subspace of a normed linear space X and $H(X)$ denote the family of all nonempty closed, bounded, and convex subsets of X.

Define $h : H(X) \times H(X) \longrightarrow \boldsymbol{R}$ by

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$$h(A, B) = \sup_{a \in A} d(a, B),$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$. But h is not a metric on $H(X)$.

DEFINITION 2.1. Define $H : H(X) \times H(X) \longrightarrow \boldsymbol{R}$ by

$$\begin{aligned} H(A, B) &= \max\{h(A, B), h(B, A)\} \\ &= \inf\{\varepsilon > 0 : A \subset B_\varepsilon(B),\ B \subset B_\varepsilon(A)\}. \end{aligned}$$

Then H is called the *Hausdorff metric* on $H(X)$.

Usually we can define the Hausdorff metric on the family of all nonempty closed and bounded subsets of X. We obtain some properties of the Hausdorff metric H.

PROPOSITION 2.2. *Let A and B be any bounded subsets of a metric space* X. *Then*

(1) $H(A, B) = H(\overline{A}, \overline{B})$.

(2) $$\begin{aligned} H(co(A), co(B)) &= H(\overline{co(A)}, \overline{co(B)}) \\ &= H(co(\overline{A}), co(\overline{B})) \\ &= H(\overline{co(\overline{A})}, \overline{co(\overline{B})}) \end{aligned}$$

where $co(A)$ is the convex hull of A and $\overline{A}$ is the closure of A.

(3) $H(A, B) \geqq H(co(A), co(B))$.

Proof. See [7].

R. B. Holmes and B. R. Kripke [5] gave the characterization of Lipschitz continuity of P_M when M is a Chebyshev subspace. Here we give the characterization of Lipschitz continuity of P_M when M is proximinal. We will use an argument similar to Holmes and Kripke's.

PROPOSITION 2.3. *Let M be a proximinal subspace of X. Suppose that for some $x \in X$ there exists $\lambda > 0$ and $\delta > 0$ such that $H(P_M(x), P_M(y)) \leqq \lambda\|x-y\|$ whenever $\|x-y\| < \delta$. Then for all $y \in X$,*

$$H(P_M(x),\ P_M(y)) \leqq \max\left(\lambda,\ 2+\frac{4\|x\|}{\delta}\right)\|x-y\|.$$

Proof. If $\|x-y\| \geqq \delta$, then $\|x\| \leqq \frac{\|x\|}{\delta}\|x-y\|$ and

$$\|y\| \leqq \|x\| + \|x-y\| \leqq \left(1+\frac{\|x\|}{\delta}\right)\|x-y\|.$$

Now for each $z \in X$,

$$\sup_{m \in P_M(z)} \|m\| \leqq \sup_{m \in P_M(z)} \|m-z\| + \|z\| \leqq 2\|z\|.$$

Therefore, we have

$$\begin{aligned} H(P_M(x), P_M(y)) &= \max\{ \sup_{m_x \in P_M(x)} d(m_x, P_M(y)), \sup_{m_y \in P_M(y)} d(m_y, P_M(x))\} \\ &\leqq \sup\{\|m_x - m_y\| : m_x \in P_M(x), m_y \in P_M(y)\} \\ &\leqq \sup\{\|m_x\| + \|m_y\| : m_x \in P_M(x), m_y \in P_M(y)\} \\ &\leqq \sup_{m_x \in P_M(x)} \|m_x\| + \sup_{m_y \in P_M(y)} \|m_y\| \\ &\leqq 2\|x\| + 2\|y\| \\ &\leqq \left(2 + \frac{4\|x\|}{\delta}\right)\|x-y\|. \end{aligned}$$

COROLLARY 2.4. *Let M be a proximinal subspace of X. Assume that there exists λ, $\delta > 0$ such that if $x \in Ker\, P_M$, $\|x\| = 1$ and $\|x-y\| < \delta$, then*

$$H(P_M(x), P_M(y)) \leqq \lambda\|x-y\|.$$

Then P_M is Lipschitz continuous.

