CLOSED SEMI-IDEALS IN A II₁-FACTOR

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1. Introduction

Throughout, $M$ will be a fixed II$_1$-factor with normalized trace $\tau$. A (nonempty) subset $S$ of $M$ is called a semi-ideal in $M$ if $xS\subset S$ for all $x, y \in M$.

In Section 2, we determine the class of all (norm) closed semi-ideals in $M$. The height $h(S)$ of a semi-ideals in $M$ is defined by

$$h(S) = \sup \{ \tau(p) : p \in P(S) \},$$

where $P(\cdot)$ denotes the set of all projections in the set $(\cdot)$, through this work. We say that $h(S)$ is accessible if there is $p \in P(S)$ such that $\tau(p) = h(S)$. Otherwise, it is called inaccessible (cf. [7] Definition 2).

In Section 3, we describe the spectrum $\sigma_t(X)$ of $x \in M$, $0 < t \leq 1$, modulo the closed semi-ideal $f_t$ where $f_t$ is uniquely determined as the closed semi-ideal in $M$ whose height $h(f_t)$ is inaccessible and $h(f_t) = t$.

2. The closed semi-ideals

For every $t \in (0, 1]$, we put

$$I_t = \{ x \in M : \tau(l(x)) < t \} \quad \text{and} \quad J_t = \overline{I_t},$$

where $l(x)$ denotes the left support (projection) of $x \in M$. It is immediate to verify that $I_t$, $J_t$ are semi-ideals in $M$.

In what follows, $H$ will be the underlying Hilbert space on which operators of $M$ act, and $P(M)$ will be abbreviated by $P$.

Proposition 2.1. Let $x \in M$, $0 < t < 1$. The following conditions are mutually equivalent.

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If \( q \in P \) and \( q(H) \subset x(H) \), then \( \tau(q) < t \).

(iii) If \( x \) is bounded below on \( p(H) \) for \( p \in P \), then \( \tau(p) < t \).

(iv) For every \( \varepsilon > 0 \), there is \( p \in P \) such that \( \|xp\| < \varepsilon \) and \( \tau(1-p) < t \).

**Proof.** (i) \( \rightarrow \) (ii). If we modify ([1] Theorem 1) slightly, we see that \( q = xy \) for some \( y \in M \). Then \( q \in J_\varepsilon \), since \( J_\varepsilon \) is a semi-ideal in \( M \). Let \( \{q_n\} \) be a sequence in \( J_\varepsilon \) such that \( \|q - q_n\| \to 0 \). One can easily check that the operator \( q_nq(H) : q(H) \to H \) is bounded below for all sufficiently large \( n \)'s. Let us fix one such \( n \). Then \( q_nq \) has the kernel \((I-q)(H)\) and has the closed range. Note that \( l(q_nq) \sim r(q_nq) \) in \( M \), where \( r(\cdot) \) denotes the right support of the element \((\cdot)\), and that \( r(q_nq) = q \). Consequently, \( \tau(q) = \tau(r(q_nq)) = \tau(l(q_nq)) \leq \tau(q_n) < t \), as desired.

(ii) \( \rightarrow \) (iii). Let \( p \in P \) be given such that \( x \) is bounded below on \( p(H) \). We put \( q = l(xp) \in P \). Then \( q(H) \subset x(H) \). By hypothesis (ii), \( \tau(q) < t \). On the other hand, \( r(xp) = p \), since \( \ker(xp) = \ker p \). It follows that \( \tau(p) = \tau(r(xp)) = \tau(l(xp)) = \tau(q) < t \).

(iii) \( \rightarrow \) (iv). Let \( E(\cdot) \) be the spectral measure of \( |x| \). If we put \( p = E[0, \varepsilon/2) \), then it is easily seen that \( p \) is a required one in (iv), by a semilar argument as in the proof of ([5] Lemma 2.5).

(iv) \( \rightarrow \) (i). Let \( \varepsilon(>0) \) be given arbitrary. By (iv), there is \( p \in P \) such that \( \|xp\| < \varepsilon \) and \( 1 - p \in I_\varepsilon \). Note that \( x(1 - p) \in I_\varepsilon \). Now \( \|x - x(1-p)\| = \|xp\| < \varepsilon \). Hence \( x \in I_\varepsilon = J_\varepsilon \).

