LIPSCHITZ CONTINUOUS METRIC PROJECTIONS 
AND SELECTIONS

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1. Introduction

Let $X$ be a normed linear space and $M$ be a subspace of $X$. The metric projection onto $M$ is the (generally set-valued) mapping $P_M : X \rightarrow 2^M$ defined by

$$P_M(x) := \{ y \in M : \| x - y \| = d(x, M) \},$$

where $d(x, M) = \inf \{ \| x - y \| : y \in M \}$, i.e., $P_M(x)$ is the set of all "best approximations" or "nearest points" to $x$ from $M$. $M$ is called a proximinal subspace if $P_M(x)$ is nonempty for each $x$. $M$ is called a Chebyshev subspace if $P_M(x)$ is a singleton for each $x$. A metric selection for $M$ is any selection $s$ for $P_M$, that is, $s : X \rightarrow M$ and $s(x) \in P_M(x)$ for each $x \in X$. The kernel of $P_M$ is the set

$$\text{Ker } P_M := \{ x \in X : 0 \in P_M(x) \}.$$

A metric selection $s$ is homogeneous if $s(\alpha x) = \alpha s(x)$ for each $x \in X$ and $\alpha \in \mathbb{R}$. A metric selection $s$ is additive modulo $M$ if $s(x + m) = s(x) + m$ for each $m \in M$ and $x \in X$.

In this article we are interested in characterizing when metric projection is Lipschitz continuous and determining when metric selections which are also Lipschitz continuous exist.

2. Lipschitz continuous metric projections

Through this section, unless otherwise specified, $M$ will denote a proximinal subspace of a normed linear space $X$ and $H(X)$ denote the family of all nonempty closed, bounded, and convex subsets of $X$.

Define $h : H(X) \times H(X) \rightarrow \mathbb{R}$ by

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\[ h(A, B) = \sup_{a \in A} d(a, B), \]
where \( d(a, B) = \inf \{ d(a, b) : b \in B \} \). But \( h \) is not a metric on \( H(X) \).

**Definition 2.1.** Define \( H : H(X) \times H(X) \longrightarrow \mathbb{R} \) by
\[
H(A, B) = \max \{ h(A, B), h(B, A) \} \\
= \inf \{ \varepsilon > 0 : A \subseteq B_{\varepsilon}(B), \ B \subseteq B_{\varepsilon}(A) \}.
\]
Then \( H \) is called the Hausdorff metric on \( H(X) \).

Usually we can define the Hausdorff metric on the family of all nonempty closed and bounded subsets of \( X \). We obtain some properties of the Hausdorff metric \( H \).

**Proposition 2.2.** Let \( A \) and \( B \) be any bounded subsets of a metric space \( X \). Then
1. \( H(A, B) = H(\overline{A}, \overline{B}) \).
2. \( H(\text{co}(A), \text{co}(B)) = H(\overline{\text{co}(A)}, \overline{\text{co}(B)}) \)
   \[
   = H(\overline{\text{co}(A)}, \overline{\text{co}(B)}) \\
   = H(\text{co}(\overline{A}), \text{co}(\overline{B}))
   \]
where \( \text{co}(A) \) is the convex hull of \( A \) and \( \overline{A} \) is the closure of \( A \).
3. \( H(A, B) \geq H(\text{co}(A), \text{co}(B)) \).

**Proof.** See [7].

R.B. Holmes and B.R. Kripke [5] gave the characterization of Lipschitz continuity of \( P_M \) when \( M \) is a Chebyshev subspace. Here we give the characterization of Lipschitz continuity of \( P_M \) when \( M \) is proximinal. We will use an argument similar to Holmes and Kripke’s.

