MAXIMAL OPERATORS CONCERNING THE DIFFERENTIABILITY OF FUNCTIONS

YOUNG-HWA HA* AND DO YOUNG KWAK

1. Introduction

In this paper we will introduce certain maximal operators in terms of which we will characterize the first order Sobolev functions. The first order Sobolev space $L^1_1(\mathbb{R}^n)$ is defined to be the set of all functions $f$ belonging to $L^1(\mathbb{R}^n)$ whose distributional derivatives $\frac{\partial f}{\partial x_j}, j=1, \ldots, n$, also belong to $L^1(\mathbb{R}^n)$. It is well known that if a function $f$ and its distributional derivative $\frac{\partial f}{\partial x_j}$ are locally integrable then $f$ (possibly modified on a set of measure zero) is in fact partially differentiable with respect to $x_j$ almost everywhere. For this and other properties of Sobolev functions we refer the readers to [1] and [4]. The differentiability of a function at almost every point in a given set has been studied by many persons. We refer the readers to Stein [3], which shows a systematic approach to the problem, and also to Neugebauer [2] for a succinct condition for the differentiability property. In their studies the even part of a function played an important role. We are, however, concerned with the odd part. The even and odd parts of a function $f$ on $\mathbb{R}^1$ at $x$ are defined to be the functions $\varphi$ and $\psi$, respectively, given by $\varphi(t) = \frac{1}{2}(f(x+t) + f(x-t))$ and $\psi(t) = \frac{1}{2}(f(x+t) - f(x-t))$.

2. Definitions

For a function $f \in C^1(\mathbb{R}^n)$ and for $j=1, \ldots, n$ and $h>0$ we define the mean difference quotient $\delta_{j,h}f(x)$ of $f$ at $x \in \mathbb{R}^n$ by the equation

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(1) \[ \delta_{j,h}f(x) = \frac{1}{h} \int_0^h \frac{f(x+te_j) - f(x-te_j)}{2t} \, dt, \]
where \( e_j \) is the \( j \)-th standard basis element of \( \mathbb{R}^n \). The maximal 

derivate \( D_j f(x) \) is then defined to be the associated maximal 

function given by the equation 

(2) \[ D_j f(x) = \sup_{0 \leq \alpha \leq 1} |\delta_{j,h} f(x)|. \]

We interpret the singular integral in (1) as the limit of the 

following integrals when \( h \to 0 \):

\[ \frac{1}{h} \int_0^h \frac{f(x+te_j) - f(x-te_j)}{2t} \, dt. \]

**Lemma 1.** Let \( f \in L^1_\text{loc}(\mathbb{R}^n) \). Then for every \( j=1, \ldots, n \) and every 

\( h>0 \), the integral defining \( \delta_{j,h} f(x) \) converges and is finite for 

a.e. \( x \in \mathbb{R}^n \), and so \( D_j f(x) \) is a well-defined measurable function.

**Proof.** Suppose first \( n=1 \) and fix \( h>0 \). The function \( \varphi_h \) defined 

to be \( 1/s \) if \( |s| \leq h \) and zero otherwise is a Calderón–Zygmund 

kernel, and so for every \( g \in L^1(\mathbb{R}^n) \) the singular integral 

\( g*\varphi_h(x) \) exists for a.e. \( x \in \mathbb{R}^n \). Now for each positive integer \( N \) let \( f_h(x) \) 

to be \( f(x) \) if \( |x| \leq N \) and zero otherwise. Then since \( f_h \in L^1(\mathbb{R}^n) \), 

\( f_h*\varphi_h(x) \) exist for a.e. \( x \in \mathbb{R}^n \). If \( |x| \leq N-h \), then \( f*\varphi_h(x) = f_h*\varphi_h(x) \), and so \( f*\varphi_h(x) \) exists for a.e. \( x \) with \( |x| \leq N-h \). Letting 

\( N \to \infty \) we now see that \( f*\varphi_h(x) \) exists for a.e. \( x \in \mathbb{R}^n \). But, 

\( \delta_{j,h} f(x) = -\frac{1}{2h} f*\varphi_h(x) \). The assertion for \( \delta_{j,h} f \) is thus proved for 

the case \( n=1 \).

