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PERIODIC-DIRICHLET BOUNDARY VALUE PROBLEM FOR NONLINEAR DISSIPATIVE HYPERBOLIC EQUATIONS AT RESONANCE

WAN SE KIM

1. Introduction

Let Z^+, Z and R be the set of all positive integers, integers and real numbers, respectively, and let $\Omega=[0, 2\pi]\times[0, \pi]$ and $I=[0, \pi]$.

Let $p\in[1, \infty[$. By $L^p(\Omega)$ we denote the space of all measurable real valued functions $u : \Omega\longrightarrow R$ for which $|u(t, x)|^p$ is Lebesgue integrable. The norm is given by

$$\|u\|_{L^p}=\left[\iint_{\Omega}|u(t, x)|^p\, dtdx\right]^{1/2}.$$

In particular, $L^2(\Omega)$ is the Hilbert space with usual inner product $(\ ,\ )$ and usual norm $\|\cdot\|_{L^2}$. Let $L^\infty(\Omega)$ be the space of measurable real valued functions $u : \Omega \longrightarrow R$ which are essentially bounded with the norm

$$\|u\|_{L^\infty}=\operatorname*{ess\,sup}_{(t,x)\in\Omega}|u(t, x)|.$$

Let $C^k(\Omega)$ be the space of all continuous functions $u : \Omega \longrightarrow R$ such that the partial derivatives up to order k with respect to both variables are continuous on Ω, while $C(\Omega)$ is used for $C^0(\Omega)$ with the usual norm $\|\cdot\|_\infty$ and we write $C^\infty(\Omega)=\bigcap_{k=0}^{\infty} C^k(\Omega)$.

Let $W^{k,2}(\Omega)$ be the Sobolev space of all functions $u : \Omega \longrightarrow R$ in $L^2(\Omega)$ such that all distributional derivatives $D_t^pD_x^q$ $(0\leqq p+q\leqq k)$ belong to $L^2(\Omega)$

$$\|u\|_{W^{k,2}}=\left[\sum_{0\leqq p+q\leqq k}\iint_{\Omega}(D_t^pD_x^q u(t, x))^2dtdx\right]^{1/2}.$$

In this work, we will investigate the existence of weak solutions

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of the periodic-Dirichlet problem for non-linear dissipative hyperbolic equations of the form

(1. 1) $\beta u_t + u_{tt} - u_{xx} - u - g(t, x, u) = h(t, x)$ in Ω,

where $\beta(\neq 0) \in R$, $u = u(t, x)$, $h \in L^2(\Omega)$ and $g : \Omega \longrightarrow R$ is a Caratheodory function; i. e., $g(.,.,u)$ is measurable on Ω for each $u \in R$ and $g(t, x, .)$ is continuous on R a. e. on Ω.

Moreover, we assume the following;

(H_1) there exists $r(t, x) \in L^2(\Omega)$, $c > 0$ such that $|g(t, x, u)| \leqq c|u| + r(t, x)$ for all $u \in R$ and a. e. in Ω,

(H_2) $\limsup_{|u| \to \infty} g(t, x, u)/u = \Gamma(t, x)$ for $(t, x) \in \Omega$, where $\Gamma(t, x) \in L^\infty(\Omega)$ satisfies $\Gamma(t, x) < 2$ a. e. in Ω and $\Gamma(t, x) \leqq 2$ on a set of positive measure in Ω. Let $g(t, x, \infty) = \liminf_{u \to \infty} g(t, x, u)$ and $g(t, x, -\infty) = \limsup_{u \to -\infty} g(t, x, u)$.

A weak solution of the periodic-Dirichlet problem for (1. 1) will be a $u \in L^\infty(\Omega)$ such that

(1. 2)
$$\iint_\Omega u(t, x)[-\beta v_t(t, x) + v_{tt}(t, x) - v_{xx}(t, x) - v(t, x)]\, dtdx - \iint_\Omega g(t, x, u(t, x))\, v(t, x)\, dtdx = \iint_\Omega h(t, x)\, dtdx$$

for every $v \in C^2(\Omega)$ satisfying boundary conditions

$$v(0, x) - v(2\pi, x) = v_t(0, x) - v_t(2\pi, x) = 0, \quad x \in [0, \pi],$$
$$v_{|\partial I} = 0, \quad t \in [0, 2\pi].$$

