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Dualizing Complex of the Blowing-up of Some one Dimensional Local Rings. *

by

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1. Introduction

Let C be an affine curve, P a closed point on C, X a nonsingular surface containing C as a closed subscheme of codimension 1.

The blowing-up X' of X with center P induces the blowing-up C' of C with center P, which is a curve on X'.

It is known that C' is affine. Let R, R' be respectively the coordinate rings of C, C'; if m is the maximal ideal in R corresponding to P, R' is called the ring *"obtained from R by blowing-up m"*.

In (4), Lipman gives a more general definition of the ring obtained from a 1-dimensional ring blowing-up an ideal. Let (A, m) be a local Cohen-Macaulay ring of dimension 1. Then the ring obtained by blowing-up m is

$$\bigcup_{n>0}(m^n : m^n) = \bigcup_{n>0}\{a \in \bar{A} \mid am^n \subset m^n\}$$

which is also equal to $A[z_1/x, \cdots, z_t/x]$ where $\{z_1, z_2, \cdots, z_t\}$ is a set of generators of m, X is a sutible element in m and $\bar{A}$ is the normalization of A.

Let $A = k[\![t^{n_1}, \cdots, t^{n_p}]\!]$ be a subring of $R = k[\![t]\!]$ where k is algebraically closed. We assume $n_1 < n_2 < \cdots < n_p$ and $(n_1, \cdots, n_p) = 1$.
R is integral over A and the quotient field of them are equal because $(n_1, n_2, \cdots, n_p) = 1$.
The injective envelope of k over R is

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$$E_R(k)=k[t^{-1}]=kf_0+kf_1+\cdots$$

where $f_i=1/t^{i+1}$ (mod R).

Then, we can show that the dualizing complex for A is

$$I^\circ : O\longrightarrow K\longrightarrow E_A(k)\longrightarrow O$$

where $K=\Phi(A)$, $E_A(k)$ the injective envelope of k over A. (6)

Proposition 1-1. Let $A=k[\![t^{n_1},\cdots,t^{n_p}]\!]$, where $n_1<n_2<\cdots<n_p$, $(n_1,\cdots,n_p)=1$. **Then the** ring obtained from A by blowing-up $m=(t^{n_1},\cdots,t^{n_p})$ is $\Lambda=k[\![t^{n_1},t^{n_2-n_1},\cdots,t^{n_p-n_1}]\!]$

Proof. According to (11), we have $\Lambda=A[t^{n_1}/x,\cdots,t^{n_p}/x]$ where $x\in m$ is such that $xm^n=m^{n+1}$ for a sutible $n>0$. We put $x=t^{n_1}$, showing that the valuation $v(x)$ is equal to n_1. Since $xm^n=m^{n+1}$, we have

$$(n+1)n_1\in\Sigma=<v(x)+nn_1,\cdots,v(x)+nn_p,\cdots,v(x)+k_1n_1+\cdots+k_pn_p,\cdots>$$

where $k_1+k_2+\cdots+k_p=n$.

If $v(x)>n_1$, for each $(k_1,\cdots,k_p)$ such that $k_1+\cdots+k_p=n$. We have $(n+1)n_1=nn_1+n_1=(k_1+\cdots+k_p)\ n_1+n_1\ <(k_1n_1+\cdots+k_pn_p)+n_1<k_1n_1+\cdots+k_pn_p+v(x)$.

$(n+1)n_1\notin\Sigma$, which is contradiction.

If follows

$$\Lambda=A[t^{n_1}/t^{n_1},\ t^{n_2}/t^{n_1},\cdots,t^{n_p}/t^{n_1}]$$
$$=k[\![t^{n_1},\ t^{n_2-n_1},\cdots,t^{n_p-n_1}]\!].$$

2. Dualizing Complex of the ring obtained from $k[t^{n_1},\cdots,t^{n_p}]$ by blowing-up $m=(t^{n_1},\cdots,t^{n_p})$

We shall recall the definitions and some properties of the Local Cohomology, the Dualizing Complex and the Matlis Duality which are used in this paper.

Let A be a noetherian local ring with maximal ideal m and M be a finitely generated A-module. Let

$$L_m(M)=\{x\in M\,|\,m^kx=0 \text{ for some } k\}$$

Then L_m is a left exact functor.

Let H_m^i be the i-th right derived functor of L_m, then

$$H_m^i(M) = \varinjlim_{v} Ext_A^i(A/m^v, M).$$

This $H_m^i(M)$ is called the i-th Local Cohomology module of the A-module M. Also it can be defined in the usual way taking an injective resolution $I^\cdot$,

$$H_m^i(M) = H^i(L_m(I^\cdot)).$$

A Dualizing Complex for A is an injective complex $I^\cdot$ in $\mathscr{D}_c^b(A)$ (that is, $I^\cdot$ is bounded and all its cohomology modules are finitely generated) with the following property; whenever $X^\cdot$ is a complex in $\mathscr{D}_c^b(A)$ then

$$\theta^\cdot : X^\cdot \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(X^\cdot, I^\cdot), I^\cdot)$$

is a quasi-isomorphism.

According to (9), an injective complex $I^\cdot \in \mathscr{D}_c^b(A)$ is a Dualizing Complex for A iff the morphism of complexes

$$\alpha^\cdot : A \longrightarrow \mathrm{Hom}_A(I^\cdot, I^\cdot)$$

is a quasi-isomorphism.

