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Some Remarks on Noetherian Local Rings

by

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1. Introduction

The theory of commutative ring has been developed, since about 1950, together with extensive researches on the Algebraic geometry and has been deeply related with Homological algebra in 1970. Theorems on theories of commutative ring and its open problems can be found in [4],[9]. In 1950, France mathematician Serre studied the theory of Cohen-Macaulay ring. Thereafter, we can see the research of this has been continued([5]). Serre discovered the fundamental theorem on regular local rings($(***)_2$ of §4) and we can find his successive study on these in [5],[8],[11] and [13].

In this paper, we study the regular local rings and stably free modules, which has been developed in the seminar performed during the last semester. In details, the contents of this thesis is as follows.

Section 2 contains some properties which will be needed in section 3 and section 4.

In section 3, we deal with the Unique factorization domain.

In section 4, we prove theorem 4.1 which is one of the main theorems in this dissertation, and by using this theorem we attempt to give a new proof of Proposition 4.3.

Theorem 4.1 says: (i) Let A be a ring and let M be an A-module. If M is stably free, then M has a finite free resolution. Conversely, if M is a finitely generated projective A-module which has a finite free resolution then M is stably free.

(ii) Let A be noetherian. If every finitely generated projective A-module is stably free, then every finitely generated A-module M with projective dimensioon$<\infty$ has a finite free resolution.

(iii) Let A be noetherian. If every finitely generated A-module has a finite free resolution, then A is a regular ring.

2. Preliminaries

Throughout this paper, by a ring we mean a commutative ring with 1.

Let A be a ring. An element $a \in A$, $a \neq 0$, is an *irreducible element* if a is not a unit and

$$a = bc \Longrightarrow b \text{ or } c \text{ is a unit.}$$

This is equal to that aA is a maximal ideal among principal ideals which are not A.

If aA is a prime ideal then a is said to be a *prime element*. A prime element is irreducible, but its converse is not true as follows.

Let a be a prime element, and we assume that $a = bc$ such that b and c are not units. Then aA is not a prime ideal because that b and c are not in aA. Therefore, at least one of b and c is a unit. This implies that a is an irreducible element.

Consider the commutative ring $Z[\sqrt{-5}] = \{m + n\sqrt{-5} \mid m, n\text{: integers}\}$ where Z is the ring of integers. It is clear that $2 \in Z[\sqrt{-5}]$. Moreover, for $n + m\sqrt{-5} \in Z[\sqrt{-5}]$ $2 = n^2 + 5m^2$ has no integral solutions for m and n. This means that 2 is an irreducible element of $Z[\sqrt{-5}]$. But $2Z[\sqrt{-5}]$ is not a prime ideal because $2 \cdot 3 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5}) \in 2Z[\sqrt{-5}]$ and $1 + \sqrt{-5}$, $1 - \sqrt{-5}$ are not in $2Z[\sqrt{-5}]$.

Let A be an integral domain. If every nonzero element of A which is not a unit can be uniquely expressed as a product of prime elements then A is called a ***unique factorization domain***, written UFD.

Proposition 2.1. An integral domain A is a UFD if and only if the following two conditions hold.

(1) Every irreducible element is a prime element.

(2) Every nonempty family $\mathscr{F}$ of principal ideals has a maximal element.

Proof. Let A be a UFD, and an element $\alpha \in A$ be an irreducible element. If α is not a prime element then we have the unique factorization of α by prime elements such that

$$\alpha = p_1^{n_1} \cdots p_r^{n_r}$$

where $p_i (i = 1, \cdots, r)$ is a prime element. Since α is irreducible there exists $n_i = 1$

$(1 \leqq i \leqq r)$ and $n_j = 0$ for $1 \leqq j \neq i \leqq r$.

Therefore, every irreducible element is a prime element. For an indexing set I we assume that $\mathscr{F} = \{a_i A \mid a_i \in A, \ i \in I\}$ is not empty. Take $a_i A$. Since there is a unique factorization of a_i by finite number of prime elements, there is only a finite number of $a_i{}^j A$ in $\mathscr{F}$ such that $a_i A = a_i{}^\circ A \subsetneqq a_i{}^1 A \subsetneqq \cdots \subsetneqq a_i{}^l A$. Of course $a_i{}^l A \subset A$. Then $a_i{}^l A$ is a maximal element in $\mathscr{F}$.

