A BASIC PRINCIPLE OF TOPOLOGICAL VECTOR SPACE THEORY

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Recently, P. Antosik and C. Swartz established an effectual matrix method in analysis ([1], [2], [3], [4]). The kernel of the matrix method is the Mikusinski-Antosik-Swartz basic matrix theorem ([2], [3]). Many important basic results of functional analysis and measure theory, such as the uniform boundedness principle, the Orlicz-Pettis theorem, Schur lemma, Phillips lemma, and the Nykodym boundedness theorem can be conveniently obtained from the basic matrix theorem ([1], [2], [3], [4]).

In this paper, we will improve the Mikusinski-Antosik-Swartz theorem. Specifically, we will get our matrix theorem only from the definition of topological vector space, though C. Swartz and P. Antosik got their theorem by the uniform structure theory of topological groups. Thus, we will show that the basic matrix theorem is the most fundamental principle in topological vector space theory.

Throughout the remainder of the paper, G will denote an Abelian topological group, N will denote the set of all symmetric zero neighborhoods of G. From the definition of G, we come to an immediate conclusion: For every neighborhood U of zero of G there is a VEN such that V + V ≤ U.

We will only use this initial property of G.

**Theorem 1.** Let xi j∈G for i, j∈N such that lim_ j xij=0 for each i∈N. The followings are equivalent.

1. lim_i xij=xj exists uniformly for j∈N;
2. lim_i xij=xj exists for each j∈N, and for every U∈N there is a sequence {pi}⊆N such that if pi≤qi∈N and {j_i}⊆N are arbitrary,
then \( \sum_{k=1}^{n} \left[ x_{q_{n_k} j_k} - x_{q_{n_k+1} j_k} \right] \in U \) for some subsequence \( \{j_{n_k}\} \subseteq \{j_n\} \), \( \{x_{q_{n_k}}\} \) \( \subseteq \{x_{q_n}\} \) and \( m, k_0 \in \mathbb{N}, m \geq k_0 \).

**Proof.** \((1) \rightarrow (2)\) Let \( U_0 \in \mathbb{N} \). Take a sequence \( \{U_n\} \subseteq \mathbb{N} \) such that
\[
U_{n+1} + U_{n+1} \in U_n \quad \text{for } n = 0, 1, 2, \ldots.
\]
The condition (1) says that for every \( i \in \mathbb{N} \) there is a \( p_i \in \mathbb{N} \) such that \( x_{p_j} - x_j \in U_i \) for all \( j \in \mathbb{N} \) and \( p_i \leq p \in \mathbb{N} \).

We may assume that, \( p_1 < p_2 < p_3 < \ldots \).

Let \( p_i \leq q_i \in \mathbb{N} \) and \( \{j_n\} \subseteq \mathbb{N} \) be arbitrary. Then,
\[
\sum_{n=1}^{5} \left[ x_{q_5 j_n} - x_{q_6 j_n} \right] = \sum_{n=1}^{5} \left[ x_{q_5 j_n} - x_{q_6 j_n} \right] \in (U_5 + U_6) + (U_5 + U_6) + (U_5 + U_6) \quad \text{for all } j \in \mathbb{N}.
\]
\[
\subseteq U_4 + U_4 + U_4 + U_4
\]
\[
\subseteq U_3 + U_3 + U_3
\]
\[
\subseteq U_2 + U_2
\]
\[
\subseteq U_1
\]
\[
\subseteq U_0.
\]

\((2) \rightarrow (1)\). Suppose not. Then there is a \( U_0 \in \mathbb{N} \) such that for any \( p \in \mathbb{N} \) there exists \( q \gg p \) and \( j \in \mathbb{N} \) such that
\[
x_{q_j} - x_j \notin U_0 \quad (\ast)
\]
Take a sequence \( \{U_n\} \subseteq \mathbb{N} \) such that \( U_n + U_n \subseteq U_{n-1} \) for all \( n \in \mathbb{N} \). Let \( \{p_n\} \) be the integer sequence such that the condition (2) ensured its existence with respect to \( U_2 \). There is a \( q_1 > p_1 \) and \( j_1 \in \mathbb{N} \) such that \( x_{q_1 j_1} - x_{j_1} \notin U_0 \) by \((\ast)\). But lim \( x_{p_j} = x_{j_1} \), there is an integer \( k_1 > \max(q_1, p_2) \) such that \( x_{k_1 j_1} - x_{j_1} \notin U_2 \) if \( k \geq k_1 \).

Since \( x_{q_1 j_1} - x_{j_1} = x_{q_1 j_1} - x_{k_1 j_1} + x_{k_1 j_1} - x_{j_1} \) and \( x_{q_1 j_1} - x_{j_1} \notin U_0 \), we have that \( x_{q_1 j_1} - x_{k_1} \notin U_1 \) if \( k \geq k_1 \).

