NOTES ON THE FRESNEL INTEGRABLE FUNCTIONS *

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I. Introductory Preliminaries

Let $H$ be a separable Hilbert space over $\mathbb{R}$. Let $M(H)$ be the collection of $\mathbb{C}$-valued, countably additive measures on $\mathcal{B}(H)$, the Borel class of $H$. $M(H)$ is a Banach algebra under the total variation norm where the convolution is taken as the multiplication. Given $\mu \in M(H)$, $\hat{\mu}$ is defined for every $r$ in $H$ by the formula

$$\hat{\mu}(r) = \int_{H} \exp\{i(r, h)\} d\mu(h).$$

Let $F(H) = \{\hat{\mu} \mid \mu \in M(H)\}$. The correspondence $\mu \rightarrow \hat{\mu}$ is injective and carries convolution into pointwise multiplication. Hence, letting $\|\hat{\mu}\| = \|\mu\|$, we have that $F(H)$ is a Banach algebra. The Fresnel integral $\mathcal{F}(\hat{\mu})$ is defined for $\hat{\mu}$ in $F(H)$ by the formula

$$\mathcal{F}(\hat{\mu}) = \int_{H} \exp\{-\frac{i}{2} \|h\|^2\} d\mu(h).$$

The space $F(H)$ plays a key role throughout the fundamental monograph [2] of Albeverio and Høegh-Krohn.

Fix $t > 0$. Let $H_t$ be the space of $\mathbb{R}$-valued functions $r$ on $[0, t]$ which are absolutely continuous with square integrable derivative $Dr$ and which satisfy $r(t) = 0$. $H_t$ is a separable Hilbert space over $\mathbb{R}$ with inner product

$$\langle r_1, r_2 \rangle = \int_{0}^{t} (Dr_1)(s)(Dr_2)(s) ds.$$
Functions on $H_t$ of the form

\begin{equation}
    g(r) = \psi(r(0)),
\end{equation}

where $\psi : \mathbb{R} \to \mathbb{C}$, are simple but crucial for the applications of the theory to quantum mechanics. A very simple result of Albeverio and Høegh–Krohn [2] shows that if $\psi = \tilde{\nu}$ where $\nu$ is in $M(\mathbb{R})$, then $g$ is in $\mathcal{F}(H_t)$, the Fresnel class of $H_t$. Because of the great usefulness of the Fresnel class, it is natural to ask if $g$ of the form (1.4) is in $\mathcal{F}(H_t)$ for some class of $\psi$'s. In their paper [6], Chang, Johnson and Skoug showed that the answer is “No”; that is, if $g$ is in $\mathcal{F}(H_t)$, then there exists $\nu$ in $M(\mathbb{R})$ such that $\psi = \tilde{\nu}$. And also Johnson proved in his paper [7] that $\mathcal{F}(H_t)$ is equivalent to the space $S$ which is a Banach algebra of analytic Feynman integrable functionals.

There is a particular Hilbert space $H_Q$ which is an extension of Albeverio and Høegh–Krohn’s $H_t$. $H_Q$ is the space in which we will be concerned throughout this paper.

Fix $p, q > 0$ and let $Q = [0, p] \times [0, q]$. Let $H_Q$ be the set of all functions $r : Q \to \mathbb{R}$ for which there exists $v$ in $L_2(Q)$ such that for all $(s, t)$ in $Q$

\begin{equation}
    r(s, t) = \int_s^p \int_t^q v(\tau_1, \tau_2) \, d\tau_1 \, d\tau_2.
\end{equation}

The inner product on $H_Q$ is defined by

\begin{equation}
    (r_1, r_2)_{H_Q} = \int_0^p \int_0^q \left( \frac{\partial^2 r_1}{\partial s \partial t} \right)(s, t) \left( \frac{\partial^2 r_2}{\partial s \partial t} \right)(s, t) \, ds \, dt.
\end{equation}

$H_Q$, equipped with this inner product, is a separable infinite dimensional Hilbert space over $\mathbb{R}$. It will be helpful to introduce the family of functions $\{r_{\tau_1, \tau_2} : (\tau_1, \tau_2) \in Q\}$ from $H_Q$;

\begin{equation}
    r_{\tau_1, \tau_2}(s, t) = \min\{p - s, p - \tau_1\} \min\{q - t, q - \tau_2\}.
\end{equation}

These functions have the reproducing property,

\[ (r, r_{\tau_1, \tau_2})_{H_Q} = r(\tau_1, \tau_2) \quad \text{for all} \quad r \quad \text{in} \quad H_Q, \]
and also $H_Q$ is the reproducing kernel Hilbert space associated with two parameter Brownian motion.

In this paper, we show that various functions belong to $\mathcal{F}(H_Q)$, the space of Fresnel integrable functions on $H_Q$. And we establish necessary and sufficient conditions for the Fresnel integrability of certain class of functions on $H_Q$.

