THE CANONICAL DECOMPOSITION OF SIEGEL MODULAR FORMS I

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Introduction

Let $M^*_n(q, \lambda)$ be the space of Siegel modular forms of degree $n$, weight $k$, level $q$, and character $\lambda$, where $n, k, q$ are positive integers and $\lambda$ is a Dirichlet character modulo $q$. The purpose of this article is to show that $M^*_n(q, \lambda)$ can be decomposed into $n+1$ subspaces which are pairwise orthogonal with respect to the so-called canonical inner product. Actually, we prove this for more general space, namely, $M^*_r(I, \lambda)$, where $I$ is any congruence subgroup of the symplectic group $Sp_n(Z)$ of level $q$.

Evdokimov[1], in 1981, gave a proof of this on the way of proving that $M^*_r(I, \lambda)$ has a simultaneous eigenbasis with respect to all the Hecke operators from a certain Hecke ring. But unfortunately, his proof contains a mistake in defining the canonical inner product, and as a consequence his proof of the existence of such eigenbasis needs a major correction.

In this article, the mistakes are corrected to get the canonical decomposition of $M^*_r(I, \lambda)$ (section 3.) and some useful theorems on the decomposition are given.

Let $Z, Q, R,$ and $C$ be the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively.

Let $M_{m,n}(A)$ be the set of all $m \times n$ matrices over $A$, a commutative ring with 1, and let $M_n(A) = M_{n,n}(A)$. Let $GL_n(A)$ and $SL_n(A)$ be the group of invertible matrices in $M_n(A)$ and its subgroup consisting of matrices of determinant 1, respectively. For $M \in M_n(A)$, $N \in M_{m,n}(A)$, let $M[N] = MN^t$, where $N^t$ is the transpose of $N$. Let $E_n$ and $0_n$
be the identity and the zero matrices in $M_n(A)$, respectively. Let $\det M$ be the determinant of $M$. For $M \in \operatorname{M}_{2n}(A)$, we let $A_M$, $B_M$, $C_M$, and $D_M$ be the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners of $M$, respectively, and write $M = (A_M, B_M; C_M, D_M)$. Let $N_m$ be the set of all semi-positive definite (eigenvalues $\geq 0$), semi-integral (diagonal entries and twice of nondiagonal entries are integers), symmetric $m \times m$ matrices, and $N_m^+$ be its subset consisting of positive definite (eigenvalues $> 0$) matrices.

Let $G_n = \operatorname{GSp}_n^+(\mathbb{R}) = \{ M \in \operatorname{M}_{2n}(\mathbb{R}) : J_n[M] = r J_n, \quad r > 0 \}$ where $J_n = (0_n, E_n; -E_n, 0_n)$ and $r = r(M)$ is a real number determined by $M$. Let $I_n^* = \operatorname{Sp}_n(\mathbb{Z}) = \{ M \in \operatorname{M}_{2n}(\mathbb{Z}) : J_n[M] = J_n \}$. Let $H_n = \{ Z = X + i Y \in M_n(\mathbb{C}) ; \quad \Im(Z), Y > 0 \}$. For $M \in G_n$ and $Z \in H_n$, we set $M(Z) = (A_M Z + B_M) (C_M Z + D_M)^{-1} \in H_n$.

For $M \in M_n(\mathbb{C})$, let $\psi(M) = \exp(2\pi i \sigma(M))$ where $\sigma(M)$ is the trace of $M$.

1. Siegel modular forms

Let $n, q$ be positive integers. We define $\Gamma_0^* = \Gamma_0^*(q) = \{ M \in \Gamma^* ; C_M \equiv 0_n \pmod{q} \}$ and $\Gamma_1^* = \Gamma_1^*(q) = \{ M \in \Gamma^* : M \equiv E_{2n} \pmod{q} \}$.

