HONAM MATHEMATICAL JOURNAL
Volume 11, Number 1. August 1989

On the Ring $R(X)$

by

Yong-Hwan Cho

Dept. of Mathematics Education, Chonbuk National University, Chonju (560—756), Korea.

1. Introduction

The *f.c.c*, *s.c.c*, and *c.c*, are conditions that are well-known to hold for a local domain of classical algebraic geometry ([26]). These terminologies are used in Nagata ([14],[15]).

Taut rings and Taut-Level rings were introduced by S. MacAdam and Ratliff ([12]). However, the concept of such a ring can clearly be traced back to I.S. Cohen ([5]).

The *o.h.c.c*, condition was introduced by M.E. Pettit Jr. and Ratliff ([19]). A number of known characterizations of semi local domain that satisfies the *o.h.c.c*, have been studied in [19] and [25].

The notion of ring of quotients, $R(X)$, was first introduced by Nagata who treated the case that S consists of polynomials whose coefficients generate the unit ideal R in R. Many properties on this ring $R(X)$ have been extensively studied by D.D. Anderson ([1],[2]), J.T. Anold ([3]), G.W. Hinkle and J.A. Huckaba ([8]) etc.

Throughout this paper, all rings are assumed to be commutative with an identity.

In section 2, we introduced some terminologies and basic properties which is needed in section 3.

In section 3, many statements are considered about the ring of quotients and the *o.h.c.c*, and a number of relationships between the statements are proved. We also will prove our main theorems (Theorem 3.7, Theorem 3.8, Theorem 3.9, Theorem 3.10, and Corollary 3.11).

2. Preliminaries

If a ring R has only one maximal ideal M, then this fact will often be displayed by the notation (R,M), and it is said that (R,M) is a quasi local ring. A local ring is a Noetherian quasi local ring, and a semi local ring is a Noetherian ring which has only a finite number of maximal ideals.

Let Spec R be the set of all prime ideals of R. If $P \subset Q$ in Spec R, then a chain

$$(*) \qquad P=P_0 \subset P_1 \subset \cdots \subset P_n=Q$$

in Spec R is a saturated chain of prime ideals between P and Q if for each $i=0,1,\cdots,n-1$ there are no $P' \in$Spec R such $P_i \subset P' \subset P_{i+1}$. ($I \subset J$ means I is a proper subset of the set J). The length of the chain (*) is n (so it is the number of joins in the chain that are counted to give its length).

A maximal chain of prime ideals in R is simply a saturated chain of prime ideals between a minimal prime ideal and a maximal prime ideal. If $P \in$Spec R, then the height of P is the supremum of the lengths of chains of prime ideals descending from P, the depth of P is defined analogously by using chains ascending from P, and the altitude of R is the supremum of the heights of the maximal ideals in R or, equivalently, the supremum of the depths of the minimal prime ideals in R.

Definition 2.1. R satisfies the first chain condition for prime ideals (*f.c.c*) in case each maximal chain of prime ideals in R has length equal to the altitude R.

Definition 2.2. R satisfies the saturated chain condition for prime ideals (or R is catenary) in case for every pair of prime ideals $P \subset Q$ in R, $(R/P)_{Q/P}$ satisfies the *f.c.c.*

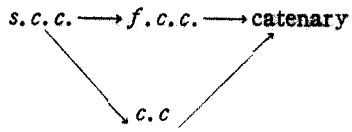
Definition 2.3. R satisfies the second chain condition for prime ideals (*s.c.c*) in case, for every minimal prime ideal Q in R and for every integral domain S, which contains and is integral over R/Q, S satisfies the *f.c.c.* and altitude $S=$altitude R.

Definition 2.4. R satisfies the chain condition for prime ideals (*c.c*) in case, for every pair of prime ideal $P \subset Q$ in R, $(R/P)_{Q/P}$ satisfies the *s.c.c.*

Remark. (1) If R satisfies the *f.c.c.* then height $P+$ depth $P=$altitude R, for each prime ideal P in R.

(2) R satisfies the *f.c.c.* if and only if for each maximal ideal M in R, R_M satisfies the *f.c.c.* and altitude $R_M=$altitude R.

(3) Let R be a ring with finite altitude. Then



Proposition 2.5. The first chain condition holds in a ring R if and only if for every minimal prime ideal P of R, altitude R=altitude R/P and the first chain condition holds in R/P.