II. Some Fresnel Integrable Functions on $H_Q$

In their paper [5], Chang, Johnson and Skoug established a main theorem. After the statement of this theorem, we find various Fresnel integrable functions on $H_Q$ as its corollaries.

**Theorem.** (1.) Let $H$ be a separable infinite dimensional Hilbert space over $\mathbb{R}$.

(2.) Let $(Y, \mathcal{Y}, \eta)$ be a measure space where $\eta$ is either a non-negative, $\sigma$-finite measure or a $\mathbb{C}$-valued measure.

(3.) Let $\theta_{i,j} : Y \to H$ be $\mathcal{Y} - \mathcal{B}(H)$ measurable for $i = 1, \ldots, l$, $j = 1, \ldots, m$.

(4.) Let $\theta : Y \times \mathbb{R}^m \to \mathbb{C}$ be given by $\theta(y, \cdot) = \hat{\nu}_y(\cdot)$ where $\nu_y$ is in $M(\mathbb{R}^m)$ for every $y$ in $Y$ and where the family $\{\nu_y : y \in Y\}$ satisfies:

(i) $\nu_y(B)$ is a $\mathcal{Y}$-measurable function of $y$ for every $B$ in $\mathcal{B}(\mathbb{R}^m)$, and

(ii) $\|\nu_y\|$ is in $L_1(Y, \mathcal{Y}, |\eta|)$.

Under these hypotheses, $f : H \to \mathbb{C}$ given by

\begin{equation}
(2.1) \quad f(r) = \int_Y \theta(y, <r, \theta_{1,1}(y)>, \ldots, (r, \theta_{l,n}(y)) >) d\eta(y)
\end{equation}

belong to $\mathcal{F}(H)$ and satisfies the inequality

\begin{equation}
(2.2) \quad \|f\| \leq \int_Y \|\nu_y\|d|\eta|(y).
\end{equation}

Further, since $\mathcal{F}(H)$ is a Banach algebra, $g$ is in $\mathcal{F}(H)$ where

\begin{equation}
(2.3) \quad g(r) = \exp\{f(r)\}.
\end{equation}

**Remarks.** (1) It suffices to assume in (4.) that $\theta(y, \cdot) = \hat{\nu}_y(\cdot)$ for $\eta$-a.e. $y$ in $Y$. 

(2) Since $\mathcal{F}(H)$ is a Banach algebra, many analytic functions of $f$ can formed. We explicitly mention the exponential function in (2.3) because it plays a central role in the quantum theory.

Our first two corollaries are the extension of simple results of Albeverio and Hoegh–Krohn’s [2].

**COROLLARY 1.** Let $\psi = \hat{\nu}$ where $\nu$ is in $M(R)$. Define $f_1 : H_Q \to C$ by

$$f_1(r) = \psi(r(0,0)).$$

Then $f_1$ belongs to $\mathcal{F}(H_Q)$.

**Proof.** Apply the above theorem after making the following choices: $H = H_Q, (Y,Y,\eta) = (Q,\mathcal{B}(Q),\eta)$ where $\eta$ is any probability measure, $l = m = 1$ and $\theta_{1,1}(s,t) = r_{0,0}(s,t)$ as in (1.7), $\theta((s,t),\cdot) = \hat{\nu}(\cdot)$. With these choices, the right hand side of (2.1) becomes

$$\int_Q \hat{\nu}((r,r_{0,0})_{H_Q}) d\eta(s,t) = \psi(r(0,0)),$$

and the result follows.

**COROLLARY 2.** Let $\theta = \hat{\nu}$ where $\nu$ is in $M(R)$. Define $f_2 : H_Q \to C$ by

$$f_2(r) = \int_Q \theta(r(s,t)) ds dt.$$

Then $f_2$ belongs to $\mathcal{F}(H_Q)$.

**Proof.** Take $H, Y, \eta, l, m,$ and $\theta$ as in the proof of Corollary 1. Let $\eta$ be Lebesgue measure on $Q$ and take $\theta_{1,1}(s,t) = r_{s,t}$ as in (1.7). Then the right hand side of (2.1) is just

$$\int_Q \theta(r,r_{s,t})_{H_Q}) ds dt = \int_Q \theta(r(s,t)) ds dt,$$

and the result follows.
Let $H_Q^{lm} = \prod_1^m H_Q$ consist of functions $r : Q \to \mathbb{R}^{lm}$ such that each component $r_{i,j}$ is in $H_Q$. Define the inner product of $r$ and $r^*$ in $H_Q^{lm}$ as the sum of the $H_Q$ inner products of the components.

