HONAM MATHEMATICAL JOURNAL
Volume 11, Number 1. August 1989

A Note on Spinorial Structures in Vector Bundles

by

Keean Lee and Hongjae Lee

Dept. of Mathematics, Chonbuk National University,
Chonju (560—756), Korea.

Throughout this paper we shall assume that X is a compact topological space and a vector bundle ξ over X is a finite dimensional real vector bundle over X. In a vector bundle ξ over X, the spinorial structure (Definition 4) and the Stieful Whitney classes of ξ are closely related ([1], [4]). Moreover, if ξ is a spin bundle (*i.e.*, ξ has the spinorial structure) then there is the Thom-Gysin isomorphism

$$K_R^n(X) \xrightarrow{\cong} K_R^{n+\dim\xi}(X^\xi),$$

where X^ξ is the Thom complex of ξ ([2]).

In this paper, we shall prove that for a vector bundle ξ over X $\xi\oplus\xi\oplus\xi\oplus\xi$ is a spin bundle (Theorem 5).

Let G be a topological group. A G-cocycle on X is given by an open cover $\{U_i\}$ of X, and continuous maps

$$g_{ji} : U_i \cup U_j \longrightarrow G$$

such that

(i) $\forall x\in U_i\cap U_j\cap U_k \quad g_{ki}(x)=g_{kj}(x)g_{ji}(x)$

(ii) $\forall x\in U_i \quad g_{ii}(x)=1_G$(identity of G)

(iii) $\forall x\in U_i\cap U_j \; g_{ji}(x)=g_{ij}(x)^{-1}$.

Let (U_i, g_{ji}) and (V_r, h_{sr}) be two G-cocycles on X. If there exist continuous maps $g_i^r : U_i\cap V_r \longrightarrow G$ such that $h_{sr}(x)=g_j^s(x)\cdot g_{ji}(x)\cdot g_i^r(x)^{-1}$ for each $x\in U_i\cap U_j\cap V_r\cap V_s$ then (U_i, g_{ji}) and (V_r, h_{sr}) are said to be *equivalent*, written $(U_i, g_{ji})\sim(V_r, h_{sr})$. Then "$\sim$" is an equivalence relation ([4]). The set of all G-cocycles over X is denoted

---

*The Present Studies were supported by the Basic Science Research Institute program, Minstry of Education, 1988.

by Cocy $(X:G)$. We shall put $H^1(X:G)=\mathrm{Cocy}(X:G)/\sim$.

Let $\Phi^R_n(X)$ be the set of isomorphism classes of real vector bundles over X with rank (dimension) n. Then $\Phi^R_n(X)$ is naturally isomorphic to $H^1(X:GL_n(\boldsymbol{R}))$ ([4]). Let a $GL_n(\boldsymbol{R})$-cocycle (U_i, g_{ji}) correspond to $[\xi=(E,p,X)]\in\Phi^R_n(X)$, and let $h_i: U_i\times\boldsymbol{R}^n\longrightarrow E|U_i=E_{U_i}$ be a trivialization of ξ. Then for each $x\in U_i\cap U_j$, $h_j(x,v)=h_i(x,g_{ji}(x)v)$ ([3]).

Lemma 1. For a vector bundle ξ over X with a metric β, let $s_1,\cdots,s_n$ be vector fields such that for each $x\in X$, $s_1(x),\cdots,s_n(x)$ are linearly independent. Then there exist vector fields $s_1^*,\cdots,s_n^*$ such that for each $x\in X$, $\beta(s_i^*(x), s_j^*(x))=\delta_{ij}$.