Proof. If $x \in M$, then for each $y \in X$

$$\begin{aligned} H(P_M(x), P_M(y)) &= H(\{x\}, P_M(y)) \\ &= \max\{d(x, P_M(y)), \sup_{m \in P_M(y)} \|m - x\|\} \\ &= \sup_{m \in P_M(y)} \|m - x\| \\ &\leqq \sup_{m \in P_M(y)} (\|x-y\| + \|y-m\|) \\ &= \|x-y\| + \sup_{m \in P_M(y)} \|y-m\| \\ &\leqq \|x-y\| + \|y-x\| = 2\|x-y\|. \end{aligned}$$

If $x \notin M$, then putting $m \in P_M(x)$ and $c = \|x-m\|$, by Proposition 2.3,

$$\begin{aligned} H(P_M(x), P_M(y)) &= H(cP_M(c^{-1}x), cP_M(c^{-1}y)) \\ &= cH(P_M(c^{-1}x), P_M(c^{-1}y)) \\ &= cH(P_M(c^{-1}(x-m)), P_M(c^{-1}(y-m))) \\ &\leqq c\max\left(\lambda, 2 + \frac{4}{\delta}\right)\|c^{-1}(x-m) - c^{-1}(y-m)\| \\ &= \max\left(\lambda, 2 + \frac{4}{\delta}\right)\|x-y\|. \end{aligned}$$

Thus P_M is Lipschitz continuous.

COROLLARY 2.5. *Let M be a proximinal subspace of a normed linear space X. Then the following statements are equivalent:*

(1) *P_M is Lipschitz continuous.*

(2) *P_M is "uniformly locally pointwise Lipschitz continuous", i.e., there exist two constants $\lambda = \lambda(M) > 0$ and $\delta = \delta(M) > 0$ such that $x \in Ker P_M$, $\|x\| = 1$, and $\|x-y\| \leqq \delta$ imply $H(P_M(x), P_M(y)) \leqq \lambda\|x-y\|$.*

Proof. (1)$\Rightarrow$(2) is obvious.

(2)$\Rightarrow$(1) follows from Corollary 2. 4.

Now we characterize Lipschitz continuity. By this characterization, we will prove that Lipschitz continuity and uniform continuity are equivalent when a set-valued mapping is positive homogeneous.

THEOREM 2. 6. *Let X and Y be normed linear spaces and a set valued mapping $F: X \longrightarrow H(Y)$ have bounded, closed, and convex images which is positive homogeneous. The following statements are equivalent*:

(1) *F is Lipschitz continuous.*

(2) *There exists $\lambda>0$ such that for each $\varepsilon>0$, for any $x, y\in X$ with $\|x-y\|<\varepsilon$, for each $a\in F(x)$ there exists $b\in F(y)$ such that $\|a-b\|<\lambda\varepsilon$.*

(3) *There exist $\lambda>0$, $\delta>0$ such that for any $x, y\in X$ with $\|x-y\|<\delta$, for each $a\in F(x)$ there exists $b\in F(y)$ such that $\|a-b\|<\lambda\delta$.*

(4) *F is uniformly continuous.*

Proof. (1)$\Rightarrow$(4) is obvious.

(4)$\Rightarrow$(3) Suppose that (4) holds. Then there exists $\delta>0$ such that $\|x-y\|<\delta$ implies $H(F(x), F(y))<1=\delta^{-1}\delta$. Set $\lambda=\delta^{-1}$. Then for any $x, y\in X$ with $\|x-y\|<\delta$, $H(F(x), F(y))<\lambda\delta$. Thus (3) holds.

(3)$\Rightarrow$(2) Suppose that (3) holds. Let $\varepsilon>0$ be given. Since $\varepsilon>0$ and $\delta>0$, there exists $\alpha>0$ such that $\varepsilon=\alpha\delta$. Let $x', y'\in X$ be with $\|x'-y'\|<\varepsilon$. Then $\left\|\frac{x'}{\alpha}-\frac{y'}{\alpha}\right\|<\frac{\varepsilon}{\alpha}=\delta$. By (3), for each $a\in F\left(\frac{x'}{\alpha}\right)$ there exists $b\in F\left(\frac{y'}{\alpha}\right)$ such that $\|a-b\|<\lambda\delta$. Thus for each $\alpha a\in F(x')$ there exists $\alpha b\in F(y')$ such that $\|\alpha a-\alpha b\|<\alpha(\lambda\delta)=\lambda(\alpha\delta)=\lambda\varepsilon$. Therefore there exists $\lambda>0$ such that for each $\varepsilon>0$ for any x, $y\in X$ with $\|x-y\|<\varepsilon$, for each $a\in F(x)$ there exists $b\in F(y)$ such that $\|a-b\|<\lambda\varepsilon$.

