SOME CHARACTERIZATIONS OF THE BLOCH SPACE IN THE BALL

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1. Introduction

Let $B=B_n$ be the unit ball in $\mathbb{C}^n$ and let $H(B)$ be the space of all holomorphic functions in the ball. For $z, \zeta \in \mathbb{C}^n$, we let $\langle z, \zeta \rangle = \sum_{j=1}^{n} z_j \bar{\zeta}_j$ be the inner product and let $\|z\| = \langle z, z \rangle^{1/2}$ be the Euclidean norm. Let $dv$ be the usual Lebesgue measure on $\mathbb{C}^n$ and let $d\sigma$ be the normalized surface measure on the boundary $\partial B$ of $B$. For $q>0$, $dv_q$ is the probability measure on $B$ defined by

$$dv_q(z) = \frac{\Gamma(n+q)}{\pi^n \Gamma(q)} (1-\|z\|^2)^{q-1} dv(z).$$

We let $A^q_v(B) = \{ f \in H(B), \|f\|_{A^q_v} = \int_B |f(z)|^q dv(z) < \infty \}$ be the weighted Bergman spaces.

**Definition** For $q>0$, we define $B^q$ to be the space of all functions $f$ in $H(B)$ such that

$$\|f\|_{B^q} = \sup_{z \in B} (1-\|z\|^2)^q \left| (R+n+1) f(z) \right| \leq \infty,$$

where $R f(z) = \sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i}(z)$ is the radial derivative introduced in [7].

The definition does not change if one replaces $R+n+1$ by $R+1$, but one can get sharper norm estimate in some other context, see [6].

When $q=1$, $B^1$ become the usual Bloch space and there are plenty of papers about it, most of them are one variable results, see [1, 2, 4, 5, 8] for example. In this paper, we give some generalizations of one variable results, with the weight $(1-\|z\|^2)^q$. Other generalizations will

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appear in [6]. The Bloch space with weight appears in [3] but we have not seen it elsewhere.

2. Equivalent norms on the Bloch spaces

Let $\phi_a(z)$ be the automorphism of $B$ given by

$$\phi_a(z) = \frac{a - P_a z - s Q_a z}{1 - \langle z, a \rangle}$$

where $P_a z = \frac{\langle z, a \rangle}{\|a\|^2} a$, $Q_a z = (I - P_a) z$ and $s = \sqrt{1 - \|a\|^2}$. It is easy to check that $\phi_a(0) = a$, $\phi_a(a) = 0$, $\phi_a^2 = I$ and the real Jacobian of $\phi_a(z)$ is $[(1 - \|a\|^2) / (1 - \langle z, a \rangle^2)]^{n+1}$. We let

$$B(a, r) = \{ z \in B : \|\phi_a(z)\| < r \}$$

be the pseudo-hyperbolic ball of radius $r$.

**Lemma 1.**

1. $1 - \|\phi_a(z)\|^2 = \frac{(1 - \|a\|^2)(1 - \|z\|^2)}{1 - \langle z, a \rangle^2}$
2. For any multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$, we have

$$\int_{\partial B} |z^\alpha|^2 d\sigma(\zeta) = \frac{(n-1)! \alpha!}{(n-1+|\alpha|)!}.$$
3. For a fixed $0 < r < 1$, the quotient $v_q(B(a, r)) / (1 - \|a\|^2)^{n+q}$ approaches a positive constant as $\|a\| \to 1$.
4. $\int_B \|z\|^2 dv_q(z) = \frac{n}{n+q}$.