Let $F$ be an arbitrary complex valued function on $H_n$, and let $M = (A, B ; C, D) \in G_n$. We set

\[(1.1) \quad (F|_k M)(Z) = \sqrt{(\det M)^{k-n+1}} \cdot (\det(CZ + D))^{-k} F(M\langle Z \rangle) \]

where $Z \in H_n$ and $k$ is a positive integer. Note that $F|_k M$ is holomorphic on $H_n$ if $F$ is. Also note that $F|_k M_1|_k M_2 = F|_k M_1 M_2$ for $M_1, M_2 \in G_n$.

Let $\chi \colon (\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{C}^*$ be a Dirichlet character modulo $q$. A function $F \colon H_n \rightarrow \mathbb{C}$ is called a Siegel modular form of weight $k$, degree $n$, level $q$, and character $\chi$ if (i) $F$ is holomorphic on $H_n$, (ii) $F|_k M = \chi(\det D_M) \cdot F$ for any $M \in \Gamma_0^*$, and (iii) if $n = 1$, $(cz + d)^{-k} F(\langle az + b \rangle (cx + d)^{-1})$ is bounded as $\Im(z) \rightarrow +\infty$ for any matrix $(a, b ; c, d) \in \Gamma^* = \operatorname{SL}_2(\mathbb{Z})$. The set of all such forms is denoted by $M^*_k(q, \chi)$. This is a finite dimensional vector space over $\mathbb{C}$. (See [2].) We define $M^*_k(q)$ by the set of all $F : H_n \rightarrow \mathbb{C}$ satisfying (i), (iii), and (ii)$'$ $F|_k M = F$ for any $M \in \Gamma_0^*$. $M^*_k(q)$ is also a finite dimensional vector space over $\mathbb{C}$. Note that $M^*_k(q, \chi) \subseteq M^*_k(q)$.

It is known [3] that every $F \in M^*_k(q)$, hence every $F \in M^*_k(q, \chi)$,
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has a Fourier expansion of the form

\begin{equation}
F(Z) = \sum_{N \in \mathbb{N}} f(N) e(NZ/q), \quad Z \in H_n.
\end{equation}

2. The Siegel operator

Let \( 0 \leq r \leq n \) and let \( G^*_r \) be the \( r \)-th Satake group ([4], Exposé 12), i.e.,

\begin{equation}
G^*_r = \left\{ M \in \Gamma^* ; M = \begin{pmatrix} A_1 & 0 & B_1 & B_{12} \\ A_{21} & A_2 & B_{21} & B_2 \\ C_1 & 0 & D_1 & D_{12} \\ 0 & 0 & 0 & D_2 \end{pmatrix} \right\}
\end{equation}

with \( A_1, B_1, C_1, D_1, D_2 \in M_r(\mathbb{Z}) \).

The matrix \( M_1 = (A_1, B_1 ; C_1, D_1) \in \Gamma^r \) and the map \( w^*_r : G^*_r \to \Gamma^r \) defined by \( w^*_r(M) = M_1 \) is a surjective group homomorphism. Note that \( w^*_i(\Gamma^*_r \cap G^*_r) = \Gamma^*_r \) for \( i = 0, 1 \).

Let \( F : H_n \to \mathbb{C} \) be an arbitrary function with Fourier expansion (1.2). The Siegel operator \( \Phi \) is defined by

\begin{equation}
(\Phi F)(Z') = \lim_{i \to \infty} F \left( \begin{pmatrix} Z' & 0 \\ 0 & i \lambda \end{pmatrix} \right) = \sum_{N \in \mathbb{N}} f \left( \begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix} \right) e(N'Z'/q)
\end{equation}

where \( Z' \in H_{n-1}, \lambda > 0 \). For \( 0 \leq r \leq n \), we define \( \Phi^r \) by \( \Phi^r = \Phi \circ \Phi^r \) for \( 1 \leq r \leq n \). Let \( M \in G^*_r \) of the form (2.1). Then it is easy to see that

\begin{equation}
(\Phi^r F)(M) = (\det D_2)^{-k} \Phi^r F |_{M_1}(w^*_r(M)).
\end{equation}