(2)$\Rightarrow$(1) Suppose that (2) holds. Let $x, y\in X$ be given and $\|x-y\|=l$. For each $n\in \boldsymbol{N}$, $\|x-y\|<l+\frac{1}{n}$. By (2), for each $a\in F(x)$ there exists $b\in F(y)$ such that $\|a-b\|<\lambda\left(l+\frac{1}{n}\right)$ for any $n\in \boldsymbol{N}$. Then $d(a, F(y))\leqq\|a-b\|<\lambda\left(l+\frac{1}{n}\right)$ for any $n\in \boldsymbol{N}$ and for any $a\in F(x)$. So

$$\sup_{a\in F(x)} d(a, F(y))\leqq\lambda l=\lambda\|x-y\|.$$

By the symmetric argument, $h(F(y), F(x))\leqq\lambda\|x-y\|$. Thus

$$H(F(x), F(y))\leqq\lambda\|x-y\|.$$

Since $x, y\in$ X were arbitrary, F is Lipschitz continuous.

Now we obtain the main result in this section.

Corollary 2.7. *Let M be a proximinal subspace of a normed linear space X. The following statements are equivalent*:

(1) *P_M is Lipschitz continuous.*

(2) *There exists $\lambda>0$ such that for each $\varepsilon>0$ for any $x, y\in X$ with $\|x-y\|<\varepsilon$, for each $a\in P_M(x)$ there exists $b\in P_M(y)$ such that $\|a-b\|<\lambda\varepsilon$, i.e., $H(P_M(x), P_M(y))<\lambda\varepsilon$.*

(3) *There exist $\lambda>0$, $\delta>0$ such that for any $x, y\in X$ with $\|x-y\|<\delta$, for each $a\in P_M(x)$ there exists $b\in P_M(y)$ such that $\|a-b\|<\lambda\delta$, i.e. $H(P_M(x), P_M(y))<\lambda\delta$.*

(4) *P_M is uniformly continuous.*

Proof. Since P_M is homogeneous, by Theorem 2.6, (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3)$\Leftrightarrow$(4).

Remark. When M is Chebyshev, (1)$\Leftrightarrow$(4) was proven by R. B. Holmes and B. R. Kripke [5].

Now we will define a set-valued mapping φ_M from X/M into $2^{P_M^{-1}(0)}$. We will find the relationship between P_M and φ_M for Lipschitz continuity.

Definition 2.8. [2] Let M be a proximinal subspace of a normed linear space X. Define a set-valued mapping $\varphi_M : X \longrightarrow 2^{P_M^{-1}(0)}$ by $\varphi_M(x+M)=x-P_M(x)$ for each $x+M\in X/M$. Then φ_M is well-defined, since P_M is additive modulo M and M is proximinal. Moreover, $\varphi_M(x+M)$ is nonempty bounded, closed, and convex subset of Ker P_M.

Theorem 2.9. *Let M be a proximinal subspace of a normed linear space X. The following statements are equivalent*:

(1) *P_M is Lipschitz continuous.*

(2) *$I-P_M$ is Lipschitz continuous.*

(3) *φ_M is Lipschitz continuous.*

Proof. (1)$\Rightarrow$(2) is clear.

(2)$\Rightarrow$(3) Suppose that (2) holds, that is, there exists $\lambda>0$ such that $H(x-P_M(x), y-P_M(y))\leqq\lambda\|x-y\|$ for any $x, y\in X$. Note that

$$\begin{aligned}\|(x+M)-(y+M)\|_{X/M}&=\|x-y+M\|\\&=\|x-y+m_0\|\end{aligned}$$

for some $m_0\in M$. Let $x'=x-m_0$ and $y'=y$. Then

$$\begin{aligned} H(\varphi_M(x+M), \varphi_M(y+M)) &= H(x-P_M(x), y-P_M(y)) \\ &= H(x'-P_M(x'), y'-P_M(y')) \\ &\leqq \lambda\|x'-y'\| \\ &= \lambda\|x-y-m_0\| \\ &= \lambda\|(x+M)-(y+M)\|_{X/M}. \end{aligned}$$

Thus (3) holds.