Suppose now \( n>1 \), and fix \( j=1, \ldots, n \). Let \( V_j \) be the hyperplane 

of \( \mathbb{R}^n \) perpendicular to \( e_j \), and for each \( x' \in V_j \) let \( f_{x'}(t) = f(x'+te_j) \), 
\( t \in \mathbb{R}^1 \). By the Fubini’s theorem it follows that \( f_{x'} \in L^1_{\text{loc}}(\mathbb{R}^1) \) for 

a.e. \( x' \in V_j \). The previous case then implies that \( f_{x'}*\varphi_h(t) \) exists 

for a.e. \( x' \in V_j \) and for a.e. \( t \in \mathbb{R}^1 \). But, \( \delta_{j,h} f(x) = \delta_{j,h} f(x'+te_j) = 

-\frac{1}{2h} f_{x'}*\varphi_h(t) \). Thus \( \delta_{j,h} f(x) \) exists for a.e. \( x \in \mathbb{R}^n \).

The measurability of \( D_j f \) follows from the equation \( \sup \{ \delta_{j,h} f(x) : 0<h \leq 1 \} = \sup \{ \delta_{j,r} f(x) : 0<r \leq 1, \ r \text{ rational} \} \), which in turn follows 

from the fact that for each fixed \( x \), \( \delta_{j,h} f(x) \) is continuous in \( h \).

Note that \( D_j f \) is well-defined in particular for every \( f \in L^1(\mathbb{R}^n) \),
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$1 \leq p \leq \infty$. The readers are referred to [4] for the Calderón-Zygmund kernel.

3. Main results

**Theorem 1.** Let $f \in \mathcal{L}_1(R^n)$, $1 \leq p \leq \infty$. Then for $j = 1, \ldots, n$

(a) if $1 < p \leq \infty$, then $D_j f \in L^p(R^n)$ and

$$\|D_j f\|_p \leq c_j \left\| \frac{\partial f}{\partial x_j} \right\|_p;$$

and

(b) if $p = 1$, then for every $r > 0$

$$| \{ x : D_j f(x) > r \} | \leq \frac{c_1}{r} \left\| \frac{\partial f}{\partial x_j} \right\|_1,$$

where $| \cdot |$ denotes the Lebesgue measure on $R^n$. The constants $c_p$ depend only on the parameters $p$ and $n$.

We need the following lemma for the proof of the above theorem.

**Lemma 2.** Let $f \in \mathcal{L}_1(R^n)$, $1 \leq p \leq \infty$. Then for $j = 1, \ldots, n$ and $h > 0$

$$\delta_{h,j} f(x) = \frac{1}{h} \int_0^h \int_{-\frac{1}{2}t}^{\frac{1}{2}t} \partial f(x + s e_j) ds dt$$

for a.e. $x \in R^n$.

**Proof.** By the Fubini’s theorem we may assume $n = 1$. Fix $h > 0$ and let

$$I(x) = \delta_{h,j} f(x) - \frac{1}{h} \int_0^h \int_{-\frac{1}{2}t}^{\frac{1}{2}t} f''(x + s) ds dt.$$

It suffices to show that for every $\varphi \in C_c^\infty(R^1)$

$$\int I(x) \varphi(x) dx = 0. \quad (3)$$

Setting

$$J(t) = \int (f(x + t) - f(x - t) - \int_{-t}^t f'(x + s) ds) \varphi(x) dx,$$

we get

$$\int I(x) \varphi(x) dx = \frac{1}{h} \int_0^h \frac{1}{2t} J(t) dt.$$

But

$$J(t) = \int f(x) (\varphi(x - t) - \varphi(x + t) + \int_{-t}^t \varphi'(x - s) ds) dx$$

$$= 0.$$
Now (3) follows from this.