**Proof.** For (1) and (2), see [7]. For (3), we use the change of variable $w=\phi_a(z)$ to get

$$\int_{B(a, r)} dv_q(w) = \int_{B(0, r)} \left( \frac{1 - \|a\|^2}{1 - \langle z, a \rangle^2} \right)^{n+1} (1 - \|\phi_a(z)\|^2)^{q-1} dv_1(z)$$

$$= \int_{B(0, r)} \left( \frac{1 - \|a\|^2}{1 - \langle z, a \rangle^2} \right)^{n+q} dv_q(z)$$

$$= (1 - \|a\|^2)^{n+q} \int_{|z| < r} |1 - \langle z, a \rangle|^{-2s-2q} dv_q(z)$$

Since $1 - r \leq |1 - \langle z, a \rangle| \leq 1 + r$, the result follows. (4) follows directly from the polar coordinate change:

$$\int_B \|z\|^2 dv_q(z) = \frac{2\Gamma(n+q)}{\Gamma(n) \Gamma(q)} \int_0^1 r^{2(n-1)} (1 - \rho^2)^{q-1} \int_{|z| < r} |\rho z|^2 d\sigma(z) d\rho$$

$$= \frac{\Gamma(n+q)}{\Gamma(n) \Gamma(q)} \int_0^1 r^n (1 - s)^{q-1} ds = \frac{n}{n+q}.$$
Lemma 2. For $1 \leq q$, we have $\|f \circ \varphi_z\|_{B_q} \leq C_q (1 - \|z\|^2)^{1 - q} \|f\|_{B_q}$, where $C_q$ depends only on $q$.

Proof. We let $w = \varphi_z(\zeta)$ and compute $R(f \circ \varphi_z)$. With $s = \sqrt{1 - \|z\|^2}$,
\[
\frac{\partial (f \circ \varphi_z)}{\partial \zeta^i} = \sum_j \frac{\partial f}{\partial \omega_j} \cdot \frac{\partial \omega_j}{\partial \zeta^i} = \frac{1}{(1 - \langle \zeta, z \rangle)^2} \left( \sum_j \frac{\partial f}{\partial \omega_j} \xi_j z_j s (1 - s) \right) - s \xi_i \sum_j \frac{\partial f}{\partial \omega_j} \xi_j - s \frac{\partial f}{\partial \omega_i} (1 - \langle \zeta, z \rangle) \right).
\]
Hence
\[
R(f \circ \varphi_z)(\zeta) = \sum_{i=1}^n \frac{s}{(1 - \langle \zeta, z \rangle)^2} \left( \langle \zeta, z \rangle \frac{\langle Pf, \bar{\zeta} \rangle}{1 + s} \right) - s \frac{\partial f}{\partial \omega_i} (1 - \langle \zeta, z \rangle).
\]
But since $\langle Pf, \varphi_z(\zeta) \rangle = \frac{1}{1 - \langle \zeta, z \rangle} \left( \frac{1 + s - \langle \zeta, z \rangle}{1 + s} \right) \langle Pf, \bar{\zeta} \rangle - s \langle Pf, \bar{\zeta} \rangle$, we solve for $\langle Pf, \bar{\zeta} \rangle$ and combine it with the above equation to obtain
\[
R(f \circ \varphi_z)(\zeta) = \frac{1}{1 - \langle \zeta, z \rangle} \langle Pf, \bar{\zeta} \rangle.
\]
Now (1) of Lemma 1 gives
\[
(1 - \|z\|^2)^{q} | R(f \circ \varphi_z)(\zeta) | \leq 2 (1 - \|\varphi_z(\zeta)\|^2)^q \langle Pf, \varphi_z(\zeta) \rangle |
\]
\[
\times \frac{1 - \langle \zeta, z \rangle}{(1 - \|z\|^2)^q} \langle Pf, \bar{\zeta} \rangle
\]
\[
\leq C_q \cdot \|f\|_{B_q} \cdot (1 - \|z\|^2)^{1 - q}.
\]

Theorem 1. Let $1 \leq q < 2$ and $0 < r < 1$ be fixed. Then the following statements are equivalent:

(A0) $f \in B_r$;

(A1) $\sup_{z \in B} (1 - \|z\|^2)^q | Pf (z) | < \infty$;

(A2) $\sup_{z \in B} (1 - \|z\|^2)^q \|Pf (z)\| < \infty$;

(A3) $\sup_{z \in B} (1 - \|z\|^2)^q | Pf, \bar{z} \rangle < \infty$;

