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ON SURJECTIVITY OF m-ACCRETIVE OPERATORS IN BANACH SPACES

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1. Introduction

Recently many authors [2, 3, 5, 6] proved the existence of zeros of accretive operators and estimated the range of m-accretive operators or compact perturbations of m-accretive operators more sharply. Their results could be obtained from differential equations in Banach spaces or iteration methods or Leray-Schauder degree theory.

On the other hand Kirk and Schönberg [9] used the domain invariance theorem of Deimling [3] to prove some general minimum principles for continuous accretive operators. Kirk and Schönberg [10] also obtained the range of m-accretive operators (multi-valued and without any continuity assumption) and the implications of an equivalent boundary conditions.

Their fundamental tool of proofs is based on a precise anlysis of the orbit of resolvents of m-accretive operator at a specified point in its domain.

In this paper we obtain a sufficient condition for m-accretive operators to have a zero. From this we derive Theorem 1 of Kirk and Schönberg [10] and some results of Morales [12, 13] and Torrejǒn [15]. And we further generalize Theorem 5 of Browder [1] by using Theorem 3 of Kirk and Schönberg [10].

2. Preliminaries

Let X be a Banach space and X^* be a dual banach space of X. We use $B(X\,;\,r)$ to denote the open ball centered at $x \in X$ with radius $r>0$ and ∂U to denote the boundary of a subset U of X. We also use the notation $|A| = \inf\{\|x\| : x \in A\}$, $A \subset X$.

The duality mapping J of X into 2^{X^*} is defined by

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$$J(x)=\{j\in X^* : (x, j)=\|x\|\,\|j\|=\|x\|^2\}.$$

Let T be a multivalued operator from X to 2^X. Then we define $D(T)=\{x\in X ; Tx\neq\phi\}$, $R(T)=\cup\{Tx : x\in D(T)\}$. T is said to be accretive if for each x, $y\in D(T)$ and $u\in Tx$, $v\in Ty$, there exists $j\in J(x-y)$ such that

$$(u-v, j)\geqslant 0.$$

It follows that T is accretive [7] if for any x, y in $D(T)$ and $r>0$, $\|x-y\|\leqslant|x-y+r(Tx-Ty)|$. An accretive operator T is said to be m-accretive iff $R(I+rT)=X$ for some $r>0$ (hence all $r>0$). Thus for m-accretive T, the resolvent $(I+rT)^{-1}=J_r$, $r>0$ is a single-valued nonexpansive mapping which is defined on all of X.

3. Main results

In the following theorem we obtain sufficient conditions for m-accretive operator to have a zero.

THEOREM 1. *Let X be a Banach space such that every nonempty closed convex bounded subset of X has the fixed point property with respect to nonexpansive self-mappings. Suppose that $T : D(T)\subset X\to 2^X$ is m-accretive operator satisfying: $|Tx_n|\to 0$ as $n\to\infty$ for some bounded sequence x_n in $D(T)\subset X$. Then T has a zero (i.e. $0\in R(T)$).*

Proof. For any n we can choose $y_n\in Tx_n$ such that $\|y_n\|\leqslant|Tx_n|+\frac{1}{n}$. Then $x_n+y_n\in x_n+Tx_n$. Let $z_n=x_n+y_n$. So $(I+T)^{-1}z_n=x_n=J_1z_n$. Since $\{x_n\}$ and $\{y_n\}$ are bounded, $\{z_n\}$ is bounded. Let $\limsup_{n\to\infty}\|z_n\|=M$ If we define

$$C=\{x\in X : \limsup_{n\to\infty}\|x-z_n\|\leqslant M\},$$

then C is a nonempty closed convex bounded subset of X. On the other hand we have

$$\|J_1x-z_n\|-\|J_1z_n-z_n\|\leqslant\|J_1x-J_1z_n\|\leqslant\|x-z_n\|,$$
$$\text{i. e. } \|J_1x-z_n\|\leqslant\|x-z_n\|+\|J_1z_n-z_n\|.$$

From $\|J_1z_n-z_n\|=\|y_n\|\leqslant|T(x_n)|+\frac{1}{n}$, we have $\lim_{n\to\infty}\|J_1z_n-z_n\|=0$. Hence C is mapped into itself by a nonexpansive mapping J_1. From the assumptions on X the resolvent J_1 of T has a fixed point in C which is a zero of T.

For the following corollary we need the following definition. A single-valued mapping $T : X \to X$ is said to be pseudo-contractive [9] if $I-T$ is accretive.