**Proposition 2.3.** Let \( M \) be a proximinal subspace of \( X \). Suppose that for some \( x \in X \) there exists \( \lambda > 0 \) and \( \delta > 0 \) such that \( H(P_M(x), P_M(y)) \leq \lambda \| x - y \| \) whenever \( \| x - y \| < \delta \). Then for all \( y \in X \),
\[
H(P_M(x), P_M(y)) \leq \max \left( \lambda, 2 + \frac{4\|x\|}{\delta} \right) \| x - y \|.
\]

**Proof.** If \( \| x - y \| \geq \delta \), then \( \| x \| \leq \frac{\| x \|}{\delta} \| x - y \| \) and
\[
\| y \| \leq \| x \| + \| x - y \| \leq \left( 1 + \frac{\| x \|}{\delta} \right) \| x - y \|.
\]
Now for each \( z \in X \),
\[
\sup_{m \in P_M(z)} \| m \| \leq \sup_{m \in P_M(z)} \| m - z \| + \| z \| \leq 2\| z \|.
\]
Therefore, we have
\[ H(P_M(x), P_M(y)) = \max \{ \sup_{m_x \in P_M(x)} d(m_x, P_M(y)), \sup_{m_y \in P_M(y)} d(m_y, P_M(x)) \} \]
\[ \leq \sup \{ \| m_x - m_y \| : m_x \in P_M(x), m_y \in P_M(y) \} \]
\[ \leq \sup \{ \| m_x \| + \| m_y \| : m_x \in P_M(x), m_y \in P_M(y) \} \]
\[ \leq \sup_{m_x \in P_M(x)} \| m_x \| + \sup_{m_y \in P_M(y)} \| m_y \| \]
\[ \leq 2\| x \| + 2\| y \| \]
\[ \leq \left( 2 + \frac{4\| x \|}{\delta} \right) \| x - y \|. \]

**Corollary 2.4.** Let \( M \) be a proximinal subspace of \( X \). Assume that there exists \( \lambda, \delta > 0 \) such that if \( x \in \text{Ker} P_M, \| x \| = 1 \) and \( \| x - y \| < \delta \), then \[ H(P_M(x), P_M(y)) \leq \lambda \| x - y \|. \]

Then \( P_M \) is Lipschitz continuous.

**Proof.** If \( x \in M \), then for each \( y \in X \)
\[ H(P_M(x), P_M(y)) = H(\{ x \}, P_M(y)) \]
\[ = \max \{ d(x, P_M(y)), \sup_{m \in P_M(y)} \| m - x \| \} \]
\[ = \sup_{m \in P_M(y)} \| m - x \| \]
\[ \leq \sup_{m \in P_M(y)} (\| x - y \| + \| y - m \|) \]
\[ = \| x - y \| + \sup_{m \in P_M(y)} \| y - m \| \]
\[ \leq \| x - y \| + \| y - x \| = 2\| x - y \|. \]

If \( x \notin M \), then putting \( m \in P_M(x) \) and \( c = \| x - m \| \), by Proposition 2.3,
\[ H(P_M(x), P_M(y)) = H(cP_M(c^{-1}x), cP_M(c^{-1}y)) = cH(P_M(c^{-1}x), P_M(c^{-1}y)) \]
\[ = cH(P_M(c^{-1}(x - m)), P_M(c^{-1}(y - m))) \]
\[ \leq c \max \left( \lambda, 2 + \frac{4}{\delta} \right) \| c^{-1}(x - m) - c^{-1}(y - m) \| \]
\[ = \max \left( \lambda, 2 + \frac{4}{\delta} \right) \| x - y \|. \]

Thus \( P_M \) is Lipschitz continuous.

**Corollary 2.5.** Let \( M \) be a proximinal subspace of a normed linear space \( X \). Then the following statements are equivalent:

(1) \( P_M \) is Lipschitz continuous.

(2) \( P_M \) is "uniformly locally pointwise Lipschitz continuous", i.e., there exist two constants \( \lambda = \lambda(M) > 0 \) and \( \delta = \delta(M) > 0 \) such that \( x \in \text{Ker} P_M, \| x \| = 1, \) and \( \| x - y \| \leq \delta \) imply \( H(P_M(x), P_M(y)) \leq \lambda \| x - y \| \).

