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Quadratic Newton-Raphson Method for DC and Transient Analyses of Electronic Circuits

(電子回路의 DC 및 過渡解析을 위한 2次 Newton-Raphson 方法)

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要 約

본 논문에서는 회로해석 중에서 DC및 과도(transient) 해석에 필요한 비선형 대수 방정식을 풀기 위한 새로운 방법으로서 Quadratic Newton Raphson Method(QNRM)을 제안한다. QNRM은 Newtok-Raphson method(NRM)에 기본을 두고 있지만, 비선형 대수 방정식의 Taylor 급수 전개에서 2차 미분항을 포함한다. 각 반복 과정에서 미지수에 관한 2차식이 되는데 해를 예측함으로서 선형화 할 수 있다. QNRM의 수렴속도를 올리기 위해서는 이 해의 정확한 예측이 매우 중요하며 그 한 방법을 제시하였다. QNRM을 DC및 과도해석에 적용한 결과 NRM을 사용한 것보다 계산시간 및 반복횟수에 있어서 25% 이상 감소됨을 보여주었다.

Abstract

In this paper we propose a new method for solving a set of nonlinear algebraic equations encountered in the DC and transient analyses of electronic circuits. This method will be called Quadratic Newton-Raphson Method (QNRM), since it is based on the Newton-Raphson Method (NRM) but effectively takes into account the second order derivative terms in the Taylor series expansion of the nonlinear algebraic equations. The second order terms are approximated by linear terms using a carefully estimated solution at each iteration. Preliminary simulation results show that the QNRM saves the overall computational time significantly in the DC and transient analysis, compared with the conventional NRM.

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I. Introduction

The Newton-Raphson Method (NRM) is perhaps the most widely used iterative method for solving a set of nonlinear equations which are first transformed approximately into a set of linear equations. This method considers only the first order terms of the Taylor series expansion of the original set of nonlinear equations around the current solution point obtained in the iterative procedure, and converges quadratically when the initially assumed solution point is close enough to the true solution point.

In this paper we take into consideration the second order terms of the Taylor series expansion in the hope of improving the approximation and hence convergence. In this method the second order terms are actually transformed approximately into linear terms, however, utilizing the currently available solution in order to linearize the whole problem. This approach to be called Quadratic Newton Raphson Method (QNRM), may then be regarded as an improved linearization of the nonlinear equations.

In applying the QNRM to the circuit simulation problem we may use a companion model for each nonlinear element as modified in accordance with the concept described in the above. Incorporating the second order effect into the companion model requires negligible computation overhead and saves the overall computation time significantly. Extensive simulation results show that with the modified companion model, the number of iterations needed is about halved and the overall computation time is saved by about 30%, in average, as compared with the conventional companion model based on the NRM. About the same improvements were obtained when the QNRM is incorporated into SPICE [4].

II. Quadratic Newton-Raphson Method

In order to give the idea of the new approach, let us first consider the simplest case, i.e., a nonlinear equation $f(x)=0$ with a single variable x. Under the assumption that $f(x)$ is differentiable three times, the Taylor series expansion of $f(x)$ about $x=x^k$ yields

$$f(x)=f(x^k)+f'(x^k)\cdot(x-x^k)+\frac{f''(x^k)}{2}\cdot(x-x^k)^2+\frac{f'''(\xi)}{6}\cdot(x-x^k)^3 \qquad (1)$$

where $x \leqslant \xi \leqslant x^k$

The NRM linearizes $f(x)$ at $x=x^k$ taking only the first order term in Eq. (1), and tries to reach the true solution iteratively. That is,

$$0=f(x)\simeq f(x^k)+f'(x^k)\cdot(x^{k+1}-x^k)$$

or

$$x^{k+1}=x^k-\frac{f(x^k)}{f'(x^k)} \qquad (2)$$

A better approximation may be obtained if we include the second order term in Eq. (1), namely

$$0=f(x)\simeq f(x^k)+f'(x^k)\cdot(x^{k+1}-x^k)+\frac{f''(x^k)}{2}\cdot(x^{k+1}-x^k)^2 \qquad (3)$$