In [9], Mawhin prove that the doubie 2π-periodic boundary value problem to the equations of the type (1. 1) has at least one weak solution provided $g(t, x, u) = \psi(u)$ is continuous and bounded in $R = [0, 2\pi] \times [0, 2\pi]$ and h satisfies the condition

$$4\pi(\psi(\infty) - \psi(-\infty)) > \iint_R h(t, x) \sin(x + \phi)\, dtdx$$

for any $\phi \in]-\infty, \infty[$. In [3], Fucik and Mawhin prove that the same problem to the equation (1. 1) has at least one weak solution if h satisfies

$$\iint_R h(t, x) \sin(x + \phi)\, dtdx = 0$$

for any $\phi \in]-\infty, \infty[$ when $g(t, x, u) = \psi(u)$ is a continuous bounded odd function, and if h satisfies

$$\sup_{\phi \in R} \left| \iint_R h(t, x) \sin(x + \phi)\, dtdx \right| < 8\pi \sup_{\xi \in R} \psi(\xi)$$

for any $\phi\in]-\infty, \infty[$ when $g(t, x, u)=\phi(u)$ is a bounded continuous, odd and expansive function. In [4], Hirano and myself prove the existence of weak solutions of periodic-Dirichlet boundary value problem to the equations of the form (1.1) if $g(t, x, u)$ is a Caratheodory function and h satisfies the condition

$$\iint_{\Omega} h(t, x)\sin x \, dtdx=0$$

which is a perpendicular condition with the first eigenfunction of the differential operator $u_{tt}-u_{xx}$ for periodic-Dirichlet problem, and we impose no boundedness and oddness on the non-linear term g.

In this study, we prove the existence of weak solutions of periodic-Dirichlet boundary value problem to the equation (1.1) when $g(t, x, u)$ is a Caratheodory function and h satisfies the Landesman-Lazer condition appearing in [8] for the study of elliptic equations, and here we also impose no boundedness, oddness and expensiveness on the non-linear term g. The argument will be carrid over by making readjustment to the method used in [4] whose main key is to find an a priori bound on solutions by using appropriate projections and the bootstrap argument.

Our result is related to that of Mckenna and Rauch in [10], and Ward in [11], [12].

2. Preliminary results

Now we consider the equation

(2.1) $\beta u_t+u_{tt}-u_{xx}=h(h, x)$ where $\beta\neq 0$, $u=u(t, x)$.

If u and $h\in L^2(\Omega)$, we may write

$$u(t, x)=\sum_{(l, m)\in Z\times Z^+} u_{lm}\exp(ilt)\sin(mx),$$

$$h(t, x)=\sum_{(l, m)\in Z\times Z^+} u_{lm}\exp(ilt)\sin(mx)$$

with $\bar{u}_{lm}=u_{-lm}$ and $\bar{h}_{lm}=h_{-lm}$ since u and h are real.

The proof of the following lemma is clear and will be omitted.

LEMMA 2.1. *$h\in L^2(\Omega)$ is a weak solution to (2.1) if and only if* $u(t, x)=\sum_{(l, m)\in Z\times Z^+}[\beta li+(m^2-l^2)]^{-1}h_{lm}\exp(ilt)\sin(mx)$.

Let Dom $L=\{u\in L^2(\Omega): \sum_{(l, m)\in Z\times Z^+}[\beta^2l^2+(m^2-l^2)^2]|u_{lm}|^2<\infty\}$ and define an operator L : Dom $L\subseteq L^2(\Omega)\longrightarrow L^2(\Omega)$ by

$(Lu)(t, x)=\sum_{(l, m)\in Z\times Z^+}[\beta li+(m^2-l^2)u_{lm}\exp(ilt)\sin(mx)$.

Then Dom L is dense in $L^2(\Omega)$, Ker $L=\{0\}$ and Im $L=L^2(\Omega)$. Hence $L^{-1}: L^2(\Omega) \longrightarrow$ Dom L exists and

$$(L^{-1}h)(t, x) = \sum_{(l,m)\in Z\times Z^+} [\beta li + (m^2 - l^2)]^{-1} h_{lm} \exp(ilt) \sin(mx).$$

Therefore, by Lemma 2.1, if $h\in L^2(\Omega)$, then u is a weak solution of the periodic-Dirichlet problem on Ω for the equation

$$\beta u_t + u_{tt} - u_{xx} = h(t, x), \quad \beta \neq 0 \text{ and } u = u(t, x),$$

if and only if $u\in$ Dom L, $Lu=h$, or $u=L^{-1}h$.