Conversely, let $a \in A$. We assume

$$a = p_1 p_2 \cdots\cdots$$

where p_i is a prime element. Then by condition (2) there exists a positive integer n such that

$$a = p_1 \cdots p_n.$$

Suppose there is another factorization of a by prime elements such that

$$a = q_1 \cdots q_m.$$

Then we can easily prove that $n = m$ and $p_1, \cdots, p_n$ is the same as $q_1, \cdots, q_m$ up to order.

Corollary 2.2. Every principal ideal domain is a unique factorization.

Proof. Let A be a principal ideal domain. Then every ideal of A is finitely generated, and thus A is a noetherian ring. Thus every non-empty set of ideals in A has a maximal element.

For an element $a \in A$, aA is contained in a maximal ideal mA, i.e., $aA \subset mA$. This implies that there exists an element b such that $mb = a$. Let us assume that a is an irreducible element. Then b is a unit. Thus $mA = aA$ which means that a is a prime element because that $aA = mA$ is a prime ideal. Therefore, by Proposition 2.1 A is a unique factorization domain. ⫽

Let A be a noetherian semi-local ring, and let $\boldsymbol{m} = \mathrm{rad}(A)$ be the intersection of all maximal ideals of A and $M \neq 0$ a finite A-module. Then we know that

$\dim(M)$ is the smallest integer r such that there exist elements
$x_1, \cdots, x_r$ in $\boldsymbol{m}$ satisfying
$l(M/x_1 M + \cdots + x_r M) < \infty$, (※)$_1$

where $l(M)$ is the *length* of M ([7],[12]). Moreover, for a noetherian ring A and a finite A-module M(i.e., M is an A-module which is finitely generated) the following conditions are equivalent ([10],[17]):

(i) M is an A-module of finite length

(ii) the ring $A/\mathrm{Ann}(M)$ is artinian (※※)$_1$

(iii) $\dim M=0$

Lemma 2.3. Let A be a noetherian ring.

(i) For an ideal $\mathfrak{a}=(a_1,\cdots,a_r)$, any minimal prime ideal $\mathscr{P}$ of $\mathfrak{a}$ has height$\leqq r$. In particular, $ht(\mathfrak{a})\leqq r$.

(ii) Let $\mathscr{P}$ be a prime ideal with height r. Then $\mathscr{P}$ is a minimal prime ideal of an ideal $\mathfrak{a}=(a_1,\cdots,a_r)$.

In this case $ht(\mathscr{P}/(a_1,\cdots,a_i))=r-i$ for $1\leqq i\leqq r$.

Proof. (i) The ideal $\mathscr{P}A_{\mathscr{P}}$ is the only prime ideal (maximal ideal) of $A_{\mathscr{P}}$ containing $\mathfrak{a}A_{\mathscr{P}}$, $ht(\mathscr{P})=ht(\mathscr{P}A_{\mathscr{P}})=ht(\mathfrak{a}A_{\mathscr{P}})=\dim A_{\mathscr{P}}$, and thus the dimension of

$$A_{\mathscr{P}}/\mathfrak{a}A_{\mathscr{P}}=A_{\mathscr{P}}/(a_1A_{\mathscr{P}}+\cdots+a_rA_{\mathscr{P}})$$

is zero. By (※※)$_1$ $A_{\mathscr{P}}/\mathfrak{a}A_{\mathscr{P}}$ is artinian. Therefore, by (※)$_1$

$$ht(\mathscr{P})=\dim(A_{\mathscr{P}})\leqq r.$$

Moreover $ht(\mathfrak{a})\leqq$r.