Let \( F \in M^*_k(q, \chi) \) and \( M_1 \in \Gamma^*_r \) and let \( M \) be any matrix in \( \Gamma^*_r \cap G^*_r \) of the form (2.1) such that \( w^*_r(M) = M_1 \). Then from (2.3) and the condition (ii) follows

\begin{equation}
(\Phi^r F)(M) = (\det D_2)^{-k} \chi(\det D_M) \Phi^r F.
\end{equation}

So \( \Phi^r F \in M^*_k(q, \chi) \) where \( \chi(\det D_1) = (\det D_2)^k \chi(\det D_M) \) which is independent of the choice of \( M \) according to (2.3). Moreover, since we can choose \( M \) such that \( \det D_M = \det D_1 \) and \( \det D_2 = 1 \), we get

\begin{equation}
\Phi^r F \in M^*_k(q, \chi) \quad \text{if} \quad F \in M^*_k(q, \chi).
\end{equation}

Similar argument shows that

\begin{equation}
\Phi^r \in M^*_k(q) \quad \text{if} \quad F \in M^*_k(q).
\end{equation}

Let \( \Gamma' \) be a congruence subgroup of level \( q \), i.e., \( \Gamma'_1 \subseteq \Gamma' \subseteq \Gamma^n \).

Let \( \chi \) be a character: \( \Gamma' \to \mathbb{C}^\times \) such that \( \chi(\Gamma'_1) = 1 \). We set \( M^*_k(\Gamma', \chi) \) to be the set of all \( F : H_n \to \mathbb{C} \) satisfying the conditions (i), (iii),
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and (ii) $F|_k M = \chi(M) F$ for any $M \in \Gamma$. This is also a finite dimensional vector space over $C$. So under this definition, Siegel modular spaces $M^*_k(q, \chi)$ and $M^*_k(q)$ can be identified with $M^*_k(\Gamma^*_0, \chi)$ and $M^*_k(\Gamma^*_0, \chi_0)$ where $\chi_0$ is the trivial character. For convenience we set $M^*_k(\Gamma, \chi) = C$, $\Gamma^* = 1$, and $H_0$ to be a single point.

Let $F \in M^*_k(\Gamma, \chi)$ and $M_1 \in \omega^*_k(\Gamma \cap \Gamma^*_0)$. Take any $M \in \Gamma \cap \Gamma^*_0$ of the form (2.1) such that $\omega^*_k(M) = M_1$. Then

$$(2.7) \quad \Phi^{\omega \rightarrow} F \in M^*_k \left( \Gamma^*_0, \chi^*_0 \right)$$

where $\Gamma^*_r = \omega^*_k(\Gamma \cap \Gamma^*_0)$ and $\chi^*_r$ is a character on $\Gamma \cap \Gamma^*_0$ defined by $\chi^*_r(M_1) = (\det D_2)^k_\gamma \chi(M)$. Again (2.3) guarantees the independence of $\chi^*_r$ under the choice of $M$.

Let $M \in \Gamma^*$. Then

$$(2.8) \quad F|_k M \in M^*_k \left( \Gamma \Gamma^{-1} M \chi^M \right)$$

where $\Gamma^M = \Gamma \Gamma^{-1} M$ and $\chi^M$ is a character on $\Gamma^{-1} \Gamma M$ defined by $\chi^M(\tilde{M}) = \chi(M \tilde{M} \Gamma^{-1})$ for $\tilde{M} \in \Gamma^{-1} \Gamma M$. Combining (2.7) and (2.8), we get

$$(2.9) \quad \Phi^{\omega \rightarrow} (F|_k M) \in M^*_k \left( \Gamma^*_0, \chi^*_0 \right)$$

where $\Gamma^*_M = \omega^*_k(\Gamma \cap \Gamma^*_0)$ and $\chi^*_M = (\chi^M)^*_r$.