**Proof.** (1) \( \Rightarrow \) (2) is obvious.
(2) $\Rightarrow$ (1) follows from Corollary 2.4.

Now we characterize Lipschitz continuity. By this characterization, we will prove that Lipschitz continuity and uniform continuity are equivalent when a set-valued mapping is positive homogeneous.

**Theorem 2.6.** Let $X$ and $Y$ be normed linear spaces and a set valued mapping $F : X \to H(Y)$ have bounded, closed, and convex images which is positive homogeneous. The following statements are equivalent:

1. $F$ is Lipschitz continuous.
2. There exists $\lambda > 0$ such that for each $\varepsilon > 0$, for any $x, y \in X$ with $\|x - y\| < \varepsilon$, for each $a \in F(x)$ there exists $b \in F(y)$ such that $\|a - b\| < \lambda \varepsilon$.
3. There exist $\lambda > 0$, $\delta > 0$ such that for any $x, y \in X$ with $\|x - y\| < \delta$, for each $a \in F(x)$ there exists $b \in F(y)$ such that $\|a - b\| < \lambda \delta$.
4. $F$ is uniformly continuous.

**Proof.** (1) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (3) Suppose that (4) holds. Then there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies $H(F(x), F(y)) < 1 = \delta^{-1}$. Set $\lambda = \delta^{-1}$. Then for any $x, y \in X$ with $\|x - y\| < \delta$, $H(F(x), F(y)) < \lambda \delta$. Thus (3) holds.

(3) $\Rightarrow$ (2) Suppose that (3) holds. Let $\varepsilon > 0$ be given. Since $\varepsilon > 0$ and $\delta > 0$, there exists $\alpha > 0$ such that $\varepsilon = \alpha \delta$. Let $x', y' \in X$ be with $\|x' - y'\| < \varepsilon$. Then $\left\| \frac{x'}{\alpha} - \frac{y'}{\alpha} \right\| < \frac{\varepsilon}{\alpha} = \delta$. By (3), for each $a \in F\left(\frac{x'}{\alpha}\right)$ there exists $b \in F\left(\frac{y'}{\alpha}\right)$ such that $\|a - b\| < \lambda \delta$. Thus for each $\alpha a \in F(x')$ there exists $\alpha b \in F(y')$ such that $\|\alpha a - \alpha b\| < \alpha (\lambda \delta) = \lambda (\alpha \delta) = \lambda \varepsilon$. Therefore there exists $\lambda > 0$ such that for each $\varepsilon > 0$ for any $x, y \in X$ with $\|x - y\| < \varepsilon$, for each $a \in F(x)$ there exists $b \in F(y)$ such that $\|a - b\| < \lambda \varepsilon$.

(2) $\Rightarrow$ (1) Suppose that (2) holds. Let $x, y \in X$ be given and $\|x - y\| = l$.

For each $n \in N$, $\|x - y\| < l + \frac{1}{n}$. By (2), for each $a \in F(x)$ there exists $b \in F(y)$ such that $\|a - b\| < \lambda \left( l + \frac{1}{n} \right)$ for any $n \in N$. Then $d(a, F(y)) \leq \|a - b\| < \lambda \left( l + \frac{1}{n} \right)$ for any $n \in N$ and for any $a \in F(x)$. So $\sup_{a \in F(x)} d(a, F(y)) \leq \lambda l = \lambda \|x - y\|$. By the symmetric argument, $h(F(y), F(x)) \leq \lambda \|x - y\|$. Thus $H(F(x), F(y)) \leq \lambda \|x - y\|$. Since $x, y \in X$ were arbitrary, $F$ is Lipschitz continuous.
Now we obtain the main result in this section.