(ii) By our assumption $\dim(A_{\mathscr{P}})=ht(\mathscr{P})=r$. Since $A_{\mathscr{P}}$ is a noetherian local ring, by (※)$_1$ there exist r elements $a_1,\cdots,a_r$ in $\mathscr{P}A_{\mathscr{P}}$ and a positive integer n such that

$$(\mathscr{P}A_{\mathscr{P}})^n\subset(a_1,\cdots,a_r)A_{\mathscr{P}}$$

Since a_i is a product of an element in $\mathscr{P}$ and a unit in $A_{\mathscr{P}}$, without loss of generality we can assume that $a_i\in\mathscr{P}$ for $i=1,\cdots,r$. Then there is no prime ideal between $\mathscr{P}$ and $\mathfrak{a}=(a_1,\cdots,a_r)$. That is, $\mathscr{P}$ is a minimal prime ideal of $\mathfrak{a}$. We put

$$\bar{\mathscr{P}}=\mathscr{P}/(a_1,\cdots,a_i) \text{ and } \bar{A}=A/(a_1,\cdots,a_i)\ (1\leqq i\leqq r).$$

Consider the canonical projection

$$\pi:\ A\longrightarrow\bar{A}$$
$$\pi(a_j)=\!=\!=\bar{a}_j(i+1\leqq j\leqq r)$$

and $ht(\overline{\mathscr{P}})=s$. Then $\overline{\mathscr{P}}$ is a minimal prime ideal of $(\overline{a_{i+1}},\cdots,\overline{a_r})$, and thus $s\leqq r-i$ by (i). Since $ht(\overline{\mathscr{P}})=s$, there are s elements $\overline{b}_1,\cdots,\overline{b}_s$ such that $\overline{\mathscr{P}}$ is a minimal prime ideal of $(\overline{b}_1,\cdots,\overline{b}_s)$. Therefore, $\mathscr{P}$ is a minimal prime ideal of the ideal $(a_1,\cdots,a_i, b_1,\cdots,b_s)$. Hence $r\leqq i+s$. Therefore $s=r-i$, that is, $ht(\mathscr{P}/(a_1,\cdots,a_i))=r-i$ $(1\leqq i\leqq r)$. ///

Lemma 2.4. Let A be a ring and $\mathfrak{a},\mathscr{P}_1,\cdots,\mathscr{P}_r$ be ideals such that

(i) $\mathscr{P}_3,\cdots,\mathscr{P}_r$ are prime ideals

(ii) $\mathfrak{a}\subsetneqq\mathscr{P}_i$ for $1\leqq i\leqq r$.

Then there exists an element $x\in\mathfrak{a}$ such that $x\notin\mathscr{P}_i$ $(i=1,\cdots,r)$ ([10]).

Proof. We may assume that there are no inclusion relations between the $\mathscr{P}_i$'s.

We will prove the assertion by induction on r.

When $r=2$ we shall assume $\mathfrak{a}\subseteq\mathscr{P}_1\cup\mathscr{P}_2$. Let

$$x\in\mathfrak{a}-\mathscr{P}_2 \text{ and } y\in\mathfrak{a}-\mathscr{P}_1.$$

Since $x\in\mathscr{P}_1$, $x+y\notin\mathscr{P}_1$. We have $x+y\in\mathscr{P}_2$. Since $y\in\mathscr{P}_2$, $x\in\mathscr{P}_2$ and we have a contradiction. Therefore $\mathfrak{a}\subsetneqq\mathscr{P}_1\cup\mathscr{P}_2$ and thus there exists an element $x\in\mathfrak{a}$ such that $x\notin\mathscr{P}_1\cup\mathscr{P}_2$.

When $r>2$ we get $\mathfrak{a}\mathscr{P}_1\cdots\mathscr{P}_{r-1}\subsetneqq\mathscr{P}_r$, and thus we take $x\in\mathfrak{a}\mathscr{P}_1\cdots\mathscr{P}_{r-1}-\mathscr{P}_r$. By induction hypothesis

$$S=\mathfrak{a}-(\mathscr{P}_1\cup\cdots\cup\mathscr{P}_{r-1})$$

is not empty. If $\mathfrak{a}\subseteq\mathscr{P}_1\cup\cdots\cup\mathscr{P}_r$ then $S\subseteq\mathscr{P}_r$. For $y\in S$, $x+y\in S$ and thus y and $x+y$ are in $S\subseteq\mathscr{P}_r$. This implies $x\in\mathscr{P}_r$.

We get a contradiction. ///

Proposition 2.5. A regular local ring is an integral domain([10],[12] and [17]).

Proof. Let $(A,\boldsymbol{m})$ be a regular local ring with $\dim A=\boldsymbol{n}$. We shall prove our proposition by induction on $\boldsymbol{n}$. When $\boldsymbol{n}=0$. $\boldsymbol{m}=(0)$ implies that A is a field.