**Corollary 3.** Let $\theta = \hat{\nu}$ where $\nu$ is in $M(\mathbb{R}^{lm})$. Define $f_3 : H_Q^{lm} \to \mathbb{C}$ by

$$f_3(r) = \int_Q \theta(r_{1,1}(s,t), \ldots, r_{l,m}(s,t)) \, ds \, dt. \quad (2.6)$$

Then $f_3$ belongs to $\mathcal{F}(H_Q^{lm})$.

**Proof.** Apply the above theorem after making the following choices: $H = H_Q^{lm}$, $(Y, \mathcal{Y}, \eta)$ as in Corollary 2, $\theta((s,t), \cdot) = \hat{\nu}(\cdot)$, $\theta_{i,j}(s,t)$ the function in $H_Q^{lm}$ which is 0 except in the $(i,j)$th component where it is $r_{s,t}$.

The next corollary is an extension of Corollary 4 in [5].

**Corollary 4.** Let $\theta : Q \times \mathbb{R} \to \mathbb{C}$ be given by $\theta((s,t), \cdot) = \hat{\nu}_{s,t}(\cdot)$ where $\nu_{s,t}$ is in $M(\mathbb{R})$ for every $(s,t) \in Q$ and where the family $\{\nu_{s,t} : (s,t) \in G\}$ satisfies: (i) $\nu_{s,t}(B)$ is a Borel measurable function of $(s,t)$ for every $B$ in $\mathcal{B}(\mathbb{R})$, and (ii) $\|\nu_{s,t}\|$ is integrable over $Q$ with respect to Lebesgue measure. Define $f_4 : H_Q \to \mathbb{C}$ by

$$f_4(r) = \int_Q \theta((s,t), r(s,t)) \, ds \, dt. \quad (2.7)$$

Then $f_4$ belongs to $\mathcal{F}(H_Q)$.

**Proof.** Make the choice of $H, Y, \ldots$ from the above theorem as in Corollary 2 except taking $\theta((s,t), \cdot) = \hat{\nu}_{s,t}(\cdot)$.

The following is as in Corollary 4 except that Lebesgue measure is replaced by a general Borel measure $\eta$ on $Q$.

**Corollary 5.** Let $\theta : Q \times \mathbb{R} \to \mathbb{C}$ be given by $\theta((s,t), \cdot) = \hat{\nu}_{s,t}(\cdot)$ where $\nu_{s,t}$ is in $M(\mathbb{R})$ for every $(s,t) \in Q$ and where the family $\{\nu_{s,t} : (s,t) \in \bar{Q}\}$ satisfies: (i) $\nu_{s,t}(B)$ is a Borel measurable function of $(s,t)$
for every \( B \) in \( \mathcal{B}(R) \), and (ii) \( \| \nu_{s,t} \| \) is integrable over \( Q \) with respect to |\( \eta \)| where \( \eta \) is a Borel measure on \( Q \). Define \( f_5 : H_Q \to \mathbb{C} \) by

\[
f_5(r) = \int_Q \theta((s,t),r(s,t))d\eta(s,t).
\]

Then \( f_5 \) belongs to \( \mathcal{F}(H_Q) \).

**Proof.** Apply the above theorem with \( H = H_Q, Y = Q, \mathcal{Y} = \mathcal{B}(Q), l = m = 1 \) and \( \theta_{1,1}(s,t) = r_{s,t} \) where \( r_{s,t} \) is given by (1.7).

**III. Necessary and Sufficient Conditions for the Fresnel Integrability on \( H_Q \)**

In this section, we establish necessary and sufficient conditions for the Fresnel integrability of certain class of functions on \( H_Q \) which are similar to those on \( H_t \).

**THEOREM 1.** Let \( 0 \leq s_1 < s_2 < \cdots < s_l < p, \ 0 \leq t_1 < t_2 < \cdots < t_m < q, \) and let \( \nu \) be in \( M(\mathbb{R}^{lm}) \). Define \( f : H_Q \to \mathbb{C} \) by

\[
f(r) = \tilde{\nu}(r(s_1,t_1),\ldots,r(s_l,t_m)).
\]

Then \( f \) is in \( \mathcal{F}(H_Q) \); in fact, there exists a unique measure \( \mu \) in \( M(H_Q) \) such that

\[
\tilde{\mu}(r) = \tilde{\nu}(r(s_1,t_1),\ldots,r(s_l,t_m))
\]

for all \( r \) in \( H_Q \).