**Corollary 2.2.** For every semi-ideal \( J_\varepsilon \), the height \( h(J_\varepsilon) \) is inaccessible.

**Proof.** Clearly \( h(J_\varepsilon) = h(I_\varepsilon) = t \). Assume contrary that \( \tau(p) = t \) for some \( p \in P(J_\varepsilon) \). By (i) \( \rightarrow \) (ii) of Proposition 2.1, \( \tau(p) < t \), which is a contradiction.

**Lemma 2.3.** A (norm) closed semi-ideal \( S \) of \( M \) is determined by the projections contained in \( S \).

**Proof.** We have to show the following: If \( S_i(i = 1, 2) \) are two closed semi-ideals in \( M \) such that \( P(S_1) = P(S_2) \), then \( S_1 = S_2 \).
Let $x \in S_1$. Let $x = u \vert x \vert$ be the left polar decomposition of $x$ and $E(\cdot)$ be the spectral measure of $\vert x \vert$. If we put $e_n = E[1/n, \infty)$, $n = 1, 2, \ldots$, then

\[ (*) \quad \| \vert x \vert - \vert x \vert e_n \| = \| \vert x \vert E[0, 1/n] \| \leq 1/n \to 0. \]

Here $\vert x \vert e_n \in S_1$, since $\vert x \vert = u^* x \in S_1$. Note that $\vert x \vert e_n$ has the closed range, so that $l(\vert x \vert e_n)(H) = (\vert x \vert e_n)(H) \subset \vert x \vert (H)$. By the modified version of the Douglas result ([1] Theorem 1) mentioned already, $l(\vert x \vert e_n) = \vert x \vert y$ for some $y \in M$. Hence $l(\vert x \vert e_n) \in S_1$, Now $l(\vert x \vert e_n) \sim r(\vert x \vert e_n) = e_n$, which implies that $e_n \in P(S_1)$, since two equivalent projections in a finite von Neumann algebra are unitarily equivalent. As we have assumed that $P(S_1) = P(S_2)$, we have $e_n \in P(S_2)$ and hence $\vert x \vert e_n \in S_2$. By $(*)$, $\vert x \vert \in S_2$ and consequently $x \in S_2$. We have shown that $S_1 \subset S_2$. The reverse inclusion is proven by the symmetric argument.

**Lemma 2.4.** If $J$ is a (norm) closed nontrivial $\{0\} \subset J \subset M$ semi-ideal of $M$ such that $h(J)$ is inaccessible, then there is a unique $t \in (0, 1]$ such that $J = J_t$.

**Proof.** To see the uniqueness, let $0 < t_1 < t_2 \leq 1$. Find $p \in P$ such that $\tau(p) = t_1$. Then $p \not\in J_{t_1}$ by Corollary 2.2, while $p \in J_{t_2}$. Hence $J_{t_1} \subset J_{t_2}$.

Now let $J$ be a closed semi-ideal of $M$ such that $\{0\} \subset J \subset M$ and $h(J)$ is inaccessible. Put $t = h(J)$. Since $P(J) \neq \{0\}$ (Lemma 2.3), we see that $t \in (0, 1]$. By inaccessibility of $h(J)$, $P(J) \subset P(l_0) \subset P(J)$, which, in turn, implies that $J \subset J$ (Lemma 2.3). To get $J \subset J$, let $p \in P(J)$. Then $\tau(p) \leq t$, as we saw already. By definition of $t$, $\tau(p) < t(q)$ for some $q \in P(J)$. Thus, $p \sim q \leq q$ for some $q_1 \in P$. Since $q_1 = q_1 q \in J$, we have that $p \in J$. Hence $P(J) \subset P(J)$, and consequently $J \subset J$ (Lemma 2.3).