If $I^\cdot$ is a Dualizing Complex for A as in (10) we define $t(m : I^\cdot)$ to be the unique integer t for which

$$H^t(\mathrm{Hom}_A\ (k, I^\cdot)) \neq 0 \text{ where } k = A/m.$$

The injective envelope of $k = A/m$ over A is denoted by $E_A(k)$. Let D denote the Matlis Duality functor

$$D(-) = \mathrm{Hom}_A(-, E_A(k)).$$

Let $I^\cdot$ be a arbitrary Dualizing Complex for A. Let $t = t(\mathrm{m}:I^\cdot)$ and M be a finitely generated A-module. Then

$$H_m^i(M) \cong D(H^{t-i}(\mathrm{Hom}_A(M. I^\cdot)))$$

for each $i \geqq 0$.

Let (A, m, k) be a one-dimensional noetherian local domain. Then the Local

Cohomology of A is $H_m^1(A)=k/A$ where K is the quotient field of R. If A is Gorenstein then $H_m^1(A)\cong E_A(k)$.

The ring obtained from $k[\![t^{n_1},\cdots,t^{n_p}]\!]$ by blowing-up

$$m=(t^{n_1},\cdots,t^{n_p}) \text{ is } \Lambda=k[t^{n_1},t^{n_2-n_1},\cdots,t^{n_p-n_1}]$$

If $2n_1<n_2$ is hold then $(n_1,n_2-n_1,\cdots,n_p-n_1)=1$ and

$$n_1<n_2-n_1<n_3-n_1<\cdots<n_p-n_1.$$

So we can obtain the Dualizing Complex for Λ.

Theorem 2-1. Let $A=k[\![t^{n_1},\cdots,t^{n_p}]\!]\subset R=k[\![t]\!]$ where

$$n_1<n_2<\cdots<n_p,\ 2n_1<n_2,\ (n_1,n_2,\cdots,n_p)=1.$$

The ring obtained from A by blowing-up $m=(t^{n_1},\cdots t^{n_p})$ is

$$\Lambda=k[\![t^{n_1},t^{n_2-n_1},\cdots,t^{n_p-n_1}]\!]$$

Then the Dualizing Complex for Λ is

$$J^\cdot: O\longrightarrow K\longrightarrow E_\Lambda(k)\longrightarrow O$$

Where $K=\Phi(\Lambda)$, $E_\Lambda(k)$ is the injective envelope k over Λ.

Let $n_1,n_2,\cdots,n_p$ be positive integers such that

$$n_1<n_2<\cdots<n_p,\ (n_1,n_2,\cdots,n_p)=1.$$

Let S be the semi-group generated by $\{n_1,\cdots,n_p\}$.
Then there is an element $t=\sup\ (N\setminus S)$
S is called symmetric if S satisfies the property;

$$s\in S \text{ iff } t-s\notin S$$

Then the subring $k[\![t^{n_1},\cdots,t^{n_p}]\!]$ is Gorenstein iff the semi-group S is symmetric (2).

Theorem 2-2. Let the assumption be the same (Th. 2-1) and S be the semi-group generated by $\{n_1, n_2-n_1, \cdots, n_p-n_1\}$ Then the Local Cohomology $H_m{}^1(A) \cong K/A$ satisfies $H_m{}^1(A) \cong D(H^\circ(J^\cdot))$, where $H^\circ(J^\cdot) \simeq R+N'$ where $N' = \sum_{i \in N \setminus S} k \cdot \frac{1}{t^{i+1}}$.

Proof. See the Theorem 4-3 (6).

Examples 2-3. Consider the subrings $A = k[\![t^2, t^5]\!]$, $B = k[\![t^3, t^7, t^8]\!]$ of $R = k[\![t]\!]$. Then the rings obtained from A, B by blowing-up

$$m_A = (t^2, t^5),\quad m_B = (t^3, t^7, t^8) \text{ is } C = k[\![t^2, t^3]\!],\quad D = k[\![t^3, t^4, t^5]\!].$$

respectively.

A, C are Gorenstein and B, D are not Gorenstein. For

$S_A = \{0, (\ ), 2, (\ ), 4, 5, \cdots\}$

$S_B = \{0, (\ ), (\ ), 3, (\ ), (\ ), 6, 7, \cdots\}$

$S_C = \{0, (\ ), 2, 3, \cdots\}$

$S_D = \{0, (\ ), (\ ), 3, 4, \cdots\}$

Here S_A, S_C are symmetric and S_B, S_D are not symmetric.

Take $N_A = kf_1 + kf_3$ a A-submodule of $E_R(k)$.

Then $H_m{}^1(A) \cong D(H^\circ(I^\cdot))$

where $H^\circ(I^\cdot) \cong R+N' \cong R+kt^{-2}+kt^{-4} \cong k+kt^2+t^4R \cong A$.

Take $N_B = kf_1 + kf_2 + kf_4 + kf_5$ a B-submodule of $E_R(k)$.

Then $H_m{}^1(B) \cong D(H^\circ(I^\cdot))$.

where $H^\circ(I^\cdot) \cong R+N' \cong R+kt^{-2}+kt^{-3}+kt^{-5}+kt^{-6} \cong k+kt+kt^3+kt^4+t^6R$

Take $N_C = kf_1$ a C-submodule of $E_R(k)$

Then $H_m{}^1(C) \cong D(H^\circ(I^\cdot))$.

where $H^\circ(I^\cdot) \simeq R+N' \cong R+k_t{}^{-2} \simeq k+t^2R \simeq C$.

Take $N_D = kf_1 + kf_2$ a D-submodule of $E_R(k)$.

Then $H_m{}^1(D) \cong D(H^\circ(I^\cdot))$.

where $H^\circ(I^\cdot) \cong R+N' \simeq R+kt^{-2}+kt^{-3} \simeq k+kt+t^3R$.

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