**Proof of Theorem 1.** Let $M_j$ be the Hardy-Littlewood maximal operator acting in the direction of $e_j$, defined by the equation

$$M_j g(x) = \sup_{t>0} \frac{1}{2t} \int_{-t}^{t} |g(x+se_j)| \, ds.$$  

Then by Lemma 2

$$|\partial_{i,h} f(x)| \leq \frac{1}{h} \int_{0}^{h} \frac{1}{2t} \int_{-t}^{t} \left| \frac{\partial f}{\partial x_j}(x+se_j) \right| \, ds \, dt$$

$$\leq \frac{1}{h} \int_{0}^{h} M_j \left( \frac{\partial f}{\partial x_j} \right)(x) \, ds = M_j \left( \frac{\partial f}{\partial x_j} \right)(x).$$

Hence $D_j f(x) \leq M_j \left( \frac{\partial f}{\partial x_j} \right)(x)$ for a.e. $x \in \mathbb{R}^n$. Now the inequalities for $D_j$ follows from the corresponding inequalities for $M_j$.

We refer readers to [4] for the properties of the Hardy-Littlewood maximal operators.

**Remark.** The weak-type boundedness of the maximal operators $D_j$ on $L^1(\mathbb{R}^n)$ is the best we can expect. There are indeed functions in $L^1(\mathbb{R}^n)$ whose maximal derivates do not belong to $L^1(\mathbb{R}^n)$. An example can be constructed as follows. For each positive integer $m$

define a function $g_m$ on $\mathbb{R}^n$ by setting $g_m(x) = m^{-2}$ for $2^{-m-2} \leq x \leq 1 + 2^{-m-2}$, $g_m(x) = 0$ for $x \leq 0$ or $x \geq 1 + 2^{-m-1}$, and linear otherwise.

Then, $\|g_m\|_1 \leq 2m^{-2}$ and $\left\| \frac{d}{dx} g_m \right\|_1 \leq 2m^{-2}$, where $\frac{d}{dx} g_m$ is the distributional derivative of $g_m$. Furthermore, if $-\frac{1}{4} \leq x \leq -2^{-m-2}$, then

$$D_1 g_m(x) \geq \delta_{1,4|x|} g_m(x) \geq \frac{1}{4|x|} \int_{2|x|}^{4|x|} \frac{g_m(x+t)}{2t} \, dt$$

$$= \frac{1}{4|x|} \int_{2|x|}^{4|x|} \frac{m^{-2}}{2t} \, dt \geq \frac{1}{16m^2|x|}.$$

Hence,

$$\int_{-1}^{0} D_1 g_m(x) \, dx \geq \frac{1}{16m^2} \int_{-1/4}^{2^{-m-2}} \frac{1}{|x|} \, dx \geq \frac{c}{m},$$

where $c = \log 2/16$. Now letting $g(x) = \sum_{n=1}^{\infty} g_m(x-4(m-1))$ we obtain a desired function. It follows that
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\[ \|g\|_1 \leq \sum_{m=1}^{\infty} \|g_m\|_1 \leq 2 \sum_{m=1}^{\infty} m^{-2} < \infty, \]

and

\[ \left\| \frac{dg}{dx} \right\|_1 \leq \sum_{m=1}^{\infty} \left\| \frac{d}{dx} g_m \right\|_1 \leq 2 \sum_{m=1}^{\infty} m^{-2} < \infty, \]

and so \( g \in L^1_4(R^d) \). But, since \( D_1 g(x) = D_1 g_m(x - 4(m-1)) \) for \( 4(m-1) - 1 \leq x \leq 4(m-1) \),

\[ \|D_1 g\|_1 \geq \sum_{m=1}^{\infty} \int_{4(m-1)-1}^{4(m-1)} D_1 g(x) dx \]

\[ \geq \sum_{m=1}^{\infty} \int_{4(m-1)-1}^{4(m-1)} D_1 g_m(t) dt \geq c \sum_{m=1}^{\infty} \frac{1}{m}. \]

Thus \( D_1 g \in L^1_4(R^d) \) since the last series diverges to \( \infty \).

An immediate consequence of Theorem 1 is

**Corollary.** Let \( f \in L^p(R^d) \), and \( j = 1, \ldots, n \). Then \( \delta_{i,h}f(x) \to \frac{\partial f}{\partial x_i}(x) \) as \( h \to 0 \) for a.e. \( x \in R^d \). Furthermore, the convergence is also in the \( L^p \)-norm provided \( 1 < p \leq \infty \).