**Corollary 2.7.** Let $M$ be a proximinal subspace of a normed linear space $X$. The following statements are equivalent:

1. $P_M$ is Lipschitz continuous.
2. There exists $\lambda > 0$ such that for each $\varepsilon > 0$ for any $x, y \in X$ with $\|x - y\| < \varepsilon$, for each $a \in P_M(x)$ there exists $b \in P_M(y)$ such that $\|a - b\| < \lambda \varepsilon$, i.e., $H(P_M(x), P_M(y)) < \lambda \varepsilon$.
3. There exist $\lambda > 0$, $\delta > 0$ such that for any $x, y \in X$ with $\|x - y\| < \delta$, for each $a \in P_M(x)$ there exists $b \in P_M(y)$ such that $\|a - b\| < \lambda \delta$, i.e., $H(P_M(x), P_M(y)) < \lambda \delta$.
4. $P_M$ is uniformly continuous.

**Proof.** Since $P_M$ is homogeneous, by Theorem 2.6, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

**Remark.** When $M$ is Chebyshev, (1) $\Leftrightarrow$ (4) was proven by R. B. Holmes and B. R. Kripke [5].

Now we will define a set-valued mapping $\varphi_M$ from $X/M$ into $2^{P_M'}(0)$. We will find the relationship between $P_M$ and $\varphi_M$ for Lipschitz continuity.

**Definition 2.8.** [2] Let $M$ be a proximinal subspace of a normed linear space $X$. Define a set-valued mapping $\varphi_M : X \rightarrow 2^{P_M'}(0)$ by $\varphi_M(x + M) = x - P_M(x)$ for each $x + M \in X/M$. Then $\varphi_M$ is well-defined, since $P_M$ is additive modulo $M$ and $M$ is proximinal. Moreover, $\varphi_M(x + M)$ is nonempty bounded, closed, and convex subset of Ker $P_M$.

**Theorem 2.9.** Let $M$ be a proximinal subspace of a normed linear space $X$. The following statements are equivalent:

1. $P_M$ is Lipschitz continuous.
2. $I - P_M$ is Lipschitz continuous.
3. $\varphi_M$ is Lipschitz continuous.

**Proof.** (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Suppose that (2) holds, that is, there exists $\lambda > 0$ such that $H(x - P_M(x), y - P_M(y)) \leq \lambda \|x - y\|$ for any $x, y \in X$. Note that

$$\|(x + M) - (y + M)\|_{X/M} = \|x - y + M\| = \|x - y + m_0\|$$

for some $m_0 \in M$. Let $x' = x - m_0$ and $y' = y$. Then
\[ H(\varphi_M(x+M), \varphi_M(y+M)) = H(x-P_M(x), y-P_M(y)) = H(x'-P_M(x'), y'-P_M(y')) \leq \lambda \|x' - y'\| = \lambda \|x - y - m_0\| = \lambda \|(x+M) - (y+M)\|_{X/M}. \]

Thus (3) holds.

(3) Suppose (3) holds, that is, there exists \( \lambda > 0 \) such that
\[ H(\varphi_M(x+M), \varphi_M(y+M)) \leq \lambda \|(x+M) - (y+M)\|_{X/M} \]
for any \( x+M, y+M \in X/M \). Since \( \|(x+M) - (y+M)\|_{X/M} \leq \|x-y\| \),
\[ H(P_M(x), P_M(y)) \leq \|x-y\| + H(x-P_M(x), y-P_M(y)) = \|x-y\| + H(\varphi_M(x+M), \varphi_M(y+M)) \leq \|x-y\| + \lambda \|(x+M) - (y+M)\|_{X/M} \leq (1+\lambda)\|x-y\|. \]

Thus \( P_M \) is Lipschitz continuous.

By using Proposition 2.2, we can verify the following two propositions.

**Proposition 2.10.** Let \( X \) and \( Y \) be normed linear spaces. If a mapping \( F : X \rightarrow 2^Y \) has bounded images and is Lipschitz continuous, and if a mapping \( G : X \rightarrow 2^Y \) is such that \( G(x) = F(x) \) for every \( x \in X \), then \( G \) is Lipschitz continuous.