When $\boldsymbol{n}=1$. Since A is a regular local ring and $\dim A=1$ we have $\boldsymbol{m}=xA$. Since $ht(\boldsymbol{m})=\dim A=1$, we can assume that there exists an prime ideal $\mathscr{P}$ of A such that $\boldsymbol{m}\subsetneqq\mathscr{P}$. Take $y\in\mathscr{P}$. Then there exists an element $a\in A$ such that $y=ax$.

Since $x \notin \mathscr{P}$ and $y=ax \in \mathscr{P}$, we have $a \in \mathscr{P}$. That is, $\mathscr{P}=x\mathscr{P}$ where $x \in \boldsymbol{m}$. By Nakayama Lemma $\mathscr{P}=(0)$. Hence A is an integral domain.

When $n>1$. Take minimal prime ideal $\mathscr{P}_1, \cdots, \mathscr{P}_r$. Then

$$\boldsymbol{m} \subsetneqq \boldsymbol{m}^2 \text{ and } \boldsymbol{m} \subsetneqq \mathscr{P}_i \text{ for } i=1, \cdots, r.$$

Therefore, by Lemma 2.4, there exists an element $x \in \boldsymbol{m}-(\boldsymbol{m}^2 \cup \mathscr{P}_1 \cup \cdots \cup \mathscr{P}_r)$. Therefore, we have a regular system of parameters $(x_1, x_2, \cdots, x_n)$ where $\dim A = \text{rank}_{A/m}(\boldsymbol{m}/\boldsymbol{m}^2)=n$. By a property of a regular local ring, A/xA is a regular local ring with its dimension $n-1$. Thus, by our induction assumption, A/xA is an integral domain. Thus xA is a prime ideal of A.

We assume there is a minimal prime ideal $\mathscr{P}_1$ of A such that $\mathscr{P}_1 \subsetneqq xA$. Take $y \in \mathscr{P}_1$. Then there is an element $a \in A$ such that $y=xa$. Since $\mathscr{P}_1$ is prime and $x \notin \mathscr{P}_1$ we have $a \in \mathscr{P}_1$. That is, $\mathscr{P}_1 = x\mathscr{P}_1$. By Nakayama Lemma $\mathscr{P}_1=\{0\}$. Thus A is an integral domain. ///

Theorem 2.6. Let A be a noetherian local ring, and let $\mathscr{P}=(a_1, \cdots, a_n)$ be a prime ideal with $ht(\mathscr{P})=n$. Then the following hold:

(i) $a_1, \cdots, a_n$ is an A-sequence

(ii) $(a_1, \cdots, a_i)$ $(1 \leqq i \leqq r)$ is a prime ideal with height i.

(iii) A is an integral domain.

Proof. We shall prove our theorem by induction on n.

(1) When $n=1$. Then $\mathscr{P}=(a_1)$ is prime and $ht(\mathscr{P})=1$. By the same method as the proof of Proposition 2.5, we can easily prove that A is an integral domain.

(2) When $n=2$: Assume that $\mathscr{P}=(a_1, a_2)$ and $ht(\mathscr{P})=2$. By (ii) of Lemma 2.3 $ht(\mathscr{P}/(a_1))=1$. Since $A/(a_1)$ is a noetherian local ring and $\mathscr{P}/(a_1)$ is a prime ideal of $A/(a_1)$ which is generated by $\bar{a}_2$, where

$$A \longrightarrow A/(a_1) \quad (a_2 \longrightarrow \bar{a}_2).$$

Thus $A/(a_1)$ is an integral domain by our induction hypothesis. This implies that $(a_1)=a_1A$ is a prime ideal. By (i) of Lemma 2.3 $ht(a_1A) \leqq 1$.

(a) When $ht(a_1A)=1$. By induction hypothesis we see that a_1 is a regular A-element and A is an integral domain. Thus, in this case our assertion holds.