(A4) $\sup_{z \in B} Q_f^q(z) < \infty$, where
\[
Q_f^q(z) = \sup_{|\zeta| = 1} (1 - \|z\|^2)^q \frac{|\langle Pf(z), \bar{\zeta} \rangle|}{\sqrt{(1 - \|z\|^2) \|z\|^2 + |\zeta, \bar{\zeta}|^2}};
\]
(B0) \( \sup_{z \in B} (1 - \|z\|^2)^{2r-1} (\|P f(z)\|^2 - |\langle P f(z), \tilde{z} \rangle|^2) \)
= \( \sup_{z \in B} (1 - \|z\|^2)^{2r-2} \|P (f \circ \phi_z) (0)\|^2 < \infty \);
(B1) \( \sup_{z \in B} (1 - \|z\|^2)^{2r-1} \|f \circ \phi_z - f(z)\|_{z, q} < \infty \);
(C) \( \sup_{z \in B} (1 - \|z\|^2)^{2q-2} \int_{B(z, r)} |f(\zeta) - f_B(z, r)|^2 d\nu_q(\zeta) < \infty \), where
\( f_B(z, r) = \frac{1}{\nu_q(B(z, r))} \int_{B(z, r)} f(\zeta) d\nu_q(\zeta) \);
(D) \( \sup_{z \in B} (1 - \|z\|^2)^{2q-2} \int_{B(z, r)} \|f(\zeta) - f(z)\|_{z, q}^2 d\nu_q(\zeta) \);

Proof. The equivalence of (A0) through (A4) follows as in [8]. To see that (B0) implies (A2), we observe that from the proof of Lemma 2
\[ \left| \frac{\partial}{\partial \xi_i} (f \circ \phi_z) (0) \right| = \left| \frac{s(1-s)}{\|z\|^2} \sum_j \frac{\partial f}{\partial z_j} z_j - s \frac{\partial f}{\partial \xi_i} \right| . \]
Hence
\[ \|P (f \circ \phi_z) (0)\|^2 = \sum \left( \frac{s^2 (1-s)^2}{\|z\|^4} \sum_j \frac{\partial f}{\partial z_j} z_j \right)^2 + \left( \frac{s^2 (1-s)}{\|z\|^2} \right)^2 - \frac{2s^2 (1-s)}{\|z\|^2} \left( \sum_j \frac{\partial f}{\partial z_j} z_j - s \frac{\partial f}{\partial \xi_i} \right)^2 \]
\[ = \frac{s^2 (1-s)^2}{\|z\|^2} |\langle P f, \tilde{z} \rangle|^2 - s^2 \|P f\|^2 \]
\[ - \frac{2s^2 (1-s)}{\|z\|^2} |\langle P f, \tilde{z} \rangle|^2 \]
\[ = (1 - \|z\|^2) (\|P f\|^2 - |\langle P f, \tilde{z} \rangle|^2) . \]
We decompose \( z \) as \( a \frac{P f(z)}{\|P f(z)\|} + b \tilde{z} \), where \( \langle P f(z), \tilde{z} \rangle = 0 \), \( \|\tilde{z}\| = 1 \).
Then \( \nu_\xi \)
\[ \|P (f \circ \phi_z) (0)\|^2 = (1 - \|z\|^2) (\|P f\|^2 - |\langle P f, \tilde{z} \rangle|^2) \]
\[ = (1 - \|z\|^2) (1 - |a|^2) \|P f\|^2 \geq (1 - \|z\|^2) \|P f\|^2 . \]
Multiplying \( (1 - \|z\|^2)^{2q-2} \) on both sides, we see that (B0) implies (A2). To see the converse, we consider the identity;
\[ \|P f\|^2 = \frac{|\langle P f(z), \tilde{z} \rangle|^2 + |\langle P f(z), \tilde{z} \rangle|^2}{\|z\|^2} , \]
where \( \langle \zeta, \xi \rangle = 0 \) and \( \|\xi\| = 1 \). Then the quantity in (B0) becomes
\[ \left( \frac{1 - \|z\|^2}{\|z\|} \right)^2 |\langle P f(z), \tilde{z} \rangle|^2 \left( \frac{(1 - \|z\|^2)^2 |\langle P f(z), \tilde{z} \rangle|^2}{\|z\|} \right)^2 + (1 - \|z\|^2)^{2r-1} |\langle P f, \tilde{z} \rangle|^2 . \]
It is clear that we only have to consider the case \( \|z\| > 1/2 \). The first
term is bounded by (A2), while the second term is bounded by (A4). This proves the implication \((A2) \Rightarrow (B0)\). Now we assume the condition \((B1)\). Then