We denote $M^*_k((\Gamma^*_0)^M, \chi^M)$ and $M^*_k((\Gamma^*_0)^M, \chi^M)$ by $M^*_k(q^M, \chi^M)$ and $M^*_k(q^M, \chi^M)$, respectively. (2.6) shows that $M^*_k((\Gamma^*_0)^M, \chi^M) = M^*_k(q, \chi)$ and $M^*_k((\Gamma^*_0)^M, \chi^M) = M^*_k(q, \chi)$. It's easy to see that $M^*_k((\Gamma^*_0)^M, \chi^M) = M^*_k(q, \chi)$ and $M^*_k((\Gamma^*_0)^M, \chi^M) = M^*_k(q, \chi)$.

### 3. The canonical decomposition

$F \in M^*_k(\Gamma, \chi)$ is called a cusp form if $\Phi(F|_k M) = 0$ for all $M \in \Gamma^*$. For $F, G \in M^*_k(\Gamma, \chi)$, we set

$$(3.1) \quad (F, G) = \int_{D(\Gamma)} F(Z) \overline{G(Z)} (\det Y)^k d\mathcal{Z}$$

where $D(\Gamma)$ is a fundamental domain of $\Gamma$ in $H_n$, $Z = X + iY \in H_n$, and $d\mathcal{Z} = (\det Y)^{-n-1} dX dY$ is the $G_\Gamma$-invariant volume element on $H_n$. If either $F$ or $G$ is a cusp form, then the pairing (3.1) is a well defined non-degenerate Hermitian inner product ([4], Exposé 7) and is called the Maass–Petersson inner product on $M^*_k(\Gamma, \chi)$. But otherwise, the pairing (3.1) is meaningless.

We now construct a positive definite Hermitian inner product which
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is meaningful on the whole space \( M^*_k(\Gamma, \chi) \).

Let \( G^*_k(\Gamma, \chi) \) be the subspace of \( M^*_k(\Gamma, \chi) \) consisting of all the cusp forms. If \( F \in M^*_k(\Gamma, \chi) \), then \( F \) can be written uniquely in the form \( F = F' + F_n \) where \( F_n \in G^*_k(\Gamma, \chi) \) and \( F' \) is contained in the orthogonal complement of \( G^*_k(\Gamma, \chi) \) in \( M^*_k(\Gamma, \chi) \) with respect to the Maass-Petersson inner product. We call \( F_n \) the cusp part of \( F \). We set

\[
(F, G) = \sum_{r=0}^{n} \sum_{M \in \Gamma^r} [\Gamma' : \Gamma^r_M]^{-1} \left( \langle \phi_{\Gamma^r_M} F, \phi_{\Gamma^r_M} G \rangle \right)_o
\]

where \( \phi_{\Gamma^r_M} F = \phi_{\gamma^{-1}}(F|_k M) \), \( \langle \cdot, \cdot \rangle_o \) is the Maass-Petersson inner product on the space \( M_k(\Gamma^r, \chi^r) \).

**Theorem 3.1.** The pairing (3.2) is a well defined positive definite Hermitian inner product on the whole space \( M^*_k(\Gamma, \chi) \), which is called the canonical inner product on the space.

**Proof.** Since \( \Gamma^r M, \chi^r \) are independent of the choice of representative \( M \) of the left coset \( \Gamma M \) with \( M \in \Gamma^n \), so are the index \( [\Gamma' : \Gamma^r_M] \) and the space \( M^*_k(\Gamma^r, \chi^r) \). The Maass-Petersson inner product \( \langle \cdot, \cdot \rangle_o \) is also independent of the choice of \( M \) because \( |\chi(M')| = 1 \) for any \( M' \in \Gamma \). So (3.2) is a well defined Hermitian inner product. The positive definiteness follows immediately from (2.2) and the obvious fact that \( \langle \cdot, \cdot \rangle_o \) is positive definite when restricted to cusp forms. The theorem is proved.