Proof. Let P be a minimal prime ideal in R. Then height P+depth P=altitude R/P=altitude R(Remark (1)). Let $0\subset P_1/P\subset\cdots\subset P_t/P=M/P$ be a maximal chain of prime ideals in R/P. Then $P\subset P_1\subset\cdots\subset P_t=M$ is a maximal chain of prime ideals in R so t=altitude R=altitude R/P. Conversely, let $0\subset P_0\subset P_1\subset\cdots\subset P_s=M$ be a maximal chain of prime ideals in R. Then $0\subset P_1/P_0\subset\cdots\subset P_s/P_0=M/P_0$ is also a maximal chain of prime ideals in R/P_0. Hence s=altitude R/P_0. By assumption, altitude R/P_0 =altitude R.

So s=altitude R.

Proposition 2.6 ([22]). Let $R\subseteq S$ be rings such that S is integral over R. Then, if R satisfies the c.c then S satisfies the c.c.

Proposition 2.7 ([20]). Let $R\subseteq S$ be rings such that S is a domain, S is integral over R,S satisfies the c.c and height M=heigh $R\cap M<\infty$ for each maximal ideal M of S. Then R satisfies the c.c.

Proposition 2.8. Let $R\subseteq S$ be rings such that S is integral over R. If S satisfies the f.c.c then R satisfies the f.c.c.

Proof. Let $P_0\subset P_1\subset\cdots\subset P_t$ be a maximal chain of prime ideals in R.

By Lying over theorem, there exists a prime ideal Q_o in S such that $Q_o\cap R=P_o$. By Going up theorem, there exists a maximal chain of prime ideals $Q_o\subset Q_1\subset\cdots\subset Q_t$. Hence t=altitude S=altitude R([4]). Therefore R satisfies the f.c.c.

Definition 2.9. Let $R\subseteq S$ be rings such that every minimal prime ideal of S contracts to a minimal prime ideal of R. In this case we will say that mpcmp holds in $R\subseteq S$.

Theorem 2.10. Let $(R,M)\subseteq(T,O)\subseteq(S,N)$ be quasi local rings such that R is local and S is integral over R. Then, T satisfies the f.c.c. and mpcmp holds in $T\subseteq S$ if and only if S satisfies the f.c.c.

Proof. ($\Longleftarrow$). Let Q_0 be a minimal prime ideal in S and $Q_0\subset Q_1\subset\cdots\subset Q_s=N$ be a maximal chain of prime ideals in S. Then, $s=$altitude S. If $Q_0\cap T$ is not a minimal prime ideal in T then altitude $T\geq s+1>s=$altitude S, contradiction, and we know that T satisfies the f.c.c (Proposition 2.8).

($\Longrightarrow$). We first prove this theorem in case R, T, and S are quasi local domains such that R is a local and S is integral over R and altitude $S=$a. Clearly, we may assume that altitude $T=a>2$. Suppose that T satisfies the f.c.c. Then, R satisfies the f.c.c (Proposition 2.8) and so height $P+$depth $P=$altitude R(Remark (1)) for each prime ideal P in R.

Hence, for each height one prime ideal P in R, depth $P=$altitude $R-1=a-1$. Therefore depth $P'=a-1$ for each height one prime ideal P' of S and we can easily show that S/P' satisfies the f.c.c by induction on a.

Let $0\subset P'\subset P'_2\subset\cdots\subset P'_t=N$ be maximal chains of prime ideals in S. Then height $P'=1$ and since S/P' satisfies the f.c.c, $t-1=$depth P'. But depth $P'=a-1$.

Hence $t=$a so S satisfies the f.c.c. In general case, let Q'' be any minimal prime ideal in S and let $Q'=Q''\cap T$, let $Q=Q'\cap R$. Then $R/Q\subseteq T/Q'\subseteq S/Q''$ are quasi local domains such that S/Q'' is integral over R/Q ([4]) and R/Q is local.

Hence, T satisfies the f.c.c and mpcmp holds in $T\subseteq S\Longrightarrow T/Q'$ satisfies the f.c.c and altitude $T/Q'=$altitude T(Proposition 2.5)$\Longrightarrow S/Q''$ satisfies the f.c.c (as in the previous paragraph) and altitude $S/Q''=$altitude $T/Q'=$altitude $T=$altitude $S\Longrightarrow S$ satisfies the f.c.c (Proposition 2.5).