(b) $ht(a_1A)=0$. Since $\mathscr{P}/a_1A$ is a prime ideal with height 1 and A/a_1A is an integral domain, we have a decomposition of $(a_1, a_2)=\mathscr{P}$ such that

$$\mathscr{P}=\mathscr{P}_0=(a_1, a_2)\supsetneqq\mathscr{P}_1=(a_1A).$$

Moreover, there is no prime ideal between $\mathscr{P}$ and $a_1A=\mathscr{P}_1$. Thus $ht(\mathscr{P})=1$, which is a contradiction. Thus $ht(a_1A)\neq 0$.

(3) When $n>2$. By the same reason as above we see that

(a) $\mathscr{P}/(a_1)=(\bar{a}_2, \cdots, \bar{a}_n)$ is a prime ideal with height $n-1$. Therefore $A/(a_1)$ is an integral domain.

(b) By the above reason we can prove that $ht(a_1A)=1$ and a_1A is a prime ideal. Thus, consequently our assertions hold. ///

3. Unique Factorization Domains

We shall describe some properties of unique factorization domains which are needed in the next section ([16]).

Lemma 3.1. Let A be a noetherian domain. Then A is a unique factorization domain if and only if every prime ideal with height 1 is a principal ideal.

Proof. Let A be a unique factorization domain, and let $\mathscr{P}$ be a prime ideal with $ht(\mathscr{P})=1$. Take an element $x\in\mathscr{P}(x\neq 0)$, then we have a factorization $x=a_1\cdots a_n$ of x by prime elements. Since $x\in\mathscr{P}$ there exists at least one element a_i such that $a_i\in\mathscr{P}$ $(1\leqq i\leqq n)$. Suppose a_iA is a prime ideal. Then we have

$$\mathscr{P}\supset a_iA\supset(0).$$

Since $ht(\mathscr{P})=1$ we have $\mathscr{P}=a_iA$, that is, $\mathscr{P}$ is a principal ideal.

Conversely, we take an element x which is not a unit and zero. We suppose that

$$x=a_1a_2\cdots\cdots,$$

where a_i is an irreducible elements. Then we have

$$a_1a_2\cdots\cdots A\subsetneqq a_2a_3\cdots\cdots A\subsetneqq a_3a_4\cdots\cdots A\subsetneqq\cdots\cdots$$

Since A is a noetherian domain, there exists a positive integer n such that $x=a_1\cdots a_n$.

Therefore, it suffices to prove that every irreducible element is a prime element.

Take an irreducible element $a \in A$, and suppose $(a) \subset A$. Then a minimal prime ideal $\mathscr{P}$ of (a) has its height 1, because $ht\mathscr{P} \leqq 1$ ([2], [3] and [10]) and A is an integral domain. Thus by our hypothesis there exists a prime element b such that $\mathscr{P} = bA$. Since $(a) \subset \mathscr{P} = bA$ there exists an element $c \in A$ such that $bc = a$. Since a is an iredducible element, c is a unit. Hence we have $aA = bA$ and thus $a = b$. //

Proposition 3.2. Let A be a noetherian domain, and let Ω be a set consisting of prime elements in A. Let S be the multiplicative closed set generated by Ω. Then

$$A_S \text{ is a UFD} \Longrightarrow A \text{ is a UFD}.$$

Proof. We take a prime ideal $\mathscr{P}$ with height 1. If we can prove that $\mathscr{P}$ is a principal ideal, then A is a unique factorization domain by Lemma 3.1.

At first, we assume $\mathscr{P} \cap S \neq \phi$. Since $\mathscr{P}$ is a prime ideal there exists a prime element $q \in \Omega$ such that $\mathscr{P} = qA$. Therefore, by Lemma 3.1, A is a unique factorization domain.

Next, assume $\mathscr{P} \cap S = \phi$. Then $\mathscr{P}A_S$ is a prime ideal with $ht(\mathscr{P}A_S) = 1$ in A_S. Since A_S is a unique factorization domain, there exists an element $a \in \mathscr{P}$ such that $\mathscr{P}A_S = aA_S$ by Lemma 3.1. Note that a is a prime element in A_S but not true in A. That is, for any $q \in \Omega \subset S$, we have $aqA_S = aA_S$. Therefore there is not an element b in Ω such that $b|a$. Take $x \in \mathscr{P}$, then we have elements $s \in S$ and $y \in A$ such that $sx = ay$. Since $q_1 \cdots q_r = s$ for $q_i \in \Omega (i = 1, \cdots, r)$ and $\mathscr{P} \cap S = \phi$, by definition of a it follows that $a \notin q_i A$ for any $i (1 \leqq i \leqq r)$. From