\[
\frac{(1-||z||^2)^{2q-2}}{v_q(B(z, r))} \int_{B(z, r)} |f(\zeta) - f(z)|^2d\nu_q(\zeta) \\
= \frac{(1-||z||^2)^{2q-2}}{v_q(B(z, r))} \int_{B(0, r)} |f \circ \varphi_x(w) - f(z)|^2 \left(\frac{1-||z||^2}{|1-\langle w, z \rangle|^2}\right)^{\pi+q}d\nu_q(w) \\
\leq C(1-||z||^2)^{2q-2}||f \circ \varphi_x - f(z)||_{Z_q}^2 
\]

by \((3)\) of Lemma 1.

Hence \((B1)\) implies \((D)\). To see that \((B1)\) implies \((B0)\), we consider, for \(f \in A_q^2(B)\)

\[
f(z) = \int_B \frac{f(\zeta)}{(1-\langle z, \zeta \rangle)^{\pi+q}}d\nu_q(\zeta).
\]

Hence \(\frac{\partial f}{\partial \overline{z}_j}(0) = (n+q)\int_B f(\zeta) \overline{z}_j d\nu_q(\zeta)\) and a use of the Cauchy–Schwarz inequality gives

\[
(1) \quad ||\nabla f(0)||^2 \leq (n+q)^2 \int_B |f(\zeta)|^2d\nu_q(\zeta) = (n+q)^2||f||_{Z_q}^2
\]

Applying this to \(f \circ \varphi_x - f(z)\), we obtain

\[
(1-||z||^2)(||\nabla f(z)||^2 - ||\nabla f(x), \overline{z}||^2) \leq (n+q)^2||f \circ \varphi_x - f(z)||_{Z_q}^2.
\]

Thus \((B1)\) implies \((B0)\). To show the converse, we observe the following:

\[
|f(z) - f(0)| \leq \int_0^1 |\nabla f(tx), \overline{\zeta}| |dt| \leq \int_0^1 ||\nabla f(tx)|||dt|
\]

Replacing \(f\) with \(f \circ \varphi_x\), we see that for \(0 < q < 2\)

\[
|f \circ \varphi_x(\zeta) - f(z)| \leq ||f \circ \varphi_x||_{B_q} \cdot \int_0^1 (1-t^2||\zeta||^2)^{-q}dt \leq C||f \circ \varphi_x||_{B_q} \times ((1-||\zeta||^2)^{1-q} - 1).
\]

Hence

\[
\int_B |f \circ \varphi_x(\zeta) - f(z)|^2d\nu_q(\zeta) \leq C||f \circ \varphi_x||_{B_q}^2 \\
\times \int_B (1-||\zeta||^2)^{2-2q}d\nu_q \leq C||f \circ \varphi_x||_{B_q}^2.
\]

The case \(q=1\) is dealt with similarly. If we multiply by \((1-||z||^2)^{2q-2}\) on both sides, we get the implication \((B0) \Rightarrow (B1)\) by Lemma 2. The implication \((D) \Rightarrow (C)\) is trivial by the triangle inequality. Finally, we complete the proof by showing that \((C)\) implies \((B0)\). Using polar coordinate and \((2)\) of the Lemma 1, we see that