REMARK 2.1. L: Dom $L\subseteq L^2(\Omega) \longrightarrow L^2(\Omega)$ is (weakly) closed.

LEMMA 2.2. *If $h\in L^2(\Omega)$, then there exists a constant $C>0$ independent of h such that $\|L^{-1}h\|_\infty \leq C\|h\|_{L^2}$. The operator $L^{-1}: L^2(\Omega) \longrightarrow C(\Omega)$ is compact.*

Proof. See [4], [6] and [7].

Combining the facts in [3], [9] and Lemma 2.2., we have the following lemma.

LEMMA 2.3. *Dom $L=L^{-1}(L^2(\Omega)) \subseteq W^{1\cdot 2}(\Omega) \cap C(\Omega)$ and $L^{-1}(W^{k\cdot 2}(\Omega)) \subseteq W^{k+1\cdot 2}(\Omega)$* for $k=0, 1, 2, 3, \ldots$.
Moreover, $\|L^{-1}h\|_{W^{1\cdot 2}} \leq C_1\|h\|_{L^2}$ where $h\in L^2(\Omega)$ and $C_1>0$ is a constant independent of h.

3. Main result

We put

$$H = \{u\in L^2(\Omega) \mid \sum_{(l,m)\in Z\times Z^+} |m^2 - l^2| \, |u_{lm}|^2 + \sum_{m=|l|} |u_{lm}|^2 < \infty\}.$$

Then H is a Hilbert space with norm

$$\|u\| = \sum_{(l,m)\in Z\times Z^+} |m^2 - l^2| \, |u_{lm}|^2 + \sum_{m=|l|} |u_{lm}|^2$$

and it is known that

(3.1) $\|u\|_{L^p} \leqq C_p\|u\|$ for all $u\in H$ and $2\leqq p<\infty$, where C_p is a constant depending only on p.

We also put

$$H_1 = \{u\in H \mid u_{lm}=0 \text{ for all } m^2 - l^2 \leq 1\},$$
$$H_2 = \{u\in H \mid u_{lm}=0 \text{ for all } m^2 - l^2 \neq 1\},$$
$$H_3 = \{u\in H \mid u_{lm}=0 \text{ for all } m^2 - l^2 \geq 1\}.$$

Let P_1, P_2 and P_3 denote the projections from $L^2(\Omega)$ onto H_1, H_2 and H_3, respectively, and let P_0 denotes the projection from $L^2(\Omega)$ onto the subspace

$$H_0=\{u\in H \mid u_{lm}=0 \text{ for all } m^2-l^2\neq 0\}.$$

Then we have

(3.2) $$(Lu, u)=\|P_1u\|^2+\|P_2u\|^2-\|P_3u\|^2+\|P_0u\|^2$$

for all $u\in H$. Moreover, we see the injection $H_1\oplus H_2\oplus H_3 \longrightarrow L^2(\Omega)$ is compact.

THEOREM. *Let $h\in W^{2,2}(\Omega)$ and suppose (H_1) and (H_2) are satisfied, and the following cvonditons (H_3), (H_4), (H_4) are satisfied.*

(H_3) $g', g''\in L^\infty(\Omega\times R)$ where g', g'' denote the first and seeond derivatives of g, respectively,

(H_4) there is a number $r>0$ such that $g(t, x, u)u\geq 0$ for all $(t, x)\in\Omega$ and for $|u|\geqq r$, and

(H_5) the following inequalities hold

$$\iint_\Omega g(t, x, -\infty)\, sinx\, dtdx<-\iint_\Omega h(t, x) sinx\, dtdx<\iint_\Omega g(t, x, \infty) sinx\, dtdx$$

where infinite values are allowed for the left and right hand integrals. Then the periodic-Dirichlet boundary value problem to the equation (1.1) has at least one weak solution.