Proof. For each $x\in X$ we put

$$s_1^*(x)=s_1(x)/\sqrt{\beta(s_1(x), s_1(x))},$$

We assume that $s_1^*,\cdots,s_{k-1}^*$ have been chosen with $\beta(s_i^*(x), s_j^*(x))=\delta_{ij}$ for $1\leq i,j\leq k-1$ and $x\in X$. For $x\in X$ we put

$$s_k^*(x)=\{s_k(x)-\sum_{1\leq j\leq k-1}\beta(s_k(x), s_j^*(x))\, s_j^*(x)\}/|\beta(s_k(x)-\sum_{1\leq j\leq k-1}\beta(s_k(x), s_j^*(x))s_j^*(x)|$$

where $|s_k(x)-\sum_{1\leq j\leq k-1}\beta(s_k(x), s_j^*(x))\, s_j^*(x)|^2=\beta(s_k(x)-\sum_{1\leq j\leq k-1}\beta(s_k(x), s_j^*(x))\, s_j^*(x), s_k(x)-\sum_{1\leq j\leq k-1}\beta(s_k(x), s_j^*(x))\, s_j^*(x))$. It is clear that $s_1^*,\cdots,s_n^*$ satisfy the desired properties. ///

Lemma 2. Let ξ be a vector bundle over X with rank n, and β be a metric of ξ. Then there exists an atlas $\{(U_i, h_i^*)\,|\,(U_i, h_i^*)$ is a trivialization of $\xi\}$ such that $(v|w)=\beta(h_i^*(x,v), h_i^*(x,w))$, where $x\in U_i$, $v=(v_1,\cdots,v_n)$, $w=(w_1,\cdots,w_n)\in\boldsymbol{R}^n$ and $(v|w)=\sum_{i=1}^{n} v_iw_i$. The cocycle $\{g_{ji}\}$ of this atlas have their values in $O(n)$.

Proof. Let $\{(U_i, h_i)\}$ be an atlas of ξ. For each $x\in U_i$ and $e_j=(0,\cdots,0,\overset{(j)}{1},0,\cdots,0)\in\boldsymbol{R}^n$ $(i=1,\cdots,n)$ we define

$$h_i(x,e_j)=s_j(x), \quad j=1,2,\cdots,n$$

Then vector fields $s_1(x),\cdots,s_n(x)$ are linearly independent. By Lemma 1, there are vector fields $s_1^*,\cdots,s_n^*$ of ξ over U_i such that for all $x\in U_i$ $\beta(s_i^*(x), s_j^*(x))=\delta_{ij}$. We define

$$h_i^*: U_i\times\boldsymbol{R}^n\longrightarrow\xi|U_i$$

by $$h_i^*(x : a_1, \cdots, a_n) = a_1 s_1^*(x) + \cdots + a_n s_n^*(x)$$

Then h_i^* is a trivialization of ξ over U_i and $\{(U_i, h_i^*)\}$ is an atlas with the desired property.

Next, for $x \in U_i \cap U_j$ and $v, w \in \boldsymbol{R}^n$

$$\begin{aligned}(v|w) &= \beta(h_j^*(x,v),\ h_j^*(x,w)) = \beta(h_i^*(x, g_{ji}(x)v),\ h_i^*(x, g_{ji}(x)w) \\ &= (g_{ji}(x)v | g_{ji}(x)w)\end{aligned}$$

and thus $g_{ji}(x) \in O(n)$. ///

Let $\xi = (E, p, X)$ be a vector bundle over X. A metric β of ξ is defined as follows.

i) $E = X \times \boldsymbol{R}^n$. $\beta((x,v),\ (x,w)) = (v|w)$.

ii) E is isomorphic to $X \times \boldsymbol{R}^n$. Let $f : E \longrightarrow \mathrm{T} = X \times \boldsymbol{R}^n$ be an arbitrary isomorphism. For $e, e' \in E_X$, we assume $f(e) = (x, v)$ and $f(e') = (x, w)$ $(v, w \in \boldsymbol{R}^n)$. Then we put $\beta(e, e') = (v|w)$.

iii) E is arbitrary. Let $\{U_i | i \in I\}$ be an open cover of X such that each $E|U_i = E_{U_i}$ is trivial and $\{U_i\}_{i \in I}$ is locally finite. Let $\{\alpha_i | i \in I\}$ be a partition of unity associated with $\{U_i | i \in I\}$. Let β_i be a metric on E_{U_i} defined as in (ii). The metrix $\beta : E \times E \to \boldsymbol{R}$ is defined by the formulas

$$\begin{cases} \beta(e, e') = \sum_{i \in I} \alpha_i(x)\, \beta_i(e, e') \text{ if } x \in U_i, e \text{ and } e' \in E_x \\ \beta_i(e, e') = 0 \text{ if } x \notin U_i,\ e \text{ and } e' \in E_x. \end{cases}$$

Proposition 3. Every vector bundle ξ over X with rank n has an $O(n)$-cocycle $\{g_{ji}\}$.