Proposition 2.11. ([15]). Assume that an integral domain S is integral over its subring R which is normal. Then for an arbitrary ideal I of S such that $I\neq S$, we have height $I=$height $(I\cap R)$.

By Lying over Theorem and Going up Theorem, we obtain the following lemma.

Lemma 2.12. Let $R\subseteq S$ be rings such that S is integral over R. For each prime ideal P in R, there is a prime ideal P' in S such that $P'\cap R=P$, height $P'=$height P, depth $P'=$depth P.

Let R' denotes the integral closure of R in its total quotient ring.

Definition 2.13. A ring R is a taut (resp., taut level) ring in case, for each prime ideal P in R, depth P+height $P\in\{$altitude R, 1$\}$ (resp., =altitude R).

Definition 2.14. A ring R satisfies the $1\frac{1}{2}$chain condition (o.h.c.c) in case, R is taut and $(R/P)'$ satisfies the c.c for each minimal prime ideal P in R.

Remark. For a local domain, the o.h.c.c condition lies somewhere intermediate to the f.c.c and the s.c.c.

Example. Every integral domain of altitude at most two is a taut ring and every ring satisfying the f.c.c is a taut level ring, so every regular local ring is taut level ring.

Theorem 2.15. ([11]) Let (R, M) be a local ring. Then, R satisfies the f.c.c if and only if R is a taut level ring.

By Theorem 2.10 and Theorem 2.15, we have the following Theorem.

Theorem 2.16. Let $R\subseteq S$ be local rings such that S is integral over R and mpcmp holds in $R\subseteq S$. Then R is taut level if and only if S is taut level.

3. The Ring $R(X)$

Let R be a ring and let $S=\{f\in R[x] \mid A_f=R\}$ where A_f denotes an ideal of R generated by the cofficients of f. In [15], Nagata showed that S is a multiplicative system of $R[x]$ which contains only regular elements, and then he defined

$$R(x)=R[x]_S.$$

Proposition 3.1. Under above notations, the following statements hold.

(1) If $\{M_s\}$ is the collection of maximal ideals of R and if $M_s[x]$ denotes the extension of M_s to $R[x]$, then $S=R[x]-\bigcup M_s[x]$.

(2) If A is an ideal of R, then $AR(x)\cap R=A$.

(3) If Q is P-primary then $QR(x)$ is $PR(x)$-primary.

(4) If $A_1,\dots,A_n$ are ideals of R, then

$$(A_1 \cap A_2 \cap \ldots \cap A_n)\ R(x) = A_1R(x) \cap \ldots \cap A_nR(x).$$

(5) If A is an ideal of R then $R(x)/AR(x) = R/A(x)$.

Proof. (1)-(3) ([7]), (4)-(5) ([15],[16]).

Proposition 3.2. There is a one to one correspondence between the maximal (minimal prime) ideals of R and the maximal (minimal prime) ideals of $R(X)$.

Proof. Let M_1 and M_2 be distinct maximal ideals of R. Then $M_1R(X)$ and $M_2R(X)$ are distinct ideals of $R(X)$ (Proposition 3.1 (2)). For any maximal ideal M' of $R(X)$, there exists a maximal ideal M of R such that $MR(X) = M'$([15]).

Lemma 3.3 ([15]). Let $R \subseteq S$ be rings such that S is integral over R. Then $S(x)$ is integral over $R(x)$.

It is well known that altitude $R(x)$=altitude R if R is a Noetherian ring ([15]). Now we generalize this Theorem as follows.

Theorem 3.4. Let $R \subseteq S$ be rings such that R is a Noetherian ring and S is integral over R. Then, altitude S=altitude $S(x)$.

Proof. Since $S(x)$ is integral over $R(x)$ (Lemma 3.3), altitude $R(x)$=altitude $S(x)$ ([15]) and altitude R=altitude S(assumption). Hence, altitude S=altitude $R(x)$= altitude $S(x)$.

Lemma 3.5. If R is a normal ring, then $R(x)$ is also a normal ring.

Proof. Let R be normal. Then $R[x]$ is also normal ([7]). $R(x) = R[x]_S$ and since $S = \{f \in R[x] \mid A_f = R\}$ is a multiplicative system ([16]), $R(x)$ is a normal ring ([10]).