There is a \( q_2 > k_1 \) and \( j_2 \) in \( \mathbb{N} \) such that
\[
x_{q_2 j_2} - x_{j_2} \notin U_0.
\]
Since \( q_2 > k_1 \), it follows that
\[
x_{q_1 j_1} - x_{q_2 j_2} \notin U_1, \quad x_{q_2 j_2} - x_{j_1} \notin U_2.
\]
Observe that \( \lim_{p} x_{p j_1} = x_{j_1} \) and \( \lim_{p} x_{p j_2} = x_{j_2} \),
there is a \( k_2 > \max(q_2, p_3) \) such that \( x_{k_2 j_1} - x_{j_1} \notin U_3 \) and \( x_{k_2 j_2} - x_{j_2} \notin U_3 \) if \( k \geq k_2 \). So, observing \( x_{q_2 j_2} - x_{j_2} = x_{q_2 j_2} - x_{k_2 j_2} + x_{k_2 j_2} - x_{j_2} \), we have that
\[
x_{q_2 j_2} - x_{k_2} \notin U_1 \quad \text{if } k \geq k_2.
\]
A basic principle of topological vector space theory

In this way, we can get integer sequences \( \{q_i\} \), \( q_i \geq p_i \), and \( \{j_i\} \) such that

\[
x_{q_{i+1} j_{i+1}} - x_{q_i j_i} \in U_1, \quad x_{q_{i+1} j_{i+1}} - x_{q_i j_i} \in U_i \quad \text{if} \quad 1 \leq i < j.
\]

(*). Set \( i_1 = 1, \quad y_{11} = x_{q_{i_1} j_{i_1}} - x_{q_{i_1+1} j_{i_1}} \). Then \( y_{11} \notin U_1 \).

Since \( \lim_{j} x_{ij} = 0 \) for each \( i \), there is an \( i_2 \geq 6 > i_1 \) such that

\[
x_{q_{i_1} j_{i_1}} - x_{q_{i_1} j_{i_1+1}} = x_{q_{i_1} j_{i_1}} - 0 + 0 - x_{q_{i_1} j_{i_1+1}} \in U_5 + U_6
\]

and, from (**),

\[
x_{q_{i_1} j_{i_1}} - x_{q_{i_1} j_{i_1+1}} = x_{q_{i_1} j_{i_1}} - x_{j_{i_1}} - x_{q_{i_1} j_{i_1+1}} \in U_{i_2} + U_{i_2+1}
\]

Thus, using the notations

\[
y_{12} = x_{q_{i_1} j_{i_1}} - x_{q_{i_1} j_{i_1+1}},
\]

\[
y_{21} = x_{q_{i_1} j_{i_1}} - x_{q_{i_1} j_{i_1+1}}, \quad \text{and}
\]

\[
y_{22} = x_{q_{i_1} j_{i_1}} - x_{q_{i_1} j_{i_1+1}}.
\]

We have that \( y_{12} \) and \( y_{21} \in U_{2+3} \), but \( y_{22} \notin U_1 \).

In this way, we have a matrix

\[
(y_{nk})_{n,k} = x_{q_{i_k} j_{i_k}} - x_{q_{i_k+1} j_{i_k}},
\]

such that \( y_{nk} \) and \( y_{kn} \in U_{n+3} \) if \( 1 \leq k \leq n \); \( y_{nn} \notin U_1 \), \( \forall n \in \mathbb{N} \). From condition (2), observing \( i_k \geq k \) and hence \( q_{i_k} \geq q_k \geq p_k \), there are \( \{j_{i_k}\} \subseteq \{j_i\} \) and integers \( m, l_0, (m \geq l_0) \) such that

\[
\sum_{i=1}^{m} y_{k_i k_i} = \sum_{i=1}^{m} (x_{q_{i_{i_k}} j_{i_k}} - x_{q_{i_{i_k}+1} j_{i_k}}) \in U_2
\]

since the sequence \( \{p_i\} \) is taken with respect to \( U_2 \). But

\[
y_{k_i k_i} = \sum_{i=1}^{m} y_{k_i k_i} - \sum_{l=1}^{l_0-1} y_{k_{l+1} k_{l}} - \sum_{l=l_0+1}^{m} y_{k_{l+1} k_{l}}
\]

\[
\in U_2 + (U_{k_{l_0+3}} + U_{k_{l_0+3}} + \cdots + U_{k_{l_0+3}})
\]

\[
+ (U_{k_{l_0+3}} + U_{k_{l_0+2}+3} + \cdots + U_{k_{l_0+3}})
\]

\[
\subseteq U_2 + U_3 + U_3 \subseteq U_2 + U_2 \subseteq U_1.
\]

This contradicts the fact \( y_{k_i k_i} \notin U_1 \).
Now we can get the Mikusinski–Antosik–Swartz basic matrix theorem ([2]), Theorem 1).