Proof. Let $\varepsilon_0>0$ be such that $\Gamma(t, x)<2-\varepsilon_0<2$ and take $0<\varepsilon<\varepsilon_0$. We put $K=L-(1+\varepsilon)I$, then $K: L^2(\Omega)\longrightarrow L^2(\Omega)$ is a continuous and compact operator. Define a substitution mapping $N: L^2(\Omega)\longrightarrow L^2(\Omega)$ by $Nu=g(.,.,u)-\varepsilon Iu+h(.,.)$, then, by Krasnosellskii's results, N is continuous and maps bounded sets into bounded sets. Then u is a weak solution to the periodic-Dirichlet problem for (1.1) if and only if $u\in$ Dom L and satisfies

(3.3) $Ku=Nu$, or equivalently

(3.4) $u=K^{-1}Nu.$

If $u\in L^2(\Omega)$ solves the operator equations (3.4), then $u\in L^2(\Omega)$ is a weak solution to the periodic-Dirichlet problem. Since K^{-1} is continuous and compact and N is continuous, and maps bounded sets into bounded sets, the composition $K^{-1}N$ is continuous and compact. By using Leray-Schauder's degree argument if all solutions u to the family of equations

(3.5) $u=\lambda K^{-1}Nu,\quad 0<\lambda<1,$

are bounded in $L^2(\Omega)$ independently of $\lambda\in]0,1[$, then (3.3) has a solution. If the pair (u,λ) solves (3.5), then the pair (u,λ) solves

(3.6) $Ku=\lambda Nu$

and u is a weak solution to the periodic-Dirichlet problem to the equation

$$\beta u_t + u_{tt} - u_{xx} - u - \varepsilon(1-\lambda)u + \lambda g(t, x, u) = \lambda h.$$

Then the proof will be completed if we show that the solution to (3. 6) for λ, $0<\lambda<1$ are bounded in $L^2(\Omega)$ independently of λ. If it is not the case, then there exists a sequence of pairs $\{(u_n, \lambda_n)\}$, $u_n \in \text{Dom } L \subseteq H$, $0<\lambda_n<1$ such that $\|u_n\|_{L^2} \longrightarrow \infty$ as $n \to \infty$ and

(3. 7) $$Lu_n - (1+\varepsilon)u_n - \lambda_n g(.,\ .,\ u_n) + \lambda_n \varepsilon u_n = \lambda_n h$$

for all $n \geqq 1$. Hence we assume that $|u_n(t, x)| \longrightarrow \infty$ a. e. in Ω and also, by (3. 1), $\|u_n\| \longrightarrow \infty$ as $n \to \infty$. Applying L^{-1} on the both sides of (3. 7), we have, from Lemma 2. 2, that $u_n/\|u_n\|$ is bounded in $L^\infty(\Omega)$ and also we may assume that $\{\lambda_n\}$ converges to λ, $0<\lambda\leqq 1$. Applying P_i, $i=1, 2, 3$, on the both sides of (3. 7), we may assume also, from Lemma 2. 2, that $P_i/\|u_n\|$, $i=1, 2, 3$, are bounded in $L^\infty(\Omega)$. Since $\|(P_1+P_3)u_n/\|u_n\|\| \leqq 1$, we have, by taking a subsequence if necessary, $\lim_{n\to\infty} \|(P_1+P_3)u_n/\|u_n\|\|$ exists.

We first suppose that $\lim_{n\to\infty} \|(P_1+P_3)u_n/\|u_n\|\| > 0$. Since H_1 is compactly embedded in $L^2(\Omega)$, we may assume that $P_1 u_n/\|u_n\| \longrightarrow v$, as $n \to \infty$, strongly in $L^2(\Omega)$ and hence we may also assume that $P_1 u_n/\|u_n\| \longrightarrow v$ a. e. in Ω. Since $u_n/\|u_n\|$, $P_1 u_n/\|u_n\|$, $P_3 u_n/\|u_n\|$ and $P_3 u_n/\|u_n\|$ are bounded in $L^2(\Omega)$, there exists a function $\phi \in L^2(\Omega)$ such that

$$-g(t, x, u_n(t, x))\tilde{u}_n(t, x)/\|u_n\|^2 \geqq \phi$$

where $\tilde{u}_n = u_n - 2(P_2+P_3)u_n$.

Then by Fatou's lemma, we have that

$$\begin{aligned}
&\liminf_{n\to\infty} (-g(.,\ .,\ u_n), \tilde{u}_n)/\|u_n\|^2 \\
&= \liminf_{n\to\infty} \iint_\Omega -g(t, x, u_n(t, x))\tilde{u}_n(t, x)/\|u_n\|^2\, dtdx \\
&\geq \iint_\Omega \liminf_{n\to\infty} [-g(t, x, u_n(t, x))/u_n(t, x)]u_n(t, x)\tilde{u}_n(t, x)/\|u_n\|^2\, dtdx \\
&\geq -\iint_\Omega \limsup_{n\to\infty} [g(t, x, u_n(t, x))/u_n(t, x)][P_1 u_n(t, x)/\|u_n\|]^2\, dtdx \\
&= -\iint_\Omega \Gamma(t, x)[v(t, x)]^2\, dtdx.
\end{aligned}$$