Proof. By the above description ξ has a metric β. By Lemma 2, ξ has an atlas $\{(U_i, h_i^*)\}$ such that the cocycle $\{g_{ji}\}$ of the atlas $\{(U_i, h_i^*)\}$ have their values in $O(n)$, *i.e.*, for all $x \in U_i \cap U_j$ $g_{ji}(x) \in O(n)$. ///

For a finite dimensional real vector space with a nondegenerate quadratic form Q, we can define the Clifford algebra $C(V, Q) = C(V)$ ([3],[4]). Moreover, there exists the canonical map $V \longrightarrow C(V)$ which is injective, and thus we identify V with its image in $C(V)$. In particular, the endomorphism $v \longmapsto -v$ of V induces an involution on $C(V)$, written $x \longmapsto \bar{x}$ $(x \in C(V))$. As well-known $C(V)$ is $\boldsymbol{Z}/2$-graded($\boldsymbol{Z}$: integers) such that $C(V) = C^0(V) \oplus C^1(V)$ ([3],[4]). If $x \in C^0(V)$ then $\bar{x} = x$ and if $x \in C^1(V)$ then $\bar{x} = -x$. We put $C^*(V) = C(V) - \{0\}$. The *twisted* Clifford group $\tilde{\Gamma}(V)$ is the set $\{x \in C^*(V) | \bar{x} V x^{-1} = V\}$, where x^{-1} is the inverse of x. Let $\tilde{\rho} : \tilde{\Gamma}(V) \longrightarrow GL(V)$ be the homomorphism

$x \longrightarrow \tilde{\rho}(x) = \tilde{\rho}_x$ where for each $v \in V$ $\tilde{\rho}_x(v) = \bar{x}\ vx^{-1}$.
We have the exact sequence of groups:

$$1 \longrightarrow \boldsymbol{R}^* \longrightarrow \tilde{\Gamma}(V) \xrightarrow{\tilde{\rho}} O(V) \longrightarrow 1$$

where $\boldsymbol{R}^* = \boldsymbol{R} - \{0\}$ ($\boldsymbol{R}$: reals).

For an element $y = v_1 \cdots v_n (v_i \in V)$ of $C(V)$ we put ${}^t y = v_n \cdots v_1$. For an element $x \in C(V)$ its *spinorial norm* $N(x)$ is defined by $N(x) = {}^t\bar{x} \cdot x \in C(V)$. Then the map

$$\tilde{\Gamma}(V) \longrightarrow \boldsymbol{R}^* \ (x \longmapsto N(x))$$

is a group homomorphism ([4]). If we put $\Gamma^0(V) = \tilde{\Gamma}(V) \cap C^0(V)$, $SO(V) = \{u \in O(V) \mid \mathrm{Det}(u) = 1\}$ ($u \in O(V) \Longrightarrow \mathrm{Det}(u) = \pm 1$) and $\rho^0 = \tilde{\rho} \mid \Gamma^0(V)$, then we have the exact sequence ([4]):

$$1 \longrightarrow \boldsymbol{R}^* \longrightarrow \Gamma^0(V) \xrightarrow{\rho^0} SO(V) \longrightarrow 1.$$

Moreover, if we put $\mathrm{Pin}(V) = \{x \in \tilde{\Gamma}(V) \mid |N(x)| = 1\}$ and $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap C^0(V)$ then there are two exact sequences:

$$1 \longrightarrow \boldsymbol{Z}/2 \longrightarrow \mathrm{Pin}\ (V) \longrightarrow O\ (V) \longrightarrow 1$$

$$1 \longrightarrow \boldsymbol{Z}/2 \longrightarrow \mathrm{Spin}(V) \longrightarrow SO(V) \longrightarrow 1$$