**Proposition 2.11.** Let \( X \) and \( Y \) be normed linear spaces. If \( F : X \rightarrow 2^Y \) is Lipschitz continuous, then \( \text{co}(F) \) is also Lipschitz continuous.

### 3. Lipschitz continuous selections for a metric projection

F. Deutsch gave the following open problem in [4].

**Open Problem 3.1.** If \( M \) is a \( n \)-dimensional subspace of \( X \) and \( P_M \) is Lipschitz continuous, must \( P_M \) have a Lipschitz continuous selection?

Let \( \langle x, y \rangle \) denote the inner product of \( x, y \) in Euclidean \( n \)-space \( E^n \).

**Definition 3.2.** [8] For \( A \in H(E^n) \), the STEINER point of \( A \) is defined by
\[ s(A) = n \int_{S^{n-1}} q(A, x) \, dx \]
where \( S^{n-1} \) is the unit sphere of \( E^n \) equipped with unit Lebesgue measure \( dx \) and \( q(A, x) = \sup_{a \in A} \langle a, x \rangle \) is the support function. The mapping \( s : H(E^n) \rightarrow E^n \) is called the Steiner map.
By using basic properties of support function and the definition of the Steiner point, K. Przesławski proved the following Theorem.

**Theorem 3.3.** [8]

1. \( s([x]) = x \) for \( x \in E^n \);
2. \( s(A + B) = s(A) + s(B) \) for every \( A, B \in H(E^n) \);
3. \( s(UA) = Us(A) \) for every \( A \in H(E^n), \ U \in O(E^n) \) where \( O(E^n) \) is the group of orthogonal operators of \( E^n \);
4. \( s(\alpha A) = \alpha s(A) \) for every \( A \in H(E^n), \ \alpha \in \mathbb{R} \).
5. \( s(A) \in A \), for every \( A \in H(E^n) \).

R. A. Vitale [9] proved: In \( E^n \),

\[
\sup_{k \neq k'} \frac{||s(K) - s(K')||}{H(K, K')} = \frac{I^n \left( \frac{n+1}{2} \right)}{\sqrt{\pi n} I^n \left( \frac{n+1}{2} \right)}.
\]

K. Przesławski [8] announced the following theorem without proof.

**Theorem 3.4.** [8, 9]. The Steiner map \( s : H(E^n) \to E^n \) is Lipschitz continuous.

**Lemma 3.5.** If \( X \) and \( Y \) are topologically isomorphic, then there is a one-to-one correspondence between \( H(X) \) and \( H(Y) \).

**Proof.** Let \( \Phi : X \to Y \) be an isomorphism from \( X \) onto \( Y \). Then

\[
||\Phi^{-1}\|^{-1} x || \leq ||\Phi(x)|| \leq ||\Phi|| x \|\text{ for each } x \in X.
\]

Let \( A \in H(X) \). Then \( \Phi(A) := \{\Phi(a) : a \in A\} \in H(Y) \). Let \( B \in H(Y) \). Then

\[
A = \Phi^{-1}(B) := \{a \in X : \Phi(a) \in B\} = \{\Phi^{-1}(b) : b \in B\} \in H(X)
\]

and \( \Phi(A) = B \). Thus \( \Phi \) is a one-to-one correspondence between \( H(X) \) and \( H(Y) \).

**Theorem 3.6.** Let \( M \) be an \( n \)-dimensional subspace of a normed linear space \( X \). Then there is a Lipschitz continuous selection from \( H(M) \) to \( M \).

**Proof.** By Theorem 3.4, there is a Lipschitz continuous selection from \( H(E^n) \) to \( E^n \). Since \( M \) and \( E^n \) are \( n \)-dimensional, \( M \) and \( E^n \) are topologically isomorphic. Thus there is a topological isomorphism \( \varphi \) from \( M \) onto \( E^n \). Furthermore, we can identify \( \varphi \) as a mapping from \( H(M) \) onto \( H(E^n) \). Let \( s' = \varphi^{-1} s : H(M) \to H(E^n) \to E^n \to M \).