Proposition 3.6 ([24]). Let R be a semi local ring and x be an indeterminate. Then, R is taut (resp, taut level) if and only if $R(x)$ is taut (resp, taut level).

Theorem 3.7. Let $R \subseteq S$ be domains such that R is normal and S is integral over R. Then, $R(x)$ is taut(resp, taut level) if and only if $S(x)$ is taut(resp, taut level).

Proof. Let P be a prime ideal in $S(x)$. Then height P=height $P \cap R(x)$ (Lemma

3.3, Lemma 3.5 and Proposition 2.11). $R(x)/P\cap R(x)=P+R(x)/P\subseteq S(x)/P$ and $S(x)/P$ is integral over $R(x)/P\cap R(x)$ ([14]). Hence

$$\begin{aligned}\text{depth } P\cap R(x) &= \text{altitude } R(x)/P\cap R(x)\\ &= \text{altitude } S(x)/P\\ &= \text{depth } P.\end{aligned}$$

Thus height P+depth P=height $P\cap R(x)$+depth $P\cap R(x)$ and altitude $R(x)$=altitude $S(x)$ (Lemma 3.3). Thus if $R(x)$ is taut (resp, taut level) then $S(x)$ is taut (resp, taut level). The converse follows from Lemma 2.12.

Theorem 3.8. Let $R\subseteq S$ be local domains such that S is integral over R. Then, the following statements are equivalent.

(1) R is a taut level ring.

(2) S is a taut level ring.

(3) $R(x)$ is a taut level ring.

(4) $S(x)$ is a taut level ring.

Proof. S is a taut level if and only if R is a taut level ring (Theorem 2.16) if and only if $R(x)$ is a taut level ring (Proposition 3.6) if and only if $S(x)$ is a taut level ring (Theorem 2.16, Lemma 3.3).

In view of Theorem 2.15, we can restate Theorem 3.8 as follows:

Theorem 3.8′. Let $R\subseteq S$ be local domains such that S is integral over R. Then, the following statements are equivalent.

(1) R satisfies the f.c.c.

(2) S satisfies the f.c.c.

(3) $R(x)$ satisfies the f.c.c.

(4) $S(x)$ satisfies the f.c.c.

Theorem 3.9. Let $R\subseteq S$ be rings such that R is semi local, S is integral over R and mpcmp holds in $R(x)\subseteq S(x)$. Then, if $R(x)$ satisfies the o.h.c.c then $S(x)$ satisfies the o.h.c.c.

Proof. Since $S(x)$ is integral over $R(x)$ (Lemma 3.3), $S(x)$ is taut. Also, if Q' is a minimal prime ideal of $S(x)$ and $Q=Q'\cap R(x)$, then Q is a minimal prime ideal of

$R(x)$. so, $(R(x)/Q)'$ satisfies the c.c. Furthermore $(S(x)/Q')'$ is integral over $S(x)/Q'$ and $S(x)/Q'$ is also integral over $R(x)/Q$ ([4]). Thus $(S(x)/Q')'$ is integral over $(R(x)/Q)'$ and so $(S(x)/Q')'$ satisfies the c.c (Proposition 2.6).

Theorem 3.10. Let $R \subseteq S$ be rings such that S is integral over R and let altitude $S(x)$ be finite. If $S(x)$ satisfies the o.h.c.c then $R(x)$ satisfies the o.h.c.c.

Proof. It readily follows from Lemma 2.12 and Lemma 3.3 that $R(x)$ is taut, since $S(x)$ is taut. Also, if P is a minimal prime ideal in $R(x)$ and P' is a minimal prime ideal in $S(x)$ such that $P' \cap R(x) = P$, then $(S(x)/P')'$ is integral over $(R(x)/P)'$. So height $M \cap (R(x)/P)' =$ height M (Proposition 2.11) is finite for all maximal ideal M in $(S(x)/P')'$. Also, $(S(x)/P')'$ satisfies the c.c and hence $(R(x)/P)'$ satisfies the c.c (Propoosition 2.7). Our proof is complete.

Corollary 3.11. Let $R \subseteq S$ be integral domains such that R is semi local and S is integral over R. Then, $S(x)$ satisfies the o.h.c.c if and only if $R(x)$ satisfies the o.h.c.c.