(3⇒(1) Suppose (3) holds, that is, there exists $\lambda>0$ such that

$$H(\varphi_M(x+M), \varphi_M(y+M)) \leqq \lambda\|(x+M)-(y+M)\|_{X/M}$$

for any $x+M$, $y+M\in X/M$. Since $\|(x+M)-(y-M)\|_{X/M}\leqq\|x-y\|$,

$$\begin{aligned} H(P_M(x), P_M(y)) &\leqq \|x-y\| + H(x-P_M(x), y-P_M(y)) \\ &= \|x-y\| + H(\varphi_M(x+M), \varphi_M(y+M)) \\ &\leqq \|x-y\| + \lambda\|(x+M)-(y+M)\|_{X/M} \\ &\leqq (1+\lambda)\|x-y\|. \end{aligned}$$

Thus P_M is Lipschitz continuous.

By using Proposition 2. 2, we can verify the following two propositions.

Proposition 2. 10. *Let X and Y be normed linear spaces. If a mapping $F: X \longrightarrow 2^Y$ has bounded images and is Lipschitz continuous, and if a mapping $G: X \longrightarrow 2^Y$ is such that $\overline{G(x)}=\overline{F(x)}$ for every $x\in X$, then G is Lipschitz continuous.*

Proposition 2. 11. *Let X and Y be normed linear spaces. If $F: X\longrightarrow 2^Y$ is Lipschitz continuous, then $co(F)$ is also Lipschitz continuous.*

3. Lipschitz continuous selections for a metric projection

F. Deutsch gave the following open problem in [4].

Open Problem 3. 1. If M is a n-dimensional subspace of X and P_M is Lipschitz continuous, must P_M have a Lipschitz continuous selection?

Let $\langle x, y\rangle$ denote the inner product of x, y in Euclidean n-space E^n.

Definition 3. 2. [8] For $A\in H(E^n)$, the STEINER point of A is defined by

$$s(A) = n\int_{S^{n-1}} q(A, x)x dx$$

where S^{n-1} is the unit sphere of E^n equipped with unit Lebesgue measure dx and $q(A, x)=\sup_{a\in A}\langle a, x\rangle$ is the support function. The mapping $s: H(E^n) \longrightarrow E^n$ is called the Steiner map.

By using basic properties of support function and the definition of the Steiner point, K. Przeslawski proved the following Theorem.

THEOREM 3.3. [8]
(1) $s(\{x\})=x$ *for* $x\in E^n$;
(2) $s(A+B)=s(A)+s(B)$ *for every* $A, B\in H(E^n)$;
(3) $s(UA)=Us(A)$ *for every* $A\in H(E^n)$, $U\in O(E^n)$ *where* $O(E^n)$ *is the group of orthogonal operators of* E^n;
(4) $s(\alpha A)=\alpha s(A)$ *for every* $A\in H(E^n), \alpha\in \boldsymbol{R}$.
(5) $s(A)\in A$, *for every* $A\in H(E^n)$.

R. A. Vitale [9] proved: In E^n,

$$\sup_{K\neq K'}\frac{\|s(K)-s(K')\|}{H(K,K')}=\frac{\Gamma\left(\frac{n}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}.$$

K. Przeslawski [8] announced the following theorem without proof.

THEOREM 3.4. [8, 9]. *The Steiner map* $s : H(E^n)\longrightarrow E^n$ *is Lipschitz continuous.*

LEMMA 3.5. *If* X *and* Y *are topologically isomorphic, then there is a one-to-one correspondence between* $H(X)$ *and* $H(Y)$.

Proof. Let $\Phi : X\longrightarrow Y$ be an isomorphism from X onto Y. Then
$$\|\Phi^{-1}\|^{-1}\|x\|\leqq\|\Phi(x)\|\leqq\|\Phi\|\|x\|$$
for each $x\in X$. Let $A\in H(X)$. Then $\Phi(A) := \{\Phi(a) : a\in A\}\in H(Y)$. Let $B\in H(Y)$. Then
$$A=\Phi^{-1}(B) := \{a\in X : \Phi(a)\in B\} = \{\Phi^{-1}(b) : b\in B\}\in H(X)$$
and $\Phi(A)=B$. Thus Φ is a one-to-one correspondence between $H(X)$ and $H(Y)$.

THEOREM 3.6. *Let* M *be an* n*-dimensional subspace of a normed linear space* X. *Then there is a Lipschitz continuous selection from* $H(M)$ *to* M.