Solution of this equation is complicated by the presence of the quadratic term, although it is manageable in the case of a single variable. However, by approximating the second order term $(x^{k+1}-x^k)^2$ by $(\hat{x}^{k+1}-x^k)\cdot(x^{k+1}-x^k)$, where $\hat{x}^{k+1}$ is an estimate of the solution obtained in some way at each iteration, we can linearize Eq. (3) as

$$0=f(x)\simeq f(x^k)+\left[f'(x^k)+\frac{f''(^k)}{2}\cdot(\hat{x}^{k+1}-x^k)\right]\cdot(x^{k+1}-x^k) \qquad (4)$$

from which a better solution is obtained as

$$x^{k+1}=x^k-\frac{f(x^k)}{f'(x^k)+\frac{f''(x^k)}{2}\cdot(\hat{x}^{k+1}-x^k)} \qquad (5)$$

The most important part of this new approach is how to estimate $\hat{x}^{k+1}$ properly, because an improper choose of $\hat{x}^{k+1}$ may yield a worse result. Several ways of estimating $\hat{x}^{k+1}$ are conceivable; we may use (a) the NRM itself, or (b) the secant method [1], or (c) the Aitken method [1], all of which are the first order methods and hence do not require much computation time. In order to treat the order of convergence in the QNRM, we state the following theorem, which is proven in the Appendix.

Theorem 1: For a given nonlinear equation $f(x)=0$, assume that $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$

are continuous and $f'(x) \neq 0$ for all x in some neighborhood of the solution. Then, if the solution method with the p-th order of convergence is used as estimation of the solution point x^{k+1} ($p \leqslant 2$) and the initial approximation is chosen sufficiently close to the solution, the iterated approximation of the QNRM will converge to the solution with the (p+1)-th order of convergence.

To summarize the steps in each iteration of the QNRM;

step 1: calculate $f(x^k)$, $f'(x^k)$, $f''(x^k)$
step 2: estimate $\hat{x}^{k+1}$
step 3: obtain a new approximate solution x^{k+1} by Eq. (5).

For the testing purpose we have chosen the following equations given in [2]:

$$p1:\ f(x)=e^x-x-1=0$$

$$p2:\ f(x)=x+2x\cdot e^x+e^{2x}=0$$

$$p3:\ f_1(x)=x_1^2-10x_1+x_2^2+8=0$$
$$f_2(x)=x_1\cdot x_2^2+x_1-10x_1+8=0$$

$$p4:\ f_1(x)=3x_1^2-\cos(x_2x_3)-0.5=0$$
$$f_2(x)=x_1^2-81(x_2+0.1)+\sin(x_3)+1.06=0$$
$$f_3(x)=e^{-x_1x_2}+20x_3+\frac{10\pi-3}{3}=0$$

We applied the three methods for estimating $\hat{x}^{k+1}$ as suggested in the above, and obtained the results as given in Table 1. We see that QNRM with the Aitken estimate of $\hat{x}^{k+1}$, the best among these estimates, improves over the NRM significantly in the number of iterations and also in the overall computation time. However, since the Aitken method is effective only when the current approximation is close to the true solution, we will choose the NRM itself to estimate $\hat{x}^{k+1}$ for which case it can be shown that the rate of convergence of the proposed scheme is of a third order.

III. Circuit Simulation with Improved companion models

In the simulation of a nonlinear circuit, it is a general practice to employ a linearized companion model based on the NRM for each nonlinear element in the circuit. We can then expect to obtain an improved companion model by applying the QNRM to each nonlinear element.

Table 1. Comparison results of the NRM and QNRM with various estimation methods.

Method \ Problem	NRM	QNRM		
		NR Estimate	Secant Estimate	Aitken Estimate
P 1	34*/0.1+	21/0.0662	27/0.0662	19/0.0667
P 2	43/0.167	25/0.126	34/0.126	17/0.1
P 3	82/0.267	65/0.366	non-converge	61/0.366
P 4	17/0.22	10/0.2	non-converge	8/0.18

*/+ : # of iterations/computing time(sec)
(i.e. P3 has different solutions in spite of the same initiall point)

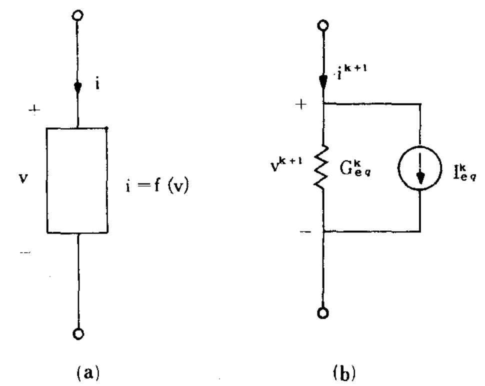

Fig.1. (a) A two terminal non-linear element (b) and its companion model.