As a converse of Theorem 1 we have

**Theorem 2.** Let \( f \in L^p(R^d), 1 \leq p \leq \infty \). If \( D_j f \in L^p(R^d) \) for some \( j = 1, \ldots, n \), then \( \frac{\partial f}{\partial x_j} \in L^p(R^d) \) and \( \left\| \frac{\partial f}{\partial x_j} \right\|_p \leq \|D_j f\|_p \). Hence, if \( D_j f \in L^p(R^d) \) for every \( j = 1, \ldots, n \), then \( f \in L^1(R^d) \).

The following lemma will be used to prove the above theorem in the case \( p = 1 \).

**Lemma 3.** Let \( \{f_k\} \) be a sequence of functions \( L^1(R^d) \), \( g \in L^1(R^d) \), and \( \mu \) a finite Borel measure on \( R^d \). Suppose \( |f_k| \leq g \) for every \( k \) and \( f_k \) converges weakly to \( \mu \), i.e., \( \int f_k \varphi \to \int \varphi d\mu \) for every \( \varphi \in C_0(R^d) \). Then \( \mu \) is absolutely continuous.

**Proof.** We may assume each of \( f_k \) and \( \mu \) is real-valued (by splitting them into the real and imaginary parts, and by applying the following arguments to each part.) It suffices to show each of \( \mu^+ \) and \( \mu^- \) is absolutely continuous. The absolute continuity of \( \mu^+ \) (or \( \mu^- \)) is obtained once we show that \( \mu^+(E) > 0 \) (or \( \mu^-(E) > 0 \)) implies \( |E| > 0 \) for every Borel set \( E \).
By the Hahn's decomposition theorem there exist Borel sets $P$ and $N$ such that $P \cap N = \emptyset$, $P \cap N = R^n$, and $\mu^+(E) = \mu(E \cap P)$ and $\mu^-(E) = -\mu(E \cap N)$ for every Borel set $E$.

To prove the absolute continuity of $\mu^+$ suppose $E$ is a Borel set and $\epsilon = \mu^+(E) > 0$. We may assume $E \subset P$ (otherwise we can consider $E \cap P$). Choose a compact set $K \subset E$ (and so $K \subset P$) with $\mu^+(E \sim K) < \epsilon/4$, and an open set $V \supset E$ with $|\mu|(V \sim E) < \epsilon/4$. Such sets can be chosen by the regularity of $\mu^+$ and $|\mu|$. Note that $\mu^+(K) > 3\epsilon/4$. Let $G$ be an arbitrary open set such that $E \subset G \subset V$, and choose $\varphi \in C_0(R^n)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $x \in K$, and $\text{supp } \varphi \subset G$. Then for $k = 1, 2, \ldots$,

$$\int g \geq \int \varphi g \geq \int |f_{x_k}| \geq \int |\varphi f_{x_k}|.$$  
Since $\int \varphi f_{x_k} \to \int \varphi d\mu$, it now follows that $\int g \geq \int \varphi d\mu$. On the other hand,

$$|\varphi d\mu| = \left| \int d\mu + \int_{E^c} \varphi d\mu \right| \geq |\mu(K)| - |\mu|(G \sim K) \geq \mu^+(K) - |\mu|(V \sim K) > \frac{3}{4} \epsilon - \frac{1}{4} \epsilon = \frac{\epsilon}{2} > 0.$$  
Thus we get $\int g \geq \epsilon/2$. Now

$$\int g = \inf \left\{ \int g : E \subset G, \ G \text{ open} \right\} = \inf \left\{ \int g : E \subset G \subset V, \ G \text{ open} \right\} \geq \frac{\epsilon}{2} > 0,$$
and this implies $|E| > 0$. We thus obtain the absolute continuity of $\mu^+$. Similarly we see that $\mu^-$ is also absolutely continuous.