Proof. By Theorem 3.4, there is a Lipschitz continuous selection from $H(E^n)$ to E^n. Since M and E^n are n-dimensional, M and E^n are topologically isomorphic. Thus there is a topological isomorphism φ from M onto E^n. Furthermore, we can identify φ as a mapping from $H(M)$ onto $H(E^n)$. Let $s'=\varphi^{-1}\circ s\circ\varphi : H(M)\longrightarrow H(E^n)\longrightarrow E^n\longrightarrow M$. Since $s(\varphi(A))\in\varphi(A)$, $s'(A)=\varphi^{-1}(s(\varphi(A)))\in A$. Then
$$\|s'(A)-s'(B)\|_M=\|\varphi^{-1}(s(\varphi(A)))-\varphi^{-1}(s(\varphi(B)))\|_M$$

$$\begin{aligned}&\leqq \|\varphi^{-1}\| \|s(\varphi(A))-s(\varphi(B))\|_{E^n}\\&\leqq \|\varphi^{-1}\|\lambda H(\varphi(A),\varphi(B))_{E^n}\\&\leqq \|\varphi^{-1}\|\|\varphi\|\lambda H(A, B)_M\end{aligned}$$

where λ is a Lipschitz constant of s since

$$\begin{aligned}H(\varphi(A),\varphi(B))_{E^n}&=\max\{\sup_{a\in A} d(\varphi(a),\varphi(B)),\ \sup_{b\in B} d(\varphi(b),\ \varphi(A))\}_{E^n}\\&\leqq \|\varphi\| \max\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)\}_M\\&=\|\varphi\| H(A, B)_M.\end{aligned}$$

Thus there is a Lipschitz continuous selection from $H(M)$ to M.

Now we can prove Open Problem 3.1 by using Theorem 3.6.

COROLLARY 3.7. *Let M be a finite dimensional subspace of a normed linear space X. If P_M is pointwise Lipschitz continuous, then P_M has a pointwise Lipschitz continuous selection which is homogeneous and additive modulo M.*

Proof. Let $x_0\in X$ be given. Since P_M is pointwise Lipschitz continuous at x_0, there exists $\lambda_{x_0}>0$ such that

$$H(P_M(x_0), P_M(x))\leqq \lambda_{x_0}\|x_0-x\|$$

for each $x\in X$. By Theorem 3.6, there is a Lipschitz continuous selection s' : $H(M)\longrightarrow M$. Thus

$$\begin{aligned}\|s'(P_M(x_0))-s'(P_M(x))\|&\leqq \lambda_s H(P_M(x_0), P_M(x))\\&\leqq \lambda_s\lambda_{x_0}\|x_0-x\|\end{aligned}$$

for each $x\in X$. Hence $p=s'\circ P_M$ is a pointwise Lipschitz continuous selection for P_M. Since P_M and s' are homogeneous and additive modulo M, p is homogeneous and additive modulo M.

COROLLARY 3.8. *Let M be a finite dimensional subspace of a normed linear space X. If P_M is Lipschitz continuous, then P_M has a Lipschitz continuous selection which is homogeneous and additive modulo M.*

Proof. We can prove it the same argument as the proof of Theorem 3.7. Define $p: X\longrightarrow M$ by $p(x)=s'(P_M(x))$ for any $x\in X$. Then p is a Lipschitz continuous selection which is homogeneous and additive modulo M since P_M and s' are homogeneous and additive modulo M.

The converse of Corollary 3.8 is false. But we can prove the following:

THEOREM 3.9. *Let M be a proximinal subspace of a normed linear space X. If $P_M|_{\mathrm{Ker}P_M}$ is Lipschitz continuous and P_M has a Lipschitz continuous selection, then P_M is Lipschitz continuous.*

Proof. Since $P_M|_{\mathrm{Ker}P_M}$ is Lipschitz continuous, there exists $\lambda_1>0$ such that $H(P_M(x), P_M(y))\leqq\lambda_1\|x-y\|$ for any $x, y\in\mathrm{Ker}\,P_M$. Let s be a Lipschitz continuous selection for P_M with the Lipschitz constant λ_2. Let $x, y\in X$. Then $x-P_M(x), y-P_M(y)\subset\mathrm{Ker}\,P_M$. Thus

$$\begin{aligned}H(P_M(x), P_M(y)) &= H(P_M(x-s(x)+s(x)), P_M(y-s(y)+s(y)))\\ &\leqq H(P_M(x-s(x)), P_M(y-s(y)))+\|s(x)-s(y)\|\\ &\leqq \lambda_1\|x-s(x)-y+s(y)\|+\lambda_2\|x-y\|\\ &\leqq(\lambda_1+\lambda_2+\lambda_1\lambda_2)\|x-y\|.\end{aligned}$$

Thus P_M is Lipschitz continuous.