$$sx = q_1 \cdots q_r x = ay$$

it follows that there exists $i (1 \leqq i \leqq r)$ such that $y \in q_i A$.
If $q_1 = s$ then $y \in sA$ (note that $1 \in A$). When $s = q_i \cdots q_{r-1}$ we assume that $y \in q_i A$ $(1 \leqq i \leqq r-1)$ implies $y \in sA$. For $s = q_1 \cdots q_r$ we assume that $y \in q_i A$ $(1 \leqq i \leqq r)$. Then we can put $y = q_i y' \in q_i A$. From

$$sx = q_1 \cdots q_i \cdots q_r x = aq_i y',$$

we get $q_1 \cdots q_{i-1} q_{i+1} \cdots q_r x = ay'$. Since $a \notin q_i A (1 \leqq i \leqq r-1)$ by the above reason we have $y' \in q_i A$ for $1 \leqq i \leqq r-1$. By induction hypothesis $y' \in q_1 \cdots q_{i-1} q_{i+1} \cdots q_r x$, and hence

$$y = q_i y' \in q_1 \cdots q_r A = sA.$$

We put $y = sb \in sA$. Then from

$$sx = asb \Longrightarrow s(x - ab) = 0$$

and that A is an integral domain it follows that $x = ab \in aA$. Therefore $\mathscr{P} = aA$ and A is a unique factorization domain by Lemma 3.1⫽

Lemma 3.3. Let A be an integral domain and let $\mathfrak{a}$ be an ideal such that

$$A^{n+1} = A^n \oplus \mathfrak{a}.$$

Then $\mathfrak{a}$ is a principal ideal.

Proof. Let $\{\xi_1, \cdots, \xi_{n+1}\}$ be a set of basis of A^{n+1}, and let $\{\eta_2, \cdots, \eta_{n+1}\}$ be a set of basis of A^n. Since $A^{n+1} \cong \mathfrak{a} \oplus A^n \subset A \oplus A^n$ for a base η_1 of A we have an isomorphism

$$\varphi:\ A^{n+1} \longrightarrow \mathfrak{a} \oplus A^n$$

such that
$$\varphi(\xi_i) = \varphi(\xi_i) = \sum_{j=1}^{n+1} a_{ij} \eta_j$$

where $a_{ij} \in A$. It is clear that $D = \det(a_{ij}) \neq 0$. Let D_i be the cofactor of a_{i1}. Then the following hold:

(i) $D = \sum_{i=1}^{n+1} a_{i1} D_i$

(ii) $0 = \sum_{i=1}^{n+1} a_{ij} D_i$ if $j \neq 1$.

Put $\xi_1' = \sum_{i=1}^{n+1} D_i \xi_i$. Then we have the following

$$\begin{aligned}\varphi(\xi_1') &= \sum_{i=1}^{n+1} D_i \varphi(\xi_i) \\ &= \sum_{i=1}^{n+1} D_i \sum_{j=1}^{n+1} a_{ij} \eta_j \\ &= \sum_{j=1}^{n+1} \left(\sum_{i=1}^{n+1} a_{ij} D_i\right) \eta_j = D\eta_1\end{aligned}$$

by (i) and (ii).

Since φ is an isomorphism and $\eta_2, \cdots, \eta_{n+1} \in A^n$ there are $\xi_2', \cdots, \xi'_{n+1} \in A^{n+1}$ such that

$$\varphi(\xi_2') = \eta_2, \cdots, \varphi(\xi'_{n+1}) = \eta_{n+1}.$$

We shall put

$$\xi_j' = \sum_{k=1}^{n+1} b_{jk}\xi_k \quad (2 \leqq j \leqq n+1, \; b_{1k} = D_k)$$

then the following holds

$$(b_{jk})(a_{ij}) = \begin{pmatrix} b_{11} \cdots\cdots b_{1n+1} \\ \vdots \\ b_{n+11} \cdots b_{n+1n+1} \end{pmatrix} \begin{pmatrix} a_{11} \cdots\cdots a_{1n+1} \\ \vdots \\ a_{n+11} \cdots a_{n+1n+1} \end{pmatrix} = \begin{pmatrix} D & 0 \cdots 0 \\ 0 & 1 \cdots 0 \\ \vdots & \ddots \; \vdots \\ 0 & 0 \cdots 1 \end{pmatrix}$$

because that

(a) $\sum_{k=1}^{n+1} a_{k1} D_k = \sum_{k=1}^{n+1} b_{ik} a_{k1} = D,$

(b) $\sum_{k=1}^{n+1} b_{1k} a_{kj} = \sum_{k=1}^{n+1} a_{kj} D_k = 0 \quad (j \neq 1)$, and