Therefore, it follows from (3. 2) that

$$\begin{aligned}
&(Lu_n - u_n - \lambda_n g(t, x, u_n) - \varepsilon(1-\lambda_n)u_n - \lambda_n h, \tilde{u}_n) \\
&\geq \|P_1 u_n\|^2 - \|P_2 u_n\|^2 + \|P_3 u_n\|^2 + \|P_0 u_n\|^2 \\
&\quad -\delta_n(\|P_1 u_n\|_{L^2}^2 - \|P_2 u_n\|_{L^2}^2 - \|P_0 u_n\|_{L^2}^2) - (h, \tilde{u}_n) - (g(.,\ .,\ u_n), \tilde{u}_n) \\
&\geq (3-\delta_n)\|P_1 u_n\|_{L^2}^2 - (h, \tilde{u}_n) - (g(.,\ .,\ u_n), \tilde{u}_n) + \|P_3 u_n\|^2.
\end{aligned}$$

because $\delta_n = 1+\varepsilon(1-\lambda_n) > 1$ and $\|P_3 u_n\|^2 \geq 3\|P_3 u_n\|_{L^2}^2$. Hence

$$\liminf_{n\to\infty} (Lu_n - u_n - \lambda_n g(.,\ .,\ u_n) - \varepsilon(1-\lambda_n)u_n - \lambda_n h, \tilde{u}_n)/\|u_n\|^2$$

$$\geqq (3-\delta)\|v\|_{L^2}^2-\iint_\Omega \Gamma(t, x)[v(t, x)]^2\, dtdx+\liminf_{n\to\infty}\|P_3u_n/\|u_n\|\|^2$$

$$\geqq \iint_\Omega [3-\delta-\Gamma(t, x)][v(t, x)]^2\, dtdx+\liminf_{n\to\infty}\|P_3u_n/\|u_n\|\|^2$$

where $\delta=1+\varepsilon(1-\lambda)$.

If $v\neq 0$, by (H_2) and the choice of $\varepsilon>0$, we have

$$\iint_\Omega [3-\delta-\Gamma(t, x)][v(t, x)]^2\, dtdx>0.$$

If $v=0$, from the assumption, we have $\liminf_{n\to\infty}\|P_3u_n/\|u_n\|\|>0$.

Therefore

$$\liminf_{n\to\infty}(Lu_n-u_n-\lambda_n g(.,.,u_n)-\varepsilon(1-\lambda_n)u_n-\lambda_n h, u_n)/\|u_n\|^2>0,$$

which contradicts (3. 7).

Next, we assume that $\lim_{n\to\infty}\|(P_1+P_3)u_n/\|u_n\|\|=0$. Then we have $\lim_{n\to\infty}P_2u_n/\|u_n\|=1$. Since H_2 is spanned by $\sin x$, $P_2u_n/\|u_n\|$ converges strongly to $\alpha\sin x$ in $L^2(\Omega)$ for some $\alpha\in R$. So we may assume that $P_2u_n/\|u_n\|$ converges to $\alpha\sin x$ a. e. in Ω.

Again, since $\lim_{n\to\infty}\|P_2u_n/\|u_n\|\|=1$, we have $\lim_{n\to\infty}\|u_n/\|u_n\|\|_{L^2}=1$ and we may see $u_n/\|u_n\|$ converges to $\alpha\sin x$ in $L^2(\Omega)$, and so we may have $u_n/\|u_n\|\to\alpha\sin x$ a. e. in Ω and $\lim_{n\to\infty}(Lu_n-u_n, u_n)=0$.

By the Lebesgue dominated convergence theorem, we may suppose, by taking a subsequence if necessary, that

$$\lim_{n\to\infty}(g(.,.,u_n), u_n)/\|u_n\|^2$$

$$=\iint_\Omega \Gamma(t, x)[\alpha\sin x]^2\, dtdx.$$

Therefore

$$\lim_{n\to\infty}(Lu_n-u_n-\lambda_n g(.,.,u_n)-\varepsilon(1-\lambda_n)u_n-\lambda_n h, u_n)/\|u_n\|^2$$

$$=-\varepsilon(1-\lambda)[\alpha\sin x]^2-\iint_\Omega \Gamma(t, x)[\alpha\sin x]^2\, dtdx.$$

Now, if $\Gamma(t, x)>0$ on a set of positive measure on Ω, then

$$\lim_{n\to\infty}(Lu_n-u_n-\lambda_n g(.,.,u_n)-\varepsilon(1-\lambda_n)u_n-\lambda_n h, u_n)/\|u_n\|^2<0$$

which contradicts (3. 7). We now suppose that $\Gamma(t, x)=0$ a. e. on Ω. Then, by using (3. 1), we may have that $\lim_{n\to\infty}g(.,.,u_n)/\|u_n\|=0$ a. e. on Ω.