Since \( s(\varphi(A)) \in \varphi(A), s'(A) = \varphi^{-1}(s(\varphi(A))) \in A \). Then

\[
\|s'(A) - s'(B)\|_M = \|\varphi^{-1}(s(\varphi(A))) - \varphi^{-1}(s(\varphi(B)))\|_M
\]
≤ ||φ⁻¹|| ||s(φ(A)) − s(φ(B))||_{E^*} ≤ ||φ⁻¹||λH(φ(A), φ(B))_{E^*} ≤ ||φ⁻¹||||φ||λH(A, B)_M

where λ is a Lipschitz constant of s since

H(φ(A), φ(B))_{E^*} = \max \{ \sup_{a \in A} d(φ(a), φ(B)), \sup_{b \in B} d(φ(b), φ(A)) \}_{E^*} ≤ ||φ|| \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \}_M ≤ ||φ||H(A, B)_M.

Thus there is a Lipschitz continuous selection from H(M) to M.

Now we can prove Open Problem 3.1 by using Theorem 3.6.

Corollary 3.7. Let M be a finite dimensional subspace of a normed linear space X. If P_M is pointwise Lipschitz continuous, then P_M has a pointwise Lipschitz continuous selection which is homogeneous and additive modulo M.

Proof. Let x_0 ∈ X be given. Since P_M is pointwise Lipschitz continuous at x_0, there exists \lambda_{x_0} > 0 such that

H(P_M(x_0), P_M(x)) ≤ \lambda_{x_0} ||x_0 − x||

for each x ∈ X. By Theorem 3.6, there is a Lipschitz continuous selection \sigma : H(M) → M. Thus

||\sigma(P_M(x_0)) − \sigma(P_M(x))|| ≤ \lambda_x H(P_M(x_0), P_M(x)) ≤ \lambda_x \lambda_{x_0} ||x_0 − x||

for each x ∈ X. Hence p = \sigma \circ P_M is a pointwise Lipschitz continuous selection for P_M. Since P_M and \sigma are homogeneous and additive modulo M, p is homogeneous and additive modulo M.

Corollary 3.8. Let M be a finite dimensional subspace of a normed linear space X. If P_M is Lipschitz continuous, then P_M has a Lipschitz continuous selection which is homogeneous and additive modulo M.

Proof. We can prove it the same argument as the proof of Theorem 3.7. Define p : X → M by p(x) = \sigma(P_M(x)) for any x ∈ X. Then p is a Lipschitz continuous selection which is homogeneous and additive modulo M since P_M and \sigma are homogeneous and additive modulo M.

The converse of Corollary 3.8 is false. But we can prove the following:

Theorem 3.9. Let M be a proximinal subspace of a normed linear space X. If P_M|_{\ker P_M} is Lipschitz continuous and P_M has a Lipschitz continuous selection, then P_M is Lipschitz continuous.
Proof. Since $P_M|_{\ker P_M}$ is Lipschitz continuous, there exists $\lambda_1 > 0$ such that $H(P_M(x), P_M(y)) \leq \lambda_1 \|x - y\|$ for any $x, y \in \ker P_M$. Let $s$ be a Lipschitz continuous selection for $P_M$ with the Lipschitz constant $\lambda_2$. Let $x, y \in X$. Then $x - P_M(x), y - P_M(y) \subset \ker P_M$. Thus

\[
H(P_M(x), P_M(y)) = H(P_M(x - s(x) + s(x)), P_M(y - s(y) + s(y))) \\
\leq H(P_M(x - s(x)), P_M(y - s(y))) + \|s(x) - s(y)\| \\
\leq \lambda_1 \|x - s(x) - y + s(y)\| + \lambda_2 \|x - y\| \\
\leq (\lambda_1 - \lambda_2 + \lambda_1 \lambda_2) \|x - y\|.
\]

Thus $P_M$ is Lipschitz continuous.