**Theorem 2.** Let \( x_{ij} \in G \) for \( i, j \in N \) satisfy
(A) \( \lim_{i} x_{ij} = x_{j} \) exists for each \( j \) and
(B) for each integer sequence \( (m_{j}) \) there is a subsequence \( (n_{j}) \) of \( (m_{j}) \) such that \( \left\{ \sum_{i=1}^{\infty} x_{ijn_{j}} \right\}_{i=1}^{\infty} \) is Cauchy.

Then \( \lim_{i} x_{ij} = x_{j} \) uniformly in \( j \in N \).

**Proof.** We claim that \( \lim_{i} x_{ij} = 0, \forall i \in N \).

Otherwise, say that \( \lim_{j} x_{ij_{0}} \neq 0 \) for some \( i_{0} \in N \), then there is \( U \in N \) and \( j_{1} < j_{2} < \cdots \) such that \( x_{i_{0}j} \notin U, \forall k \in N \).

Thus, \( \sum_{k=1}^{\infty} x_{i_{0}kj} \) has no such convergent subseries. This contradicts (B).

Now let \( U \in N \). Take \( U_{1}, U_{2} \in N \) such that \( U_{2} + U_{2} \subseteq U_{1}, U_{1} + U_{1} \subseteq U \).

Set \( \{p_{i}\} = \{1, 2, 3, \cdots \} \). Then for every \( p_{i} \leq q_{i} \in N \) and \( \{j_{n}\} \subseteq N \) there is a \( \{j_{n_{k}}\} \subseteq \{j_{n}\} \) such that \( \left\{ \sum_{i=1}^{\infty} x_{iq_{ij_{n_{k}}}} \right\}_{i=1}^{\infty} \) is a Cauchy, by (B). Thus, there is an \( i_{0} \in N \) such that
\[
\sum_{k=1}^{\infty} x_{iq_{ij_{n_{k}}}} - \sum_{k=1}^{\infty} x_{iq_{i+1j_{n_{k}}}} \in U_{2} \text{ for each } i \geq i_{0}.
\]

Take a \( k_{0} \in N \) such that \( n_{k_{0}} \geq i_{0} \). Then there is an \( m \geq k_{0} \) such that
\[
\sum_{k=m}^{\infty} x_{iq_{ik_{n_{k}}}} \in U_{2}, \sum_{k>m}^{\infty} x_{iq_{ik_{n_{k}}}} \in U_{2}.
\]

Thus,
\[
\sum_{k=1}^{m} x_{iq_{ik_{n_{k}}}} - \sum_{k=1}^{m} x_{iq_{ik_{n_{k}}}} - \left[ \sum_{k=1}^{\infty} x_{iq_{ik_{n_{k}}}} - \sum_{k=m}^{\infty} x_{iq_{ik_{n_{k}}}} \right] \n= \sum_{k=1}^{\infty} x_{iq_{ik_{n_{k}}}} - \sum_{k=m}^{\infty} x_{iq_{ik_{n_{k}}}} + \sum_{k=m}^{\infty} x_{iq_{ik_{n_{k}}}} - \sum_{k=m}^{\infty} x_{iq_{ik_{n_{k}}}}
\subseteq U_{2} + U_{2} \subseteq U_{1} + U_{1} \subseteq U.
\]

Thus, \( \{x_{ij}\} \) satisfies the condition (2) of Theorem 1. So the desired result is obtained from Theorem 1.
The conditions (A) and (B) of Theorem 2 are sufficient for the uniform convergence of columns but are not necessary.

**Example.** Consider the matrix \((1/i)\, e_j\) in \((c_0, \text{weak})\), where \(e_j = (0, 0, \cdots, 0, 1, 0, 0, \cdots), \ j \in \mathbb{N}\).

Clearly, each row tends to zero. But this matrix fails to keep the condition (B) because for each \(i\) the series \(\sum_{j=1}^{\infty} (1/i) e_j\) has no such convergent subseries. So we can not get the uniform convergence of the columns from Theorem 2. Clearly, we get it from Theorem 1. Of course, we can get it from \((1/i)\, e_j\) itself directly.

To show the forces of the matrix theorem, we will give a very general version of the uniform boundedness principle which is due to P. Antosik and C. Swartz ([5]). But we would like to give a more general version.

**Definition 3.** ([3]) Let \((E, \tau)\) be a topological vector space. A sequence \(\{x_i\}\) in \(E\) is a \(\tau\)-\(\kappa\)-convergent sequence if each subsequence \(\{x_{i_k}\}\) of \(\{x_i\}\) has a subsequence \(\{x_{i_{k_s}}\}\) such that the series \(\sum_{s=1}^{\infty} x_{i_{k_s}}\) is \(\tau\)-convergent to an element \(x\). A subset \(B \subseteq E\) is \(\tau\)-\(\kappa\)-bounded if for each \(\{x_j\} \subseteq B\) and \(\{t_j\} \subseteq c_0\) the sequence \(\{t_jx_j\}\) is \(\tau\)-\(\kappa\)-convergent.