(c) $\eta_j = \varphi(\xi_j') = \sum_{k=1}^{n+1} b_{jk} \varphi(\xi_k) = \sum_{k=1}^{n+1} b_{jk} (\sum_{l=1}^{n+1} a_{kl} \eta_l)$ and $b_{1k} = D_k$

imply

$$\sum_{k=1}^{n+1} b_{jk} a_{jk} = 1 (j \neq 1) \text{ and } \sum_{k=1}^{n+1} b_{jk} a_{lk} = 0 (j \neq l).$$

Therefore, from

$$\det(b_{jk}) \cdot \det(a_{ij}) = D \neq 0$$

it follows that $\det(b_{jk}) = 1$. This means that $\{\xi_1', \cdots, \xi'_{n+1}\}$ is a set of basis of A^{n+1}.

Since $A^{n+1} \cong \mathfrak{a} \oplus A^n$ and

$$\varphi(A\xi_2' \oplus \cdots \oplus A\xi'_{n+1}) = A\eta_2 \oplus \cdots \oplus A\eta_{n+1} = A^n.$$

we have

$$\varphi(A\xi_1') \cong \mathfrak{a} = \mathfrak{a}A = \mathfrak{a}A\eta_1 = \mathfrak{a}\eta_1.$$

Hence

$$\mathfrak{a}\eta_1=\varphi(A\xi_1')=DA\eta_1$$

and $\mathfrak{a}=DA$ is a principal ideal. ///

Lemma 3.4. Let A be a unique factorization domain.

Then for $a_i\in A$, $\bigcap_i a_iA$ is a principal ideal.

Proof. Let us put

$$a_i=\prod_\alpha p_\alpha^{t(i,\alpha)}$$

where p_α is a prime element such that $\alpha\neq\beta$ implies that $p_\alpha A\neq p_\beta A$. We put

$$\alpha=\prod p_\alpha^{max(t(i,\alpha))}$$

then it is clear that $\bigcap_i a_iA=\alpha A$. ///

Theorem 3.5. Let A be a unique factorization domain and let $q_1,\cdots q_r$ be primary ideals with $ht(q_i)=1$ for $i=1,\cdots r$. Then $q_1\cap\cdots\cap q_r$ is a pincipal ideal.

Proof. Let $r(q_i)$ be the radical of q_i for $i=1,\cdots,r$ Then $r(q_i)$ is the smallest prime ideal containing q_i. Thus we have $ht(r(q_i))=1$ for $i=1,\cdots,r$. Since A is a unique factorization domain and $ht(r(q_i))=1$, by Lemma 3.1 each $r(q_i)$ is a principal ideal, that is, we can put

$r(q_i)=x_iA$, where x_i is a prime element.

Therefore, it follows that

$$r(\bigcap_{i=1}^r q_i)=\bigcap_{i=1}^r r(q_i)=x_1\cdots x_rA.$$

For each $i(1\leqq i\leqq r)$ $(r(q_i)=x_iA$ we have two cases as follows.

(i) $x_i\in q_i$ and (ii) $x_i\notin q_i$.

In case (i), $r(q_i)=q_i=x_iA$. In case (ii), there exists the least small positive integer n_i such $x_i^{n_i}\in q_i$ and $x_i^m\notin q_i$ if $0\leqq m\leqq n_i$. Then we have $x_i^nA=q_i$. In fact, for $y\notin x_iA$ $x_i^my\notin q_i$ $(m<n_i)$ because that if $x_i^my\in q_i$ then $x_i^m\notin q_i$ implies that these exists

an positive integer $n>0$ such that $y^n \in q_i$. This is impossible because that x_i is a prime element. Hence we have

$$\bigcap_{i=1}^{r} q_i = x_i^{n_1} \cdots x_r^{n_r} A. \quad /\!\!/$$

4. Main Theorems

Let A be a ring and let M be an A-module. If there is a projective resolution over M

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where $F_i (i=0, 1, \cdots, n)$ is a finitely generated free A-module, then we say that M has a *finite free resolution*. If there exist two finitely generated free A-modules F and F' such that

$$M \oplus F \cong F'$$

then M is said to be *stably free*.