Remark. In Theorem 3. 9, neither of the conditions that P_M has a Lipschitz continuous selection or $P_M|_{\mathrm{Ker}P_M}$ is Lipschitz continuous can be dropped. We have the following examples.

Examples 3. 10. (1) (There are examples that $P_M|_{\mathrm{Ker}\,P_M}$ is Lipschitz continuous but P_M is not Lipschitz continuous). In $C[a, b]$, by Cline's results [1], for any 2-dimensional Haar subspace M of $C[a, b]$, P_M is not Lipschitz continuous but $P_M|_{\mathrm{Ker}\,P_M}\equiv 0$.

(2) [3] (There is an example that P_M has a Lipschitz continuous selection, but is not Lipschitz continuous). Let $X=\boldsymbol{R}^3$ be with the norm $\|x\|=|x(1)|+\sqrt{x(2)^2+x(3)^2}$, $x=(x(1), x(2), (3))$ and $x_1=(1, 1, 0)$. Then $P_{[x_1]}$ has a Lipschitz continuous (even linear!) selection without P_M being Lipschitz continuous.

Corollary 3. 11. *Let M be a finite dimensional subspace of a normed linear space X. The following statements are equivalent*:

(1) *P_M is Lipschitz continuous.*

(2) *$P_M|_{\mathrm{Ker}\,P_M}$ is Lipschitz continuous and P_M has a Lipschitz continuous selection.*

Proof. It follows from Corollary 3. 8 and Theorem 3. 9.

4. Applications

Let M be a closed subspace of a Banach space X. D. Yost [19] clarified the relationship between approximative properties of M and intersection properties of balls pertaining to M.

Definition 4. 1. [10] We say that M has the $1\frac{1}{2}$*-ball property* in X if the conditions $m\in M$, $x\in X$, $r_1, r_2>0$, $M\cap B(x, r_2)\neq\phi$, and $\|x-m\|$

$< r_1 + r_2$ imply that

$$M \cap B(m, r_1) \cap B(x, r_2) \neq \phi.$$

Here $B(x, r) := \{y \in X : \|x - y\| \leqq r\}$ is a closed ball.

In [10] and [11], we can find lots of examples of subspaces which have the $1\frac{1}{2}$-ball property.

THEOREM 4.2. [10] *Suppose that M has the $1\frac{1}{2}$-ball property in X. Then* (1) *M is proximinal.*
(2) *P_M is Lipschitz continuous.*

D. Yost gave the following.

THEOREM 4.3. [10] *If M has the $1\frac{1}{2}$-ball property in X, then there is a continuous selection which is homogeneous and additive modulo M.*

The following strengthens Yost's theorem when M is finite-dimensional.

THEOREM 4.4. *If M is a finite-dimensional subspace which has the $1\frac{1}{2}$-ball property in X, then P_M has a Lipschitz continuous selection which is homogeneous and additive modulo M.*

Proof. By Theorem 4.3, P_M is Lipschitz continuous. By Corollary 3.10, P_M has a Lipschitz continuous selection which is homogeneous and additive modulo M.

DEFINITION 4.5. [6] Let M be a proximinal subspace of X. The metric projection P_M is said to be uniform Hausdorff strongly unique if there exists a constant $r > 0$ such that

$$\|x - m\| \geqq d(x, M) + r d(m, P_M(x))$$

for each $x \in X$ and $m \in M$.

In [6], we showed that if M has the $1\frac{1}{2}$-ball property, then P_M is uniform Hausdorff strongly unique, but the converse is not true.

THEOREM 4.6. [6] *Let M be a proximinal subspace of X. If P_M is uniform Hausdorff strongly unique, then P_M is Lipschitz continuous.*

COROLLARY 4.7. *Let M be a finite-dimensional subspace of a normed linear space X. If P_M is uniform Hausdorff strongly unique, then P_M has a Lipschitz continuous selection.*

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