Proof of Theorem 2. Suppose first $1 < p \leq \infty$, and choose $q$ such that $1/p + 1/q = 1$. From the hypothesis we see that each $\delta_{i,k} f$ belongs to the ball of radius $\|D_j f\|_p$ in the space $L^q(R^n)$, which is the dual space of $L^p(R^n)$. By the weak-compactness of balls in dual spaces it then follows that there exists a function $g \in L^q(R^n)$ with $\|g\|_q \leq \|D_j f\|_p$, and a sequence $\{h_k\}$ with $h_k \to 0$ as $k \to \infty$ such that

$$\int \delta_{i,k} f(x) \varphi(x) dx \to \int g(x) \varphi(x) dx$$
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as \( k \to \infty \) for every \( \varphi \in L^p(\mathbb{R}^n) \). This is a fortiori true for \( \varphi \in C^\infty_c(\mathbb{R}^n) \). Then, for \( \varphi \in C^\infty_c(\mathbb{R}^n) \) it is easy to see that

\[
\int \delta_{i,h}f(x)\varphi(x)\,dx = -\int f(x)\delta_{i,h}\varphi(x)\,dx.
\]

Since \( \varphi \in L^\infty(\mathbb{R}^n) \), Theorem 1 implies \( D\varphi \in L^\infty(\mathbb{R}^n) \). Observe that \( D\varphi \) has compact support. Hence \( D\varphi \in L^r(\mathbb{R}^n) \) for every \( r \) with \( 1 \leq r \leq \infty \). In particular, \( D\varphi \in L^p(\mathbb{R}^n) \). Thus we get

\[
|f(x)\delta_{i,h}\varphi(x)| \leq |f(x)D\varphi(x)| \in L^1(\mathbb{R}^n)
\]

for every \( k \), and

\[
f(x)\delta_{i,h}\varphi(x) \to f(x)\frac{\partial \varphi}{\partial x_i}(x)
\]
as \( k \to \infty \). It now follows from the Lebesgue's dominated convergence theorem that

\[
\int f(x)\delta_{i,h}\varphi(x)\,dx \to \int f(x)\frac{\partial \varphi}{\partial x_i}(x)\,dx
\]
as \( k \to \infty \), from (4), (5), and (6) we now get

\[
\int g(x)\varphi(x)\,dx = -\int f(x)\frac{\partial \varphi}{\partial x_i}(x)\,dx
\]

for every \( \varphi \in C^\infty_c(\mathbb{R}^n) \), which indicates \( \frac{\partial f}{\partial x_i} = g \in L^s(\mathbb{R}^n) \) and completes the proof for the case \( 1 < p \leq \infty \).

Suppose next \( p = 1 \). Considering \( L^1(\mathbb{R}^n) \) as a subspace of the space of all finite Borel measures on \( \mathbb{R}^n \), which is the dual space of \( C_0(\mathbb{R}^n) \) consisting of all continuous functions vanishing at infinity, and applying the weak-compactness argument as above, we get a sequence \( \{h_k\} \) with \( h_k \to 0 \) as \( k \to \infty \) and a finite Borel measure \( \mu \) on \( \mathbb{R}^n \) with \( \|\mu\| \leq \|Df\|_1 \) such that for every \( \varphi \in C_0(\mathbb{R}^n) \)

\[
\int \delta_{i,h}f(x)\varphi(x)\,dx \to \int \varphi(x)d\mu(x)
\]
as \( k \to \infty \). But, \( \delta_{i,h}f, Df \in L^1(\mathbb{R}^n) \) and \( |\delta_{i,h}f| \leq Df \) Hence it follows from Lemma 3 that \( \mu \) is absolutely continuous, that is, there exists a function \( g \in L^1(\mathbb{R}^n) \) such that \( d\mu = g \, dx \) We thus obtain, by the same argument as above, \( \frac{\partial f}{\partial x_i} = g \in L^1(\mathbb{R}^n) \) and \( \left\| \frac{\partial f}{\partial x_i} \right\|_1 = \|\mu\| \leq \|Df\|_1 \), and finish the proof.
References


Ajou University
Suwon 440-749, Korea
and
Korea Institute of Technology
Taejon 302-338, Korea