([4]). Therefore, we let Spin (n) denote the group Spin(V) when $V = \boldsymbol{R}^n$, provided with the quadratic from Q such that

$$x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n \Longrightarrow Q(x) = \sum_{i=1}^{n} x_i^2$$

We have the exact sequence ([4]):

$$1 \longrightarrow \boldsymbol{Z}/2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\rho^0} SO(n) \longrightarrow 1. \quad (※)$$

Definition 4. Let $\xi = (V, p_V, X)$ be a vector bundle over X with rank n. An *orientation* of ξ is an element $\alpha \in H^1(X : SL_n(\boldsymbol{R}))$ such that under the natural map $\varphi : H^1(X : SL_n(\boldsymbol{R})) \longrightarrow H^1(X : GL_n(\boldsymbol{R}))$ $\varphi(\alpha)$ contains ξ. A *spinorial structure* (or *spin* *structure*) of ξ is an element $\alpha \in H^1(X:\mathrm{Spin}(n))$ such that under the natural map $\psi : H^1(X:\mathrm{Spin}(n)) \longrightarrow H^1(X: GL_n(\boldsymbol{R}))$ $\psi(\alpha)$ contains ξ. It ξ has a spin structure then ξ is called a ***spin bundle.***

Let $\{g_{ji}\}$ be G-cocycle on $X=\bigcup_{i\in I} U_i$(open cover), where G is a topological group. We introduce the equivalence relation "$\sim$" on the disjoint union $P=\dot{\bigcup}_{i\in I} U_i\times G(I$: indexing set) such that

$$\text{for } (x_i, g_i),\ (x_j, g_j)\in \dot{\bigcup_{i\in I}} U_i\times G=P,\ (x_i,g_i)\sim(x_j,g_j) \text{ iff}$$
$$x_i=x_j\in U_i\cap U_j \text{ and } g_j=g_{ji}(x)\ g_i.$$

The group G acts on the right on P such that $(x_i,g_i)g=(x_i,g_ig)$.
Since this action is free $X\approx P/G$. Let F be a n-dimensional real vector space such that G acts on the left on F. We put

$$E=P\times_G F=P\times F/\sim$$

where $(p,f)\sim(pg,g^{-1}f)$ for $g\in G$ and $(p,f)\in P\times F$. Then we have the assertion([4]):
E is a vector bundle over X associated with the cocycle $\{g_{ji}\}$ (※※).

Theorem 5. Let $\xi=(E,p,X)$ be a vector bundle over X with rank n. Then $\xi\oplus\xi$ has an orientation and $\xi\oplus\xi\oplus\xi\oplus\xi$ is a spin bundle, where $\oplus$ means the Whitney sum of bundles.

Proof. Our proof is divided into the following three steps.

Step I. By Proposition 3, there is an $O(n)$-cocycle $\{g_{ji}\}$ which is associated with the vector bundle ξ. By the above descriptions(※※), there exists a principal bundle P such that

$$E\cong P\times_{O(n)}\boldsymbol{R}^n.$$

Step II. We shall prove that $E\oplus E$ has an orientation. The principal bundle P in step I is associated with the cocycle $\{g_{ji}\}$, where for all $x\in U_i\cap U_j$, $g_{ji}(x)\in O(n)$. Then the bundle $E\oplus E$ (or $\xi\oplus\xi$) may be written as $P'\times_{O(2n)}\boldsymbol{R}^{2n}$, where P' is the principal bundle associated with the cocycle

$$h_{ji}(x)=\begin{pmatrix} g_{ji}(x) & 0 \\ 0 & g_{ji}(x)\end{pmatrix}$$

for all $x \in U_i \cap U_j$, ($X = \bigcup_{i \in I} U_i$ is an open cover). Therefore

$$\text{Det } (h_{ji}(x)) = \text{Det } (g_{ji}(x)) \cdot \text{Det } (g_{ji}(x)) = 1$$

and thus $h_{ji}(x) \in SO(2n)$. By Definition 4, $\xi \oplus \xi$ is an oriented bundle.