For every $t \in [0, 1]$, let us define

\[ K_t = \{ x \in M : \tau(l(x)) \leq t \}. \]

One can easily show that $K_t$ is a norm closed semi-ideal of $M$ and that $h(K_t)$ is accessible. The converse holds as in the following lemma. We omit the proof, as it is dealt with the similar way as the case of $J_t$'s.
Lemma 2.5. K is a closed semi-ideal of M whose height \( h(K) \) is accessible if and only if there is a unique \( t \in [0, 1] \) such that \( K = K_t \).

In ([3] Definition 2.1), the \( t \)-th singular number of \( \tau \)-measurable operator \( T \) is defined by
\[
\mu_t(T) = \inf \{ \| Tp \| : p \in P \text{ and } \tau(1-p) \leq t \},
\]
where \( t \in (0, \infty) \).

When \( x \in M \), \( t \in [0, 1] \), Proposition 2.4 of [3] implies that
\[
\mu_t(x) = \text{dist}(x, K_t),
\]
where dist denotes the distance.

For \( x \in M \), \( t \in (0, \infty) \), let us define
\[
\nu_t(x) = \inf \{ \| xp \| : p \in P \text{ and } \tau(1-p) < t \}.
\]

The next two propositions are analogues of Proposition 2.2 and Proposition 2.4 in [3], respectively. We shall omit their proofs which go parallel to the corresponding ones in [3].

**Proposition 2.6.** For \( x \in X \), \( t \in (0, \infty) \), we have
\[
\nu_t(x) = \inf \{ s \geq 0 : \lambda_s(x) < t \},
\]
where \( \lambda_s(x) = \tau(E(s, \infty)) \) and \( E(\cdot) \) is the spectral measure for \( |x| \).

**Proposition 2.7.** For \( x \in M \), \( t \in (0, 1] \), we have
\[
\nu_t(x) = \text{dist}(x, J_t).
\]

**Remark 2.8.** For every \( t \in (0, \infty) \), \( x \in M \), it is clear that \( \mu_t(x) \leq \nu_t(x) \). When \( t \in (0, 1] \), \( \nu_t(x) \) is right continuous at \( t \) if and only if \( \nu_t(x) = \mu_t(x) \). Because of this fact and similarity between definitions of \( \mu_t(x) \) and \( \nu_t(x) \), many assertions in [3], for example, Lemma 2.5 and Proposition 2.7 there, can be formulated in terms of \( \mu_t(x) \).

3. Invertibility modulo \( J \),

Let \( S \) be a closed semi-ideal of \( M \) and \( x \in M \). We say that \( x \) is left invertible in \( M \) modulo \( S \) if there is \( y \in M \) such that \( yx - I \in S \). An element \( x \in M \) is called invertible in \( M \) modulo \( S \) if there is \( y \in M \) such that \( yx - I \in S \) and \( xy - I \in S \).

If \( K \) is a closed subspace of \( H \) and \( p \) is the projection onto \( K \) such that \( p \in M \), we shall also write \( K \in M \) and \( \tau(K) \) to mean
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$\tau(p)$. The next two lemmas are von Neumann algebra version of Lemma 1.1 and 1.2 in [2], respectively. To prove Lemma B, one has to apply the parallelogram law for projections in $M$. We omit the obvious proofs.

**Lemma A.** Let $x \in M$. For every $\varepsilon > 0$, there is a closed subspace $K$ of $H$ such that kernel$(x) \subset K$, $K \in M$,

\[ ||x\xi|| < \varepsilon ||\xi||, \text{ for all } \xi \in K \text{ and} \]
\[ ||x\xi|| \geq \varepsilon ||\xi||, \text{ for all } \xi \in K^\perp. \]

(When $K = \{0\}$, the first inequality is vacuous.)

**Lemma B.** Let $x \in M$. For $\varepsilon > 0$, suppose that $K$ is a closed subspace of $H$ such that $K \in M$, $||x\xi|| < \varepsilon ||\xi||$ for all $\xi \in K$ with $\xi \neq 0$, and that $L$ is a closed subspace of $H$ such that $L \in M$, $||x\xi|| \geq \varepsilon ||\xi||$ for all $\xi \in L^\perp$. Then

\[ \tau(K) \leq \tau(L), \]
\[ \tau(L^\perp) \leq \tau(K^\perp). \]

**Proposition 3.1.** For $x \in M$, $t \in (0, 1]$, the following conditions are mutually equivalent.

(i) $x$ is left invertible modulo $I_t$.