Proof. ($\Leftarrow$). It is trivial (Theorem 3.9).

($\Rightarrow$). Since R is semi local ring, altitude $S(x)$ ($=$altitude $R(x) =$altitude R) is finite.

Hence $R(x)$ satisfies the o.h.c.c (theorem 3.10).

References

1. D.D. Anderson, *Some remarks on the ring* $R(x)$, Comment. Math. Univ. St. Pauli XXUI-2 (1977), 137~140.
2. D.D. Anderson, *Multiplication ideals, Multiplication rings, and the ring* $R(x)$, Can. J. Math. **28** (1976), 760~768.
3. J.T. Anold, *On the theory of the Kronecker function ring and the domain* $D(X)$, Can. J. Math. **21** (1969), 558~563.
4. M.F. Atyah and I.G. Mcdonald, "Introduction to Commutative algebra," Addison Wesley Pub. Co., 1969.
5. I.H. Cohen, *Length of prime ideals*, Amer. J. Math. **76** (1954), 654~668.
6. J. Dieudonne, "Topics in Local Algebra," Univ. of Notre Dame Press, Notre Dame, 1967.
7. R. Gilmer, "Multiplicative ideal theory," Marcel Dekker, Inc., New-York, 1972.
8. G.W. Hinkle and J.A. Huckaba, *The generalized Kronecker function ring and* $R(X)$, J. Reine Angew. Math. **292** (1977), 25~36.
9. E.G. Huston, *Rank in Noetherian rings*, J. of Algebra **37** (1975), 64~73.

10. I. Kaplansky, "Commutative rings," Allyn and Bacon, Inc., Boston, 1970.
11. S. McAdam, *Saturated chains in a Noetherian rings*, Indiana Univ. Math. J. ***23***(1974), 719～728.
12. S. McAdam and L. J. Ratliff, Jr., *Semi local taut rings*, Indiana Univ. Math. J. ***26*** (1977), 73～79.
13. M. Nagata, *Basic theorems on general commutative rings*, Memoirs of the college of science, Univ. of kyoto series A Vol. XXIX, (1955), 59～76.
14. M. Nagata, *On the chain problem of prime ideals*, Nagoya Math. J. ***10*** (1956), 51～46.
15. M. Nagata, "Local rings, Interscience Tracts 13," Interscience Publishers New-York, 1962.
16. M. Nagata, "Commutative rings (In Japanese)," Kinokuniya shoten, 1979.
17. M. Nishi, *On the dimension of local rings*, Memoirs of the college of science, Univ. of kyoto series A Vol. XXIX, (1955), 7～9.
18. M. E. Pettit, Jr. and L. J. Ratliff, Jr., *Characterizations of H_i local rings and of C_i local rings*, Amer. J. Math. ***100*** (1978), 1213～1234.
19. M. E. Pettit, Jr., *Properties of H_i-rings*, Annali di Mathematica Pura ed applicata (IV) Vol. XXI (1979), 1～23.
20. L. J. Ratliff, Jr., *On quasi unmixed local domains, the altitude formula, and the chain condition for prime ideals, (I)*, Amer. J. Math. ***91*** (1969), 508～528.
21. L. J. Ratliff, Jr., *On quasi unmixed local domains, the altitude formula, and the chain condition for prime ideals (II)*, Amer. J. Math. ***92*** (1970), 99～144.
22. L. J. Ratliff, Jr., *Characterizations of catenary rings*, Amer. J. Math. ***93*** (1971), 1070～1108.
23. L. J. Ratliff, Jr., *Catenary rings and the altitude formula*, Amer. J. Math. ***94*** (1972), 458～466.
24. L. J. Ratliff, Jr., *Characterizations of taut semi local rings*, Ann. Math. Pura Appl. ***112*** (1977), 151～192.
25. L. J. Ratliff, Jr., *Notes on a new chain condition for prime ideals*, J. of pure and applied Algebra ***12*** (1978), 159～175.
26. L. J. Ratliff, Jr., "Chain conjectures in ring theory," Springer Verlag, New-York, 1978.
27. L. J. Ratliff, Jr., *Four notes on Saturated Chains of prime ideals*, J. Algebra ***39*** (1976), 75～93.
28. O. Zariski and P. Sauel, "Commutative Algebra, Vol. II," Van Nostrand, New-York, 1960.