Remark. In Theorem 3.9, neither of the conditions that $P_M$ has a Lipschitz continuous selection or $P_M|_{\ker P_M}$ is Lipschitz continuous can be dropped. We have the following examples.

Examples 3.10. (1) (There are examples that $P_M|_{\ker P_M}$ is Lipschitz continuous but $P_M$ is not Lipschitz continuous). In $C[a, b]$, by Cline’s results [1], for any 2-dimensional Haar subspace $M$ of $C[a, b]$, $P_M$ is not Lipschitz continuous but $P_M|_{\ker P_M} \equiv 0$.

(2) [3] (There is an example that $P_M$ has a Lipschitz continuous selection, but is not Lipschitz continuous). Let $X = \mathbb{R}^3$ be with the norm $\|x\| = |x(1)| + \sqrt{x(2)^2 + x(3)^2}$, $x = (x(1), x(2), (3))$ and $x_1 = (1, 1, 0)$. Then $P_{[x_1]}$ has a Lipschitz continuous (even linear!) selection without $P_M$ being Lipschitz continuous.

Corollary 3.11. Let $M$ be a finite dimensional subspace of a normed linear space $X$. The following statements are equivalent:

(1) $P_M$ is Lipschitz continuous.

(2) $P_M|_{\ker P_M}$ is Lipschitz continuous and $P_M$ has a Lipschitz continuous selection.

Proof. It follows from Corollary 3.8 and Theorem 3.9.

4. Applications


Definition 4.1. [10] We say that $M$ has the $1\frac{1}{2}$-ball property in $X$ if the conditions $m \in M$, $x \in X$, $r_1, r_2 > 0$, $M \cap B(x, r_2) \neq \emptyset$, and $\|x - m\|
\( \langle r_1 + r_2 \rangle \) imply that 
\[
M \cap B(m, r_1) \cap B(x, r_2) \neq \emptyset.
\]
Here \( B(x, r) := \{ y \in X : \| x - y \| \leq r \} \) is a closed ball.

In \([10]\) and \([11]\), we can find lots of examples of subspaces which have the \( \frac{1}{2} \)-ball property.

**Theorem 4.2.** \([10]\) Suppose that \( M \) has the \( \frac{1}{2} \)-ball property in \( X \). Then

1. \( M \) is proximinal.
2. \( P_M \) is Lipschitz continuous.

D. Yost gave the following.

**Theorem 4.3.** \([10]\) If \( M \) has the \( \frac{1}{2} \)-ball property in \( X \), then there is a continuous selection which is homogeneous and additive modulo \( M \).

The following strengthens Yost's theorem when \( M \) is finite-dimensional.

**Theorem 4.4.** If \( M \) is a finite-dimensional subspace which has the \( \frac{1}{2} \)-ball property in \( X \), then \( P_M \) has a Lipschitz continuous selection which is homogeneous and additive modulo \( M \).

**Proof.** By Theorem 4.3, \( P_M \) is Lipschitz continuous. By Corollary 3.10, \( P_M \) has a Lipschitz continuous selection which is homogeneous and additive modulo \( M \).

**Definition 4.5.** \([6]\) Let \( M \) be a proximinal subspace of \( X \). The metric projection \( P_M \) is said to be uniform Hausdorff strongly unique if there exists a constant \( r > 0 \) such that 
\[
\| x - m \| \geq d(x, M) + rd(m, P_M(x))
\]
for each \( x \in X \) and \( m \in M \).

In \([6]\), we showed that if \( M \) has the \( \frac{1}{2} \)-ball property, then \( P_M \) is uniform Hausdorff strongly unique, but the converse is not true.

**Theorem 4.6.** \([6]\) Let \( M \) be a proximinal subspace of \( X \). If \( P_M \) is uniform Hausdorff strongly unique, then \( P_M \) is Lipschitz continuous.

**Corollary 4.7.** Let \( M \) be a finite-dimensional subspace of a normed linear space \( X \). If \( P_M \) is uniform Hausdorff strongly unique, then \( P_M \) has a Lipschitz continuous selection.
References


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