Consider a two-terminal nonlinear element defined by $i=f(v)$ as shown in Fig 1(a). The associated companion model is depicted in Fig 1(b) where, in the case of the NRM,

$$G^k_{eq}=f'(v^k) \tag{6a}$$
$$I^k_{eq}=f(v^k)-G^k_{eq}\cdot v^k \tag{6b}$$

and, in the case of the QNRM,

$$G^k_{eq}=f'(v^k)+\frac{f''(v^k)}{2}\cdot(\hat{v}^{k+1}-v^k) \tag{7a}$$
$$I^k_{eq}=f(v^k)-G^k_{eq}\cdot v^k \tag{7b}$$

where v^k and $\hat{v}^{k+1}$ represent the current and the estimated node voltages at the k-th iteration, respectively. Note that Eq. (7) as well as Eq. (6)

represents a linear companion model

The calculation of f'' (v^k) may be trivial in some cases (as in the diode case since $i=f(v)=I_s[\exp(v/v_T) - 1]$ and $f''(v)=f'(v)/v_T$ and similarly in the bipolar transistor case), but not in other cases. In the latter cases, $f''(v^k)$ can be calculated approximately, for example, by the following formula;

$$f''(v^k) \approx \frac{f(\hat{v}^{k+1}) - f(v^k)}{\hat{v}^{k+1} - v^k} \tag{8}$$

As will be shown, a good estimate of $\hat{v}^{k+1}$ can be obtained, instead of solving the whole circuit by, say, the NRM, by using only those values related to the particular nonlinear element under consideration and obtained at the k-th iteration. Consider Fig. 1 once again. At the k-th iteration with the QNRM,

$$i^{k+1} = G^k_{eq} \cdot v^{k+1} + I^k_{eq} \tag{9}$$

A good estimate of $\hat{v}^{k+1}$ is obtained by substituting i^k for i^{k+1} in Eq. (9):

$$i^k = G^k_{eq} \cdot \hat{v}^{k+1} + I^k_{eq} \tag{10}$$

or

$$\hat{v}^{k+1} = \frac{i^k - I^k_{eq}}{G^k_{eq}} \tag{11}$$

It turn out that estimate of $\hat{v}^{k+1}$ by this formula is almost as good as that by the NRM. Since computation overhead of $\hat{v}^{k+1}$ by Eq. (11) is minimal, the overall computation time will be almost proportional to the total number of iterations. Henceforth this model with Eq. (11) substituted in Eq. (7) will be referred to as "the simple companion model by the QNRM".

IV. Transient Analysis of Dynamic Networks

As is well known, the concept of the companion model can be extended to the transient analysis of dynamic networks [3]. In the case of a linear capacitance, it is transformed to a fixed conductance in shunt with a current source which must be updated at each time point. For a nonlinear capacitance, it is transformed to a nonlinear conductance in shunt with a current source. Together with the companion model for the nonlinear resistive elements in the circuit, the whole resistive circuit is solved iteratively to get a solution at each time point. The simple companion model by the QNRM as introduced in the previous section fits this scheme readily, and was incorporated into SPICE to handle transient analysis of dynamic linear (or nonlinear) networks.

V. Simulation Results

Table 2 compares the simulation results of typical BJT circuits (bench mark circuits given in the SPICE manual [4]) by the NRM and by the QNRM. The QNRM reduces not only the number of iterations but also the overall computation time by about 20% as compared with the conventional NRM. In particular, simulation by the simple companion model by the QNRM reduced the computation time approximately in proportion to the number of iterations. Reduction of computation time by the simple companion model by the QNRM, due to the reduced number of linerization processes, was not significant for relatively small circuits. In Table 2, in case of the differential amplifier the conventional NRM failed to converge because of the overflow

Table 2. Simulation results of BJT circuits by the NRM and by the QNRM.