(ii) $x$ is left invertible modulo $J_t$.

(iii) $x$ is bounded below on $p(H)$ for some $p \in P$ with $\tau(1 - p) < t$.

(iv) The nullity $\nu(x) < t$, where $\nu(x) = \tau(\text{kernel}(x))$.

(v) $x$ is invertible modulo $I_t$.

(vi) $x$ is invertible modulo $J_t$.

**Proof.** (i) $\rightarrow$ (ii). Trivial, since $I_t \subset J_t$.

(ii) $\rightarrow$ (iii). Assume that $yx - I \in J_t$ for some $y \in M$. Note that $y \neq 0$. By Lemma A, there is a closed subspace $K$ of $H$ such that $K \in M$,

\[ ||yx\xi|| < (1/2)||\xi|| \text{ for } \xi \in K \text{ with } \xi \neq 0 \]

and

\[ ||yx\xi|| \geq (1/2)||\xi|| \text{ for all } \xi \in K^\perp. \]

Thus $||x\xi|| \geq (1/2||y||)||\xi||$, for all $\xi \in K^\perp$. It suffices to show that $\tau(K) < t$. For all $\xi \in K$, we have

\[ ||(I - yx)\xi|| \geq ||\xi|| - ||yx\xi|| \geq ||\xi|| - (1/2)||\xi|| = (1/2)||\xi||, \]

which shows that $I - yx$ is bounded below on $K$. By Proposition 2.1, $\tau(K) < t$, as desired.
Let \( x \) be bounded below on \( p(H) \) for some \( p \in P \) with \( \tau(1-p) < t \). Choose \( \varepsilon > 0 \) such that \( ||x\xi|| \geq \varepsilon ||\xi|| \) for all \( \xi \in p(H) \). Put \( L = (1-p)(H) \). With these \( \varepsilon \) and \( L \), let \( K \) be a closed subspace satisfying Lemma A. By Lemma B, we have \( \tau(K) \leq \tau(L) < t \). Since \( \ker(x) \subseteq K \) (See Lemma A), we get the desired conclusion.

(iv) \( \Rightarrow \) (iii). Assume that \( \nu(x) < t \). Let \( E(\cdot) \) be the spectral measure of \( |x| \). Since \( \lim_{\varepsilon \to 0} \tau(E[0, \varepsilon)) = \nu(x) \), there is a positive real number \( \varepsilon \) such that \( \tau(E[0, \varepsilon)) < \frac{1}{2} \). We put \( p = E[\varepsilon, \infty) \). Then \( \tau(1-p) < t \), while \( x \) is bounded below on \( p(H) \).

(iii) \( \Rightarrow \) (i) Let \( p \in P \) be as in (iii). We can find \( y \in M \) such that
\[
yx\xi = \xi, \quad \text{for all} \quad \xi \in p(H)
\]
and
\[
y_n = 0, \quad \text{for all} \quad n \in \mathbb{Z}.
\]
Then
\[
yxp = p, \quad \text{so} \quad yx - I = (yx - I)(I - p) \subseteq I,
\]
since \( I - p \in I_1 \).

(v) \( \Rightarrow \) (vi) and (vi) \( \Rightarrow \) (ii) are clear.

It remains to prove the implication (iv) \( \Rightarrow \) (v). As in the proof of (iv) \( \Rightarrow \) (iii) we put \( p = E[\varepsilon, \infty) \), where \( \varepsilon < 0 \), \( E(\cdot) \) is the spectral projection of \( |x| \) and \( \tau(1-p) < t \). Let us find \( y \in M \) just as in the proof of (iii) \( \Rightarrow \) (i) so that \( yx - I \in I_1 \). We have to show that this \( y \) also satisfies that \( xy - I \in I_1 \).