Step Ⅲ. As in step Ⅱ, since $\xi \oplus \xi$ has an orientation there exists a principal bundle P over X with structure group $SO(2n)$ such that

$$E \oplus E \cong P \times_{SO(2n)} \boldsymbol{R}^{2n}.$$

Suppose the composite homomorphisms

$$\begin{array}{ccccc} \eta : SO(2n) & \longrightarrow & SO(2n) \times SO(2n) & \longrightarrow & SO(4n) \\ \Downarrow & & \Downarrow & & \Downarrow \\ \alpha & \longmapsto & \alpha \times \alpha & \longmapsto & \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}. \end{array}$$

Note that the map $\boldsymbol{R}^{2n} \longrightarrow \boldsymbol{R}^{2n} \oplus \boldsymbol{R}^{2n} = \boldsymbol{R}^{4n}$ ($x \longmapsto x \oplus 0$) induces the homomorphism $i_1 : \text{Spin}(2n) \longrightarrow \text{Spin}(4n)$ and the map $\boldsymbol{R}^{2n} \longrightarrow \boldsymbol{R}^{2n} \oplus \boldsymbol{R}^{2n}$ ($x \longmapsto 0 \oplus x$) induces the homomorphism $i_2 : \text{Spin}(2n) \longrightarrow \text{Spin}(4n)$.

Define the homomorphism $D : \text{Spin}(2n) \times \text{Spin}(2n) \longrightarrow \text{Spin}(4n)$ by

$$D(\alpha_1, \alpha_2) = i_1(\alpha_1) i_2(\alpha_2)$$

for each $(\alpha_1, \alpha_2) \in \text{Spin}(2n) \times \text{Spin}(2n)$. For $\varepsilon = \pm 1$ and $\eta = \pm 1$ it is clear that

$$D(\varepsilon \alpha_1, \eta \alpha_2) = \varepsilon \eta \ D(\alpha_1, \alpha_2)$$

If $u \in SO(2n)$ and $(\rho^0)^{-1}(u) = \{-\tilde{u}, +\tilde{u}\}$ (see (※) above) in $\text{Spin}(2n)$ then $D(\tilde{u}, \tilde{u}) = D(-\tilde{u}, -\tilde{u}) = v$ is a well defined element of $\text{Spin}(4n)$. We define the homomorphism $j : SO(2n) \longrightarrow \text{Spin}(4n)$ by $u \longmapsto v (j(u) = v)$. Then we have the commutative diagram

$$\begin{array}{ccc} & SO(2n) & \\ {\scriptstyle j} \swarrow & \copyright & \searrow {\scriptstyle \eta} \\ \text{Spin}(4n) & \xrightarrow{\rho^0} & SO(4n) \end{array}$$

Let $\{g'_{ji}\}$ be the $SO(2n)$-cocycle associated with the oriented bundle $E \oplus E$ (or $\xi \oplus \xi$). The $\text{Spin}(4n)$-cocycle $\{h'_{ji}\}$ associated with $E \oplus E \oplus E \oplus E$ is defined by the commutative diagram:

$$\begin{array}{ccc} U_i \cap U_j & \overset{h'_{ji}}{\cdots\cdots\rightarrow} & \mathrm{Spin}(4n) \\ & {\scriptstyle g'_{ji}} \searrow \quad \copyright \quad \nearrow {\scriptstyle j} & \\ & SO(2n) & \end{array}$$

Thus, by Definition 4 $\xi \oplus \xi \oplus \xi \oplus \xi$ is a spin bundle. ///

References

1. M.F. Atiyah, R. Bott and A. Shapiro; Clifford Modules, Topology 3, pp.3～38, (1964).
2. M.F. Atiyah and F. Hirzebruch: Riemann-Roch theorem for differentiable manifolds. Bull. *Amer. Math. Soc.*, **65** pp.276～281, (1959).
3. D.Husemoller; Fiber Bundles, Springer-Verlag, (1966).
4. M. Karoubi: K-Theory (An Introduction), Springer-Verlag (1978).