COROLLARY 1. *Suppose X is a Banach space such that every nonempty closed convex bounded subset of X has the fixed point property with respect to nonexpansive self-mappings and suppose $f : X \to X$ is a continuous pseudo-contractive mapping. If $x_n - f(x_n) \to 0$ as $n \to \infty$ for some bounded sequence $\{x_n\}$ in X, then f has a fixed point in X.*

Proof. It is known that every continuous accretive operator defined on all of X is m-accretive [11]. Hence $I-f$ is m-accretive. By applying Theorem 1 we have the conclusions.

Kirk and Schönberg [9] proved Corollary 1 with an additional assumption that X is reflexive. From Theorem 1 we also get Theorem 6 of Morales [13]. A single-valued mapping $T : X \to X$ is said to be demicontinuous if $\{x_n\}$ converges to x implies that $\{Tx_n\}$ converges weakly to Tx.

COROLLARY 2 [13]. *Let X be a Banach space with uniformly convex dual X^* such that every nonempty closed convex bounded subset of X has the fixed point property with respect to nonexpansive self-mappings. Suppose $T : D(T) \subset X \to X$ is a demicontinuous accretive operator satisfying: $|Tx_n| \to 0$ as $n \to \infty$ for some bounded sequence $\{x_n\}$ in X. Then T has a zero.*

Proof. Kenmochi [8] proved that in case the dual space X^* of X is uniformly convex, a single-valued demicontinuous accretive operator T with open domain $D(T)$ is m-accretive if and only if $x_n \to x$ implies $\|Tx_n\| \to \infty$ for every $x \in \partial D(T)$, $x_n \in D(T)$. Hence T is m-accretive and the conclusion of Corollary 2 can be obtained by applying Theorem 1.

We have also the following corollary [10, Theorem 1] by applying Theorem 1.

COROLLARY 3 [10]. *Let X be a Banach space for which each nonempty bounded closed convex subset has the fixed point property for nonexpansive self-mappings, and let $T : D(T) \subset X \to 2^X$ be m-accretive operator. Then the following are equivalent:*

(1) $0 \in R(T)$.

(2) $\liminf_{\lambda\to\infty}\|J_\lambda x\|<\infty$ *for some* $x\in X$.

Proof. Obviously (1) $\Rightarrow$ (2). Let $x\in X$ satisfying (2) and suppose $\lambda_n\in(0,\infty)$ is a sequence for which $\lambda_n\to\infty$ and $\{J_{\lambda_n}x\}$ is bounded. Since J_{λ_n} is nonexpansive for every λ_n, we have

$$\|J_{\lambda_n}x-J_{\lambda_n}0\|\leqslant\|x\|.$$

Hence $\{J_{\lambda_n}0\}$ is also bounded. Put $x_n=J_{\lambda_n}0$. Then

$$x_n+\lambda_n Tx_n\ni 0,\ \text{ i. e. } -\frac{1}{\lambda_n}x_n\in Tx_n.$$

Hence $|Tx_n|\leqslant\frac{1}{\lambda_n}\|x_n\|$ and $\lim_{n\to\infty}|Tx_n|=0$. By applying Theorem 1, the conclusion follows.

We let $[y,x]_+=\sup\{(y,j):j\in J(x)\}$.
It can be shown that $[ax,x]_+=a\|x\|^2$ for all real number a.

COROLLARY 4. *Let X be a Banach space such that every nonempty closed convex bounded subset of X has the fixed point property with respect to nonexpansive self-mappings, let $T:D(T)\subset X\to 2^X$ be m-accretive, and*

$$\liminf_{\substack{\|x\|\to\infty\\ x\in D(T)}}\frac{[y,x]_+}{\|y\|^q}>-\infty,\quad q\in[1,2)$$

where $y\in Tx$. Then $0\in R(T)$.

Proof. Suppose $\{\lambda_n\}\subset(0,\infty)$ is a sequence for which $\lambda_n\to\infty$. On the contrary we suppose that $\|x_n\|=\|J_{\lambda_n}0\|\to\infty$ as $n\to\infty$. Then we can choose $y_n\in Tx_n$ such that $x_n+\lambda_n y_n=0$. Then

$$\lim_{n\to\infty}\frac{[y_n,x_n]_+}{|y_n|^q}=\lim_{n\to\infty}-\frac{|\lambda_n|^{-1}\|x_n\|^2}{|\lambda_n|^{-q}\|x_n\|^q}=-\lim_{n\to\infty}\frac{\|x_n\|^{2-q}}{|\lambda_n|^{1-q}}=-\infty.$$

It is a contradiction. By applying Corollary 3 we have conclusion.

In Corollary 4 we have Corollary 1 of Morales [12] and Theorem 2.1 of [15] by replacing T by $T-z$. By applying Corollary 3 we have Corollary 3 of Morales [13] directly.