We now decompose \( M^*_k(\Gamma, \chi) \) into \( n+1 \) mutually orthogonal subspaces with respect to the canonical inner product. For \( 0 \leq r \leq n \), we set

\[
M^*_k(\Gamma, \chi) = \{ F \in M^*_k(\Gamma, \chi); \phi_{\Gamma^r_M} F \text{ is a cusp form for every } M \in \Gamma^n \text{ such that } \left( F, \sum_{i=r+1}^{n} M^*_k(\Gamma, \chi) \right) = 0 \text{ if } r \neq n \}.
\]

Observe that \( M^*_k(\Gamma, \chi) = M^*_k(\Gamma, \chi) \).

**Theorem 3.2.** For \( 0 \leq r \leq n-1 \),

\[
M^*_k(\Gamma, \chi) = \{ F \in M^*_k(\Gamma, \chi); \phi_{\Gamma^r_M} F \in \sum_{i=r}^{n-1} M^*_k(\Gamma^{r-i}, \chi^{r-i}) \text{ such that } \left( F, \sum_{i=r+1}^{n} M^*_k(\Gamma, \chi) \right) = 0 \}.
\]
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Proof. Let the right sides of (3.3) and (3.4) be A and B. We use induction on \( n \). For \( n=1 \) it is well known [5] that the lemma holds. Let \( n>1 \). It is clear that \( B \subset A \). Suppose \( F \in A-B \). Then there exists \( M \) such that \( \Phi_M F \) is not contained in \( \sum_{r'=0}^{n-1} (\Gamma^M_{n-r-1}, \chi^M_{n-r-1}) \). But then from the induction hypothesis and (2.3) follows that \( \Phi_M F \) is not a cusp form for some \( M' \in \Gamma^n \), which is impossible, whence \( A \subset B \).

When \( n=1 \), \( M_1^{(0)}(\Gamma, \chi) \) and \( M_1^{(1)}(\Gamma, \chi) \) coincide with the subspaces of classical Eisenstein series [5] and cusp forms [6]. Furthermore, they are orthogonal to each other with respect to Maass-Petersson inner product as well as to the canonical inner product. This can be generalized for arbitrary \( n \). More precisely.

Theorem 3.3. The space \( M_1^n(\Gamma, \chi) \) is decomposed into \( n+1 \) subspaces \( M_1^{(r)}(\Gamma, \chi), 0 \leq r \leq n \), which are pairwise orthogonal with respect to the canonical inner product.

Proof. From Theorem 3.2. and induction on \( n \), the theorem follows.

Maass[7] proved this for \( M_1(\Gamma, \chi_0) \).

We write \( M_1^n(\Gamma, \chi) = \bigoplus_{r=0}^{n} M_1^{(r)}(\Gamma, \chi) \) and call it the canonical decomposition of \( M_1^n(\Gamma, \chi) \). The subspace \( M_1^{(r)}(\Gamma, \chi) \) is called the \( r \)-th canonical subspace of \( M_1^n(\Gamma, \chi) \) for each \( r=0, \ldots, n \).

4. Some theorems

Let \( F, G \in M_1^n(\Gamma, \chi) \) such that at least one of which is a cusp form, say, \( G \). Since \( \Phi_{k'0} G = 0 \) for \( s>0 \),

\[
(4.1) \quad (F, G) = \sum_{M \in \Gamma^n} \left[ \Gamma^n : \Gamma^M \right]^{-1} \left( (F|_k M)_n, (G|_k M)_n \right).
\]

Theorem 4.1. If \( F \in M_1^n(\Gamma, \chi) \), then \( F'|_k M = (F|_k M)' \) and \( F_n|_k M = (F|_k M)_n \) for any \( M \in \Gamma^n \).
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Proof. The second equality follows from the definition of a cusp form. For \( F, G \in M^\ast_\chi(\Gamma, \chi) \), at least one of which is a cusp form, it is easy to see that \( (F|_kM, G|_kM)_o = (F, G)_o \) where the former pairing is the Mass–Petersson inner product on \( M^\ast_\chi(\Gamma^M, \chi^M) \) and the latter is that on \( M^\ast_\chi(\Gamma, \chi) \). The first equality follows.