Circuit \ Method	Method 1	Method 2	Method 3	Method 4	Method 5
Simple Diode	38*	20	20	5	4
One Stage Tr.Amp	40*/0.368⁺	22/0.35	22/0.22	7/0.12	6/0.1
Schmitt Triger	35/0.79	18/0.71	20/0.48	7/0.2	7/0.2
Differential Amp	non-converge	21/0.38	non-converge	6/0.2	5/0.16
Cascaded RTL INV	37/0.467	20/0.39	20/0.25	6/0.13	5/0.11

* : # of iterations
*/⁺ : # of iterations/computing time(sec)
Method1 : Newton Raphson Method
Method2 : QNRM (estimate of V^{k+1} by NRM)
Method3 : QNRM (estimate of V^{k+1} by simplified companion model)
Method4 : Modified NRM (both I and V axes are explored as in SPICE)
Method5 : QNRM (estimate of V^{k+1} or i^{k+1} by method4)

of the exponential term in the diode equation, although it converges when both current and voltage axes are explored during linerization. With the simple companion model by the QNRM, this cumbersome exploration was not needed for convergence.

The QNRM with the simple companion model was incorporated into SPICE2, and the result is given in Table 3. Again the bench mark circuits given in a SPICE manual were used. Table 3 shows that in the cases of 741 OP amp and 74LS inverter, in particular, 25% reduction of the computation time is obtained with the QNRM. Table 4 shows the results of transient analysis of 74LS inverter. For the first input waveform (a pulse with a rising rate 5v/3ns) SPICE with the NRM failed to converge, whereas SPICE with the QNRM did converge. For the second input waveform (a pulse with a rising rate 5v/5ns) both methods yielded covergence. But with the QNRM the transient solution was obtained with less time points (less computation time) and greater accuracy than with the NRM, due to the time step control in SPICE by the local trunction error.

Table 4. Simulation results of transient analysis with the QNRM.

Circuit \ Method		SPICE SPICE	SPICE with the companion model by QNRM	Reduction ratio
74LS	Input 1	non-convergence	74sec	
Inverter	Input 2	94.27sec	81.25	14%

VI. Conclusion

The proposed QNRM when applied to circuit simulation requires less number of iterations, less overall computation time and yields greater accuracy as compared with the NRM.

The QNRM can be implemented in a simple way if we use the conventional circuit simulator (such as SPICE) which uses the companion model for nonlinear elements and dynamic elements, since only the associated companion model need be modified according to the proposed scheme.

In the present paper the QNRM was applied only to the BJT circuits. Its application to the MOS circuits is expected to encounter some difficulty in calculating the second order derivative of the drain current as a function of the drain-source voltage and the gate voltage. This problem will be investigated in the future. Another topic is a better estimate of $\hat{v}^{k+1}$ for convergence acceleration other than that proposed in this paper.

Table 3. Simulation results of the QNRM with the simple companion model.

Circuit \ Method	# of elements	# of nodes	SPICE CPU time for DC sol.	SPICE with the on simplified companion model by QNRM CPU time for DC sol.	Reduction ratio
Diode Circuit	10Diodes	12	10/1.15	99/1.05	16%
RTL Inverter	2BJT's	6	10/0.68	9/0.61	11%
Schmitt Triger	4BJT's	8	66/3.65	38/2.35	36%
Diff Amp	4BJT's	8	10*/1.15+	9/1.05	9%
74LS Inverter	5BJT's 3Diodes	15	17/3.48	11/2.63	25%
Shottky Inverter	8BJT's 11Diodes	28	28/8.6	24/7.28	16%
741OP Amp	28BJT's	27	21/11.21	13/8.383	26%
Active LPF 1	128BJT's	171	13/47.67	8/39.28	18%
Active LPF 2	175BJT's	240	17/84.48	11/73.48	13%
Active BPF	312BJT's	291	52/377.7	27/287.0	24%

/ : # of iterations/computing time(sec)

Appendix

The proof of Theorem 1:

For the notational convenience, let x_n denote the n-th approximation of the QNRM and $\hat{x}_{n+1}$ the estimation of x_n ($n \geqslant 0$). Then, the second order Taylor approximation of f(x), expanded about x_n, is

$$0 = f(x_n) + f'(x_n)\cdot(\alpha - x_n) + \frac{1}{2} f''(x_n)\cdot(\alpha - x_n)^2 + \frac{1}{6} f'''(\xi_n)\cdot(\alpha - x_n)^3 \quad \text{(A1)}$$

where α is the solution point of f(x) and $x_n \leqslant \xi_n \leqslant \alpha$

The n-th QNRM approximation is

$$0 = f(x_n) + \left[f'(x_n) + \frac{1}{2} f'(x_n)\cdot(\hat{x}_{n+1} - x_n) \right]\cdot (x_{n+1} - x_n) \quad \text{(A2)}$$

and hence,

$$x_{n+1} = x_n - \frac{f'(x_n)}{f'(x_n) + \frac{1}{2} f''(x_n)\cdot(\hat{x}_{n+1} - x_n)} \quad \text{(A3)}$$