COROLLARY 5 [13]. *Let X be a Banach space with uniformly convex dual X^* such that every nonempty closed convex bounded subset of X has the fixed point property with respect to nonexpansive self-mappings. Suppose $T:X\to X$ is a demicontinuous accretive operator and the set*

$$\{x\in X : Tx=tx \text{ for some } t<0\}$$

is bounded. Then $0\in R(T)$.

Proof. Since T is m-accretive and the resolvent $J_{-\frac{1}{t}}0(t<0)$ is bounded, it follows that $0\in R(T)$ by applying Corollary 3.

In what follows We apply Theorem 3 of Kirk and schönberg [10] to the results of Browder and Morales and Kirk-Schönberg.

THEOREM K-S [10]. *Let X be a Banach space for which the closed unit ball has the fixed point property for nonexpansive self-mappings, let $T: D(T)\subset X\to 2^X$ be m-accretive, and suppose for some $x_0\in D(T)$,*

$$|Tx_0|<r\leqslant \liminf_{\substack{\|x\|\to\infty \\ x\in D(T)}} |Tx|.$$

Then $B(0;r)\subset R(T)$.

We apply the above theorem to generalize Theorem 5 of Browder [1].

THEOREM 2. *Let X be an uniformly convex Banach space with its dual space X^* also uniformly convex and let T and T_0 be accretive with domain and range in X. Suppose that*

(*a*) *The range of $T+I$ is all of X. $D(T)$ is dense in X.*

(*b*) *T_0 is defined and demicontinuous on all of X.*

(*c*) *For some $x_0\in D(T)$*

$$\|(T+T_0)x_0\|<r\leqslant \liminf_{\|x\|\to\infty} \|(T+T_0)x\|$$

Then $B(0;r)\subset R(T+T_0)$.

Proof. By(a) T is m-accretive and from (b) T_0 is m-accretive. By corollary of Prüß [14], we conclude that $T+T_0$ is m-accretive. By Theorem 3 of Kirk and Schönberg the conclusion holds.

COROLLARY 6. *Under the same assumptions (a) and (b), except for*

$$\lim_{\|x\|\to\infty} \|(T+T_0)x\|=\infty$$

Then $R(T+T_0)=X$.

REMARK. (1) From Corollary 6 we have Theorem 5 of Browder [1] where he obtained the conclusion by assuming in addition to the above hypotheses, that T_0 maps bounded subsets of X into bounded subsets of X. His proof is base on the fact that $-(T+T_0)$ is the infinitesimal generator of a semigroup of nonexpansive mappings in the Banach space X.

(2) Instead of (b) in Theorem 2 assuming the following condition: T_0 is single-valued and demicontinuous accretive on an open domain $D(T)$ satisfying

$$x_n\to x \text{ implies } \|Tx_n\|\to\infty \text{ for every } x\in\partial D(T),$$

$x_n\in D(T)$, we have the same conclusion from the results of Kenmochi

[8] and Prüß [14].

Corollary 7. *Let X be a Banach space such that the closed unit ball of X has the fixed point property with respect to nonexpansive self-mappings and let $T: D(T)\subset X\to 2^X$ be m-accretive operator. Suppose for some $\delta>0$ the set*

$$K\{x\in D(T): \|y\|<\delta \text{ for some } y\in Tx\}$$

in nonempty and bounded. Then $B(0;\delta)\subset R(T)$.

Proof. Since K is nonempty and bounded, we choose $x_0\in K$ and $|Tx_0|<\delta\leqslant \liminf_{\|x\|\to\infty} |Tx|$. Hence the proof follows from Theorem K-S.

We note that Corollary 1 of Morales [13] can also be obtained directly by applying Theorem K-S and Corollary 3 of Morales [12] is obtained by Corollary 7. In Corollary 3 of Morales [12] he assumed that every nonempty closed convex bounded subset of X has the fixed point property for nonexpansive self-mappings.

If $T: X\to X$ is a continuous pseudo-contractive mapping, then $I-T$ is m-accretive. Hence from Corollary 7 we have the following corollary which implies Theorem 2 of Kirk and Schönberg [9].

Corollary 8. *Let X be a Banach space for which the closed unit ball has the fixed point property for nonexpansive self-mappings, $T: X\to X$ a continuous pseudo-contractive mapping and suppose that for some $\delta>0$ the set*

$$\{x\in X: \|x-f(x)\|\leqslant\delta\}$$

is nonempty and bounded, then $B(0;\delta)\subset R(I-T)$ (in particular T has a fixed point in X).

Remark. The coercivity of $T(T^{-1}$ maps bounded subsets into bounded subsets [4]) implies the boundedness of the set K in Corollary 7 for any $\delta>0$. Therefore coercivity of T implies the surjectivity of T.

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