Let \(X\) be a linear space, \(Y\) be a topological vector space and \(\Gamma\) be a family of linear maps from \(X\) to \(Y\). Let \(\tau(\Gamma)\) be the weakest topology on \(X\) such that each member of \(\Gamma\) is continuous. Clearly, \(\tau(\Gamma)\)

\[
x_j \xrightarrow{\tau} 0 \text{ if and only if } T(x_j) \xrightarrow{\tau} 0, \ \forall T \in \Gamma.
\]

It is easy to check that \(\tau\)-\(\kappa\)-boundedness implies \(\tau\)-boundedness but they are same in Frechet space case ([3]).

**Theorem 4.** Let \(X\) be a linear space, \(Y\) be a topological vector space and \(\Gamma\) be a family of linear maps from \(X\) to \(Y\). If \(\Gamma\) is pointwise bounded, i.e., \(\{T(x) : T \in \Gamma\}\) is bounded in \(Y\) for each \(x \in X\), then \(\Gamma\) is uniformly bounded on \(\tau(\Gamma)\)-\(\kappa\) bounded sets and \(\tau(\Gamma)\)-\(\kappa\) convergent sequences.

**Proof.** Let \(B \subseteq X\) be a \(\tau(\Gamma)\)-\(\kappa\) bounded set. We have to prove that \(A = \{T(x) : T \in \Gamma, x \in B\}\) is a bounded set in \(Y\). If this is not true, then there is a balanced neighborhood \(U_0 \subseteq N(Y)\) and a sequence \(\{t_i\}\)
\( \in c_0 \) and a sequence \( \{T_i(x_i)\} \subseteq A \) such that \( t_iT_i(x_i) \in U_0 \) for all \( i \).
Since \( U_0 \) is balanced, if \( |t_i|T_i(x_i) \in U_0 \) then \( t_iT_i(x_i) \in U_0 \). Thus, we can assume that \( t_i \geq 0 \).
Consider the matrix \( (\sqrt{t_i}T_i(\sqrt{t_j}x_j))_{i,j} \). Since \( B \) is \( \tau(I')-k \) bonuded and \( \sqrt{t_j} \to 0 \), the sequence \( \{\sqrt{t_j}x_j\} \) is \( \tau(I')-k \) convergent. Hence, if \( \{j_n\} \subseteq \mathbb{N} \) then there is a \( \{j_n\} \subseteq \{j_n\} \) such that \( \sum_{i=1}^{\infty} \sqrt{t_{j_n}}x_{j_n} \) is \( \tau(I') \)-convergent, i.e., there is an \( x \in X \) such that
\[
\sum_{i=1}^{\infty} T(\sqrt{t_{j_n}}x_{j_n}) = \lim_{m \to \infty} \sum_{i=1}^{m} T(\sqrt{t_{j_n}}x_{j_n}) = T(x), \quad \forall T \in I.
\]

The sequence \( \{\sum_{i=1}^{\infty} \sqrt{t_i}T_i(\sqrt{t_{j_n}}x_{j_n})\}_{i=1}^{\infty} = \{\sqrt{t_i}T_i(x)\} \) is a Cauchy sequence in \( Y \) because \( \{T_i(x)\}_{i=1}^{\infty} \) is bounded and \( \sqrt{t_i} \to 0 \). In fact, \( \sqrt{t_i}T_i(x) \to 0 \) and \( \lim \sqrt{t_i}T_i(\sqrt{t_j}x_j) = 0 \) for each \( j \in \mathbb{N} \). Now, by THEOREM 2, \( \lim_{i} \sqrt{t_i}T_i(\sqrt{t_j}x_j) = 0 \) uniformly in \( j \in \mathbb{N} \), there is an \( i_0 \) such that if \( i \geq i_0 \) then \( \sqrt{t_i}T_i(\sqrt{t_j}x_j) \in U_0 \) for each \( j \in \mathbb{N} \).
Thus, \( t_iT_i(x_i) = \sqrt{t_i}T_i(\sqrt{t_i}x_i) \in U_0, \forall i \geq i_0 \).
This is a contradiction. The second result can be obtained by similar arguments.

**Corollary 5.** Let \( X \) be a Frechet space, \( Y \) be a topological vector space and \( I \) be a family of continuous linear operators from \( X \) to \( Y \). Then if \( I \) is pointwise bounded it is uniformly bounded on bounded sets. Furthermore, \( I \) is equicontinuous.

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**References**


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