Using Eqs. (A1) and (A3),

$$(\alpha - x_{n+1}) = -\frac{\frac{f''(x_n)}{2f'(x_n)}(\alpha - \hat{x}_{n+1})\,(\alpha - x_n) + \frac{f'''(\xi_n)}{6f'(x_n)}(\alpha - x_n)^3}{1 + \frac{f''(x_n)}{2f'(x_n)}\cdot(\hat{x}_{n+1} - x_n)} \quad \text{(A4)}$$

If the estimation has the p-th order of convergence, the relation between x_n and $\hat{x}_{n+1}$ can be written as

$$(\alpha - \hat{x}_{n+1}) = E_n\cdot(\alpha - x_n)^p \quad \text{(A5)}$$

where E_n is the error formula factor of the estimation method and is bounded by E for all n (i.e.$|E_n| < E$ for $n \geqslant 0$) [1]. In addition, the estimation method in Eq. (A5) leads $\hat{x}_{n+1}$ to α in itself, which implies that

$$|E_n\cdot(\alpha - x_n)^{p-1}| < 1 \quad \text{(A6)}$$

Without loss of generality, assume the estimation method in Eq. (A5) can find the solution α for any initial approximation in the interval I, where the interval $I=[\alpha-\epsilon,\ q+\epsilon]$ satisfies the assumptions of the theorem and ϵ is some positive number. And let

$$M = \frac{\underset{x\in I}{\mathrm{Max}}\,|f''(x)|}{2\underset{x\in I}{\mathrm{Min}}\,|f'(x)|} \quad \text{(A7)}$$

Substituting Eq. (A5) into Eq. (A4), we obtain

$$(\alpha - x_{n+1}) = Q_n\cdot(\alpha - x_n)^{p+1} \quad \text{(A8)}$$

$$Q_n = -\frac{\frac{f''(x_n)}{2f'(x_n)}\cdot E_n + \frac{f'''(\xi_n)}{6f'(x_n)}(\alpha - x_n)^{2-p}}{1 + \frac{f''(x_n)}{2f'(x_n)}\cdot[1 - E_n(\alpha - x_n)^{p-1}]\cdot(\alpha - x_n)} \quad \text{(A9)}$$

In (A8), Q_n is bounded by some fixed value Q with choosing $|\alpha\text{-}x_n| < \frac{1}{2M}$ which implies that $|\alpha\text{-}x_{n+1}| < Q.|\alpha\text{-}x_n|^{p+1}$. Pick an initial approximation x_0 such that $|\alpha\text{-}x_0| \leqslant \epsilon$, $M.|\alpha\text{-}x_0| < \frac{1}{2}$ and $Q.|a\text{-}x_0|^p < 1$. Then, from (A8), $|\alpha\text{-}x_1| \leqslant \epsilon$, $M.|\alpha - x_1| < \frac{1}{2}$ and $Q.|\alpha\text{-}x_1|^p < 1$ and $|\alpha\text{-}x_1|x<|\alpha\text{-}x_0|$. We can apply the same argument to $x_1, x_2, \ldots$, inductively, showing that $|a\text{-}x_n| \leq \epsilon, M.|\alpha\text{-}x_n| < \frac{1}{2}$, $Q.|\alpha\text{-}x_n|^p < 1$ and $|\alpha\text{-}x_n| < |\alpha\text{-}x_{n-1}|$ for all $n \geqslant 1$. Consequently, we obtain

$$|\alpha - x_n| < \frac{1}{Q^{1/p}}\cdot[Q^{1/p}\,|\alpha - x_0|]^{(p+1)^n} \quad \text{(A10)}$$

Since $Q.|\alpha\text{-}x_0|^p < 1$, (A10) shows that $x_n \to \alpha$ as $n \to \infty$ Also, (A8) represents that the the convergence order of the QNRM is equal to (p+1). This completes the proof.

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