Let us put \( L = x(E[\varepsilon, \infty)(H)) \), which is a closed subspace of \( H \) such that \( L \subseteq M \) and \( \tau(L) = \tau(p) = 1 - t \). For every \( \eta \in H \), we write \( \eta = \eta_1 \oplus \eta_2 \), where \( \eta_1 \subseteq L \) and \( \eta_2 \subseteq L^\perp \). Thus, \( \eta_1 = x\xi \) for some \( \xi \in p(H) \), and \( xy\eta = xy\eta_1 + xy\eta_2 = xyx\xi \) (noticing \( y \) in the proof of (iii) \( \Rightarrow \) (i) vanishes on \( L^\perp \), while \( yx\xi = \xi \)). This implies that \( xyq = q \), where \( q \) is the projection onto \( L \). It follows that \( xy - I = (xy - I)(I - q) \in I_1 \), since \( \tau(q) = \tau(p) \) and hence \( \tau(1-q) = \tau(1-p) < t \).

For \( x \in M, t \in (0, 1] \), let us put
\[
\sigma_t(x) = \{ \lambda \in C : \nu(x-\lambda) \geq t \}.
\]
By Proposition 3.1, \( \lambda \in \sigma_t(x) \) if and only if \( x - \lambda \) is not invertible modulo \( I_t \). In particular, \( x \) has no eigenvalue if and only if \( x - \lambda \) is invertible modulo \( I_t \), for every \( t \in (0, 1] \) and any \( \lambda \in C \).

**Proposition 3.2.** The function \( x \in M \mapsto \nu(x) \in [0, 1] \) is upper semi-continuous with respect to the norm topology of \( M \).

**Proof.** To prove the contraposition, let \( t \in [0, 1], \{ x_n \} \subseteq M \),
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$v(x_n) \geq t$ and $x_n \to x$ in norm. It suffices to prove that $v(x) \geq t$.

Assume contrary that $v(x) < t$. By Lemma 3.1, there is $p \in P$ and a positive number $\varepsilon$ such that $\tau(1-p) < t$ and $\|xp\xi\| \geq \varepsilon\|p\xi\|$ for all $\xi \in H$. Then, for all $\xi \in H$,

$$\|xp\xi\| - \|x_n - x\|p\xi\| \geq \varepsilon\|p\xi\| - \|x - x\|\|p\xi\|$$

$$= (\varepsilon - \|x_n - x\|)\|p\xi\|,$$

which shows that $x_n$ is bounded below on $p(H)$ for a sufficiently large integer $n$. By Lemma 3.1 again, we then have $v(x_n) < t$, for such $n$, which is a contradiction as desired.

**Lemma 3.3.** Let $x \in M$, $t \in (0,1]$ and $\lambda \in \mathbb{C}$. If $v_t(x) < |\lambda|$, then $x - \lambda$ is invertible in $M$ modulo $J_n$.

**Proof.** Since $v((1/|\lambda|)x) < 1$, we may prove the following: If $v(x) < 1$, then $x - I$ is bounded below on $p(H)$ for some $p \in P$ with $\tau(1-p) < t$ (Proposition 3.1). Since $v(x) = \inf\{\|xp\| : p \in P : \tau(1-p) < t\}$, there is $p \in P$ such that $\|xp\| < 1$ and $\tau(1-p) < t$. Then for all $\xi \in p(H)$ with $\|\xi\| = 1$,

$$\|(x - I)\xi\| \geq \|\xi\| - \|x\|\xi - \|xp\xi\|$$

$$\geq \|\xi\| - \|xp\|\|\xi\|$$

$$= (1 - \|xp\|)\|\xi\|,$$

while $1 - \|xp\| > 0$. By Proposition 3.1, $x - I$ is invertible in $M$ modulo $J_n$, as desired.

**Corollary 3.4.** For $x \in M$, $t \in (0,1]$, $\sigma_t(x)$ is a compact subset of $\mathbb{C}$ contained in the closed disk about the origin with radius $v_t(x)$.

**Proof.** It is immediate from Proposition 3.2 and Lemma 3.3.

**References**

3. T. Fack and H. Kosaki, *Generalized s-numbers of $\tau$-measurable operators*,