Theorem 4.2. For \( F, G \in M^\ast_\chi(\Gamma, \chi) \), at least one of which is a cusp form, the canonical inner product coincide with the Maass–Petersson inner product.

Proof. Let \( G \) be a cusp form. Since \( ((F|_kM)_o, (G|_kM)_o)_o = (F|_kM, G|_kM)_o = (F, G)_o + (F', G)_o = (F, G)_o \), from (4.1) we have
\[
(F, G) = \sum_{[M \in \Gamma^\ast]} [I^n : \Gamma^M]^{-1} (F, G)_o.
\]
where \( [I^n : \Gamma^M] = [I^n : \Gamma] \) for any \( M \in \Gamma^\ast \). The theorem is proved.

Let \( M \in \Gamma^\ast \) be given. Let \( T_M : M^\ast_\chi(\Gamma, \chi) \to M^\ast_\chi(\Gamma^M, \chi^M) \) be a homomorphism defined by \( T_M(F) = F|_kM \). It is easy to see that \( T_M \) is an isomorphism that preserves the canonical inner product and hence the canonical decomposition, i.e.,
\[
(F, G) = (F|_kM, G|_kM) = (T_MF, T_MG)
\]
where the left canonical inner product is on \( M^\ast_\chi(\Gamma, \chi) \) and the right is on \( M^\ast_\chi(\Gamma^M, \chi^M) \), and
\[
T_M(M^\ast_\chi(\Gamma, \chi)) = M^\ast_\chi(\Gamma^M, \chi^M).
\]
Let \( \Gamma' \) be a congruence subgroup contained in \( \Gamma \) and let \( \chi' \) be the restriction of \( \chi \) to \( \Gamma' \). Then \( M^\ast_\chi(\Gamma, \chi) \subseteq M^\ast_\chi(\Gamma', \chi') \).

Theorem 4.3. For \( F, G \in M^\ast_\chi(\Gamma, \chi) \),
\[
(F, G) = [\Gamma : \Gamma']^{-1} (F, G)'
\]
where the left canonical inner product is on \( M^\ast_\chi(\Gamma, \chi) \) and the right is on \( M^\ast_\chi(\Gamma', \chi') \).

Proof. From (3.1) follows that \( (F, G)_o = [\Gamma : \Gamma']^{-1} (F, G)_o' \) where \((-!, -)_o'\) is the Maass–Petersson inner product on \( M^\ast_\chi(\Gamma', \chi') \). Let \( \{N_i\}_{i=1,2,...,m} \subseteq \Gamma', \{M_j\}_{j=1,2,...,l} \subseteq \Gamma^\ast \) be full sets of left coset representatives of \( \Gamma' \setminus \Gamma \). Then \( \{N, M_j\} \) is a full set of left coset representatives of \( \Gamma' \setminus \Gamma^\ast \). For each \( 0 \leq r \leq n, N \in \{N_i\}, M \in \{M_j\} \), we have \( (\Phi^{N}_{M}F)_r = (\Phi^{r}F|_kNM)_r = \chi(N) (\Phi^{r}F)_r \). Similarly, \( (\Phi^{N}_{M}G)_r \),

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We have

$$\chi(N) = \text{trace}(\Phi_M^r G, \tau).$$

Since \((-\cdot, -)\) is Hermitian and \(|\chi(N)| = 1,

$$\left(\Phi_{Nz}^r F, \left(\Phi_{Nz}^r G, \tau\right)\right)' = \left(\Phi_{z}^r F, \left(\Phi_{z}^r G, \tau\right)\right)' \circ.$$ So

$$[I^r : (I^r)^{-1}]^{-1}(\Phi_{Nz}^r F, \left(\Phi_{Nz}^r G, \tau\right))' = [I^r : I^r]^{-1}(\Phi_{z}^r F, \left(\Phi_{z}^r G, \tau\right))'$$

and hence from (3.2) follows \((F, G)' = m(F, G) = [I^r : I^r] (F, G)\) which proves the theorem.