To finish our proof, we will make use of bootstrap argument. For this purpose, we consider, by using (3. 7), the following equality:

(3. 8) $(P_1+P_2)u_n=L^{-1}[(P_1+P_3)(\delta_n u_n+\lambda_n g(.,.u_n)+\lambda_n h)/\|u_n\|]$.

Since $\lim_{n\to\infty} \|(P_1+P_2)u_n/\|u_n\|\|=0$, from Lemma 2. 3 and (3. 8), we have $(P_1+P_3)u_n \in W^{1\cdot 2}$ and $\lim_{n\to\infty} \|(P_1+P_3)u_n/\|u_n\|\|_{W^{1\cdot 2}}=0$. Furthermore, using Lemma 2. 3 (H_3) and (3. 8) repeatly, we have $(P_1+P_3)u_n \in W^{3\cdot 2}$ and $\lim_{n\to\infty} \|(P_1+P_3)u_n/\|u_n\|\|_{W^{3\cdot 2}}=0$. Since $W^{3\cdot 2}(\Omega)$ is compactly embedded in $C^1(\Omega)$, we obtain that $(P_1+P_3)u_n \in C^1$ and $\lim_{n\to\infty} \|(P_1+P_3)u_n/\|u_n\|\|_{C^1}=0$. Therfore, we have $u_n/\|u_n\| \longrightarrow \alpha\sin x$ in C^1.

Now suppose that $\alpha>0$. Then since $u_n/\|u_n\| \longrightarrow \alpha\sin x$ in C^1, we obtain that $u_n/\|u_n\|>0$ a. e. in Ω for all large n, and thus also $u_n>0$ a. e. in Ω for all large n. Take the inner product with $\alpha\sin x$ on the both sides of (3. 7) to obtain

$$(Lu_n-(1+\varepsilon)u_n-\lambda_n g(.,.,u_n)+\lambda_n\varepsilon u_n, \alpha\sin x)=(\lambda_n h, \alpha\sin x),$$

or equivalently

$$\varepsilon(1-\lambda_n)\iint_\Omega u_n(t,x)\sin x\, dtdx+\lambda_n\iint_\Omega g(t,x,u_n(t,x))\sin x\, dtdx$$
$$=-\lambda_n\iint_\Omega h(t,x)\sin x\, dtdx.$$

Now, for large n, we have $u_n(t,x)\sin x>0$ in Ω. Since $\varepsilon(1-\lambda_n)>0$ and $\lambda_n>0$, we obtain

(3. 9) $\iint_\Omega g(t,x,u_n(t,x))\sin x\, dtdx < -\iint_\Omega h(t,x)\sin x\, dtdx$ for large n. Since $u_n(t,x) \longrightarrow \infty$ a. e. on Ω, by Fatou's lemma, we get, using also (H_4),

$$\liminf_{n\to\infty}\iint_\Omega g(t,x,u_n)\sin x\, dtdx$$
$$=\liminf_{n\to\infty}\iint_{|u_n(t,x)|\geq r} g(t,x,u_n(t,x))\sin x\, dtdx$$
$$+\iint_{|u_n(t,x)|<r} g(t,x,u_n(t,x))\sin x\, dtdx$$
$$\geq\iint_\Omega g(t,x,\infty)\sin x\, dtdx.$$

By (3. 9), we obtain

$$\iint_\Omega g(t,x,\infty)\sin x\, dtdx \leq -\iint_\Omega h(t,x)\sin x\, dtdx$$

which contradicts the right hand inequality in (H_5). If $\alpha<0$, then this lead a contradiction to the left hand inequality in (H_5).

This shows that the solution to (3. 7) are bounded in $L^2(\Omega)$ independently of $\lambda \in]0,1[$. By the Leray-Schauder degree theory (1. 1) has a solution. This proves theorem.

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Dong-A University
Pusan 604-714, Korea