**Proof.** Let \( \theta : \mathbb{R}^{lm} \to H_Q \) be defined by

\[
\theta(a_{1,1},\ldots,a_{l,m}) = a_{1,1}r_{1,1} + \cdots + a_{l,m}r_{l,m}
\]

where \( r_{i,j} \equiv r_{s_i,t_j} \) as in (1.7) for \( i = 1,\ldots,l, \ j = 1,\ldots,m. \) Let \( \mu = \nu \circ \theta^{-1} \). Then \( \mu \) is in \( M(H_Q) \). By the linear independence of \( r_{1,1},\ldots,r_{l,m} \)
and the change of variable formula, we can write, for any $r$ in $H_Q$,

\[ \hat{\mu}(r) = \int_{H_Q} \exp\{i(r, h)\} d\mu(h) \]

\[ = \int_{H_Q} \exp\{i(r, h)\} d(\nu \circ \theta^{-1})(h) \]

\[ = \int_{R^{lm}} \exp\{i(r, \theta(a_{1,1}, \ldots, a_{l,m}))\} d\nu(a_{1,1}, \ldots, a_{l,m}) \]

\[ = \int_{R^{lm}} \exp\{i(r, a_{1,1}r_{1,1} + \cdots + a_{l,m}r_{l,m})\} d\nu(a_{1,1}, \ldots, a_{l,m}) \]

\[ = \int_{R^{lm}} e^{i((r, a_{1,1}), \ldots, (r, r_{l,m}))(a_{1,1}, \ldots, a_{l,m})} d\nu(a_{1,1}, \ldots, a_{l,m}) \]

\[ = \check{\hat{\nu}}(r(s_1, t_1), \ldots, r(s_l, t_m)). \]

Finally the uniqueness of a measure $\mu$ satisfying (3.2) is a consequence of the fact that the map $\mu \to \hat{\mu}$ is one-one.

Note that Corollary 1 in Section 2 is just the special case of Theorem 1 with $l = m = 1$ and $s_1 = t_1 = 0$.

**THEOREM 2.** Let $0 \leq s_1 < s_2 < \cdots < s_l < p$, $0 \leq t_1 < t_2 < \cdots < t_m < q$, and let $\psi : R^{lm} \to C$. Suppose that there exists $\mu$ in $M(H_Q)$ such that for all $r$ in $H_Q$.

\[ \hat{\mu}(r) = \psi(r(s_1, t_1), \ldots, r(s_l, t_m)). \]

Then there exists a measure $\nu$ in $M(R^{lm})$ such that $\psi = \check{\hat{\nu}}$ on $R^{lm}$.

**Proof.** Let $r_{i,j} \equiv r_{s_i, t_j}$ as in (1.7) for $i = 1, \ldots, l$, $j = 1, \ldots, m$, and let $[r_{1,1}, \ldots, r_{l,m}]$ be the span of $r_{1,1}, \ldots, r_{l,m}$. By the linear independence of $r_{1,1}, \ldots, r_{l,m}$, we know $\dim[r_{1,1}, \ldots, r_{l,m}] = lm$, and hence $\{((r, r_{1,1}), \ldots, (r, r_{l,m})): r \in [r_{1,1}, \ldots, r_{l,m}]\} = R^{lm}$.

By the Gram–Schmidt process, we get an orthonormal set $\{e_{1,1}, \ldots, e_{l,m}\}$ which is a basis for $[r_{1,1}, \ldots, r_{l,m}]$. For each $i = 1, \ldots, l$, $j = 1, \ldots, m$,

\[ r_{i,j} = (r_{i,j}, e_{1,1})e_{1,1} + \cdots + (r_{i,j}, e_{l,m})e_{l,m}. \]
Hence
\[
\hat{\mu}(r) = \psi((r, r_{1,1}), \ldots, (r, r_{\ell,m})) \\
= ((r_{1,1}, e_{1,1})(r, e_{1,1}) + \cdots + (r_{1,1}, e_{\ell,m})(r, e_{\ell,m}), \\
\ldots \ldots \ldots, \\
(r_{\ell,m}, e_{1,1})(r, e_{1,1}) + \cdots + (r_{\ell,m}, e_{\ell,m})(r, e_{\ell,m})) \\
= B((r, e_{1,1}), \ldots, (r, e_{\ell,m}))
\]
where \( B \) is the linear map from \( \mathbb{R}^{lm} \) onto \( \mathbb{R}^{lm} \) sending \( ((r, e_{1,1}), \ldots, (r, e_{\ell,m})) \) to \( ((r, r_{1,1}), \ldots, (r, r_{\ell,m})) \).

By Proposition 6 in [6], there exists \( \eta \) in \( M(\mathbb{R}^{lm}) \) such that \( \hat{\eta} = \psi \circ B \) on \( \mathbb{R}^{lm} \). Applying Lemma 7 in [6] with \( T = B^{-1} \), we see that \( \psi = (\psi \circ B) \circ B^{-1} \) is the Fourier transform of some measure \( \nu = \eta \circ B^t \) in \( M(\mathbb{R}^{lm}) \), that is, \( \psi = \hat{\nu} \) for some \( \nu \) in \( M(\mathbb{R}^{lm}) \).

References


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