**Theorem 4.4.** For each \(0 \leq r \leq n\), we have

\[
M^r \cdot (I, \chi) = M^r \cdot (I', \chi') \cap M^r \cdot (I, \chi),
\]

Proof. We use induction on \(s = n - r\). For \(s = 0\) \((n = r)\), (4.5) follows immediately from (3.3) and Theorem 4.3. Let \(F\) be in the right side of (4.5) and \(G \in \sum_{r'=1}^s M^r \cdot (I', \chi')\) for \(r' < n\). From induction hypothesis \((F, G) = [I^r : I^r]^{-1}(F, G)' = 0\). So from (3.3), \(F \in M^r \cdot (I, \chi)\). To show the reverse inclusion, let \(F \in M^r \cdot (I, \chi)\). Then \(F \in \sum_{r'=1}^s M^r \cdot (I', \chi')\). Write \(F = F_r + G\) where \(F_r \in M^r \cdot (I', \chi')\) and \(G \in \sum_{r'=1}^s M^r \cdot (I', \chi')\).

From induction hypothesis \(0 = (F, G)' = (F_r + G, G)' = (G, G)'\). So \(G = 0\) and \(F = F_r \in M^r \cdot (I', \chi')\). The theorem follows.

According to equalities (4.2), (4.3) and Theorems 4.3., 4.4., when one needs to prove a certain property related to the canonical inner product and decomposition on \(M^r \cdot (I, \chi)\), in particular, on \(M^r \cdot (q, \chi)\), it suffices to prove it for \(M^r \cdot (q)\).

Finally, we prove the invariance of the \(r\)-th canonical subspace of \(M^r \cdot (I, \chi)\) under the action \(\chi M\) for \(M \in \Gamma^r\).

**Theorem 4.5.** If \(F\) is in \(M^r \cdot (I, \chi)\), then so is \(F |_M \) for any \(M \in \Gamma^r\).

Proof. As the remark above, it is enough to show the theorem for \(F \in M^r \cdot (q)\). It is clear that \(F |_M \in M^r \cdot (q)\). Again we use induction on \(s = n - r\). \(\Phi^r_M F\) is a cusp form for each \(M' \in \Gamma^r\). If \(M'\) runs over \(\Gamma^r\), then so does \(MM'\). So \(\Phi^r_{MM'F} = \Phi^r_{F |_M} (F |_M)\) is also a cusp form for each \(M \in \Gamma^r\) and hence it suffices to show

\[
(F |_M, G) = 0 \quad \text{for any } G \in \sum_{r'=1}^s M^r \cdot (q) \quad \text{for } 0 \leq r < n.
\]

From (3.2) we get

\[
(F |_M, G) = \sum_{r=1}^n \sum_{M' \in \Gamma^r} \left(\Phi^r_{M'F} (F |_M)\right)' \circ \left(\Phi^r_{M'G} \right)'.
\]

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where \( \langle \cdot, \cdot \rangle \) is the Maass–Petersson inner product on \( M'_{\mathbb{I}}(q) \), because \( (F^*)^\mathbb{I}' = F' \). If \( M' \) runs over a full set of representatives of \( \mathbb{I}' \setminus \mathbb{I}'' \), then so does \( M^{-1}M' \). Substitution of \( M' \) by \( M^{-1}M' \) in (4.7) yields \( (F|_{kM}, G) = (F, G|_{kM^{-1}}) \). From induction hypothesis \( (F, G|_{kM^{-1}}) = 0 \). So (4.6) and hence the theorem follows.

References


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