\[
\int_{|\zeta| < r} f(\zeta) \overline{\zeta}d\nu_q(\zeta)
\]
$$\frac{\partial f}{\partial z_i}(0) \int_{B(0, r)} |\zeta_1|^2 d\nu_q(\zeta) = \frac{2\Gamma(n+q)}{\Gamma(n)\Gamma(q)} \cdot \frac{\partial f}{\partial z_i}(0) \int_0^r r^{n-1}(1-r^2)^{q-1} d\sigma(\zeta)$$

$$= \frac{\Gamma(n+q)}{\Gamma(n)\Gamma(q)} \cdot \frac{\partial f}{\partial z_i}(0) \int_0^r r^{n} (1-s)^{q-1} ds = C(r, q) \frac{\partial f}{\partial z_i}(0),$$

where $C(r, q) \rightarrow \frac{n}{n+q}$ as $r \rightarrow 1^-$. Hence we have

$$||f(0)||^2 \leq \frac{r^2}{C(r, q)^2} \int_{|z| < r} |f(\zeta)|^2 d\nu_q(\zeta).$$

Now fix $z \in B$ and replace $f$ by $f^o\phi_z - f_B(z, r)$ to obtain

$$(1-||z||^2)(||f||^2 - |\langle f(z), \bar{z} \rangle|^2)$$

$$\leq \frac{1}{C(r, q)^2} \int_{|w| < r} |f^o\phi_z(\zeta) - f_B(z, r)|^2 d\nu_q(\zeta)$$

$$= \frac{1}{C(r, q)^2} \int_{B(z, r)} |f(w) - f_B(z, r)|^2 \left(\frac{1-||z||^2}{1-\langle w, z \rangle} \right)^{n+q} d\nu_q(w)$$

$$\leq \frac{C}{\nu_q(B(z, r))} \int_{B(z, r)} |f(w) - f_B(z, r)|^2 d\nu_q(w)$$

where the last inequality is due to Lemma 1. Now we multiply by $(1-||z||^2)^{2q-2}$ on both sides to see that (C) implies (B0).

The above proof works word by word when $B^q$ is replaced by the “Little Bloch” space $B^{q}_0$ which is defined in an obvious way:

$$B^{q}_0 = \{ f \in H(B) : (1-||z||^2)^q |(R+n+1)f(z)| \rightarrow 0 \text{ as } ||z|| \rightarrow 1^- \}.$$

**Theorem 2.** Let $1 \leq q < 2$ and $0 < r < 1$ be fixed. Then the following statements are equivalent:

(A0) $f \in B^{q}_0$;

(A1) $\lim_{|z| \rightarrow 1^-} (1-||z||^2)^q |Rf(z)| = 0$;

(A2) $\lim_{|z| \rightarrow 1^-} (1-||z||^2)^q ||f(z)|| = 0$;

(A3) $\lim_{|z| \rightarrow 1^-} (1-||z||^2)^q \langle f(z), \bar{z} \rangle = 0$;

(A4) $\lim_{|z| \rightarrow 1^-} Qf'(z) = 0$;

(B0) $\lim_{|z| \rightarrow 1^-} (1-||z||^2)^{2q-1}(||f(z)||^2 - |\langle f(z), \bar{z} \rangle|^2) = 0$;

(B1) $\lim_{|z| \rightarrow 1^-} (1-||z||^2)^{q-1} ||f^o\phi_z - f(z)||_{2, q} = 0$;

(C) $\lim_{|z| \rightarrow 1^-} (1-||z||^2)^{2q-2} \int_{B(z, r)} |f(\zeta) - f_B(z, r)|^2 d\nu_q(\zeta) = 0$;

(D) $\lim_{|z| \rightarrow 1^-} (1-||z||^2)^{2q-2} \int_{B(z, r)} |f(\zeta) - f(z)|^2 d\nu_q(\zeta) = 0$. 
Remark. The above results for the case $n=1$ and $q=1$ are due to S. Axler [2]. We used basically the same method to generalize them to several variables. The restriction $1 \leq q$ is necessary only to prove the implication $(B0) \Rightarrow (B1)$, because we use Lemma 2. For other case, the poor works for $0 < q < 2$. We thank the referee for pointing out this.

References


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