Theorem 4.1. (i) Let A be a ring and let M be an A-module. If M is stably free, then M has a finite free resolution.

Conversely, if M is a finitely generated projetive A-module which has a finite free resolution then M is stably free.

(ii) Let A be noetherian. If every finitely generated projective A-module is stably free, then every finitely generated A-module M with projective dimension $<\infty$ has a finite free resolution.

(iii) Let A be noetherian. If every finitely generated A-module has a finite free resolution, then A is a regular ring, i.e., for every prime ideal $\mathscr{P}$ of $A, A_{\mathfrak{p}}$ is a regular local ring.

Proof. (i) Since M is stably free there exist two finitely generated free A-modules F and F' such that $M \oplus F' \cong F$. Thus

$$0 \longrightarrow F' \longrightarrow F \longrightarrow M \longrightarrow 0$$

is a finite free resolution over M.

Conversely, let M be a finitely generated projective A-module and let

$$0 \longrightarrow F_n \xrightarrow{d_n} \cdots\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0 \quad (※)_1$$

be a finite free resolution over M, where F_0 is a finitely generated free A-module. We shall put

$$d_1(F_1)=M_0, \cdots, d_n(F_n)=M_{n-1},$$

then the following hold:

(1) M is projective $\Longrightarrow F_0 \cong M \oplus M_0 \Longrightarrow M_0$ is projective

(2) $0 \longrightarrow d_2(F_2)=M_1 \longrightarrow F_1 \longrightarrow d_1(F_1)=M_0 \longrightarrow 0$

is exact and M_0 is projective $\longrightarrow d_2(F_2)=M_1$ is projective

(3) Similarly $d_{n-1}(F_{n-1})=M_{n-2}$ is projective and $F_n \oplus M_{n-2} \cong F_{n-1} \Longrightarrow M_{n-2}$ is stably free

(4) $M_{n-2} \oplus M_{n-3} \cong F_{n-2} \Longrightarrow F_n \oplus M_{n-2} \oplus M_{n-3} \cong F_n \oplus F_{n-2} \Longrightarrow F_{n-1} \oplus M_{n-3} \cong F_n \oplus F_{n-2}$ $\Longrightarrow M_{n-3}$ is stably free

(5) Similarly, we get that M is stably free.

(ii) Let M be a finitely generated A-module. Then there exists a finitely generated free A-module F_0 and an epimorphism d_0: $F_0 \longrightarrow M$. Since A is a noetherian ring and F_0 is a noetherian A-module, $\mathrm{Ker}(d_0: F_0 \longrightarrow M)$ (a submodule of F_0) is a finitely generated A-module. Since projdim(M) (projective dimension of M) is finite by our assumption, by repeating the above auguments we get a projective resolution over M

$$0 \longrightarrow P_n \longrightarrow F_{n-1} \longrightarrow \cdots\cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0 \quad (※)_2$$

such that

(1) $F_i\,(0 \leqq i \leqq n-1)$ is a finitely generated free A-module, and

(2) P_n is a finitely projective A-module ([1], [6]).

By our hypothesis, P_n is stably free and thus there exist finitely generated free A-modules F' and F_n such that

$$P_n \oplus F' \cong F_n.$$

We can rewrite $(※)_2$ above as follows.

$$0 \longrightarrow F_n \xrightarrow{d_n'} F_{n-1} \oplus F' \xrightarrow{d_{n-1}'} F_{n-2} \xrightarrow{d_{n-2}} F_{n-3} \longrightarrow \cdots\cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0 \quad (※※)_2$$

where

$$d_n'|P_n=d_n,\ d_n'|F'=1_{F'},\ d_n'_{-1}|F_{n-1}=d_{n-1},\ d_n'_{-1}|F'=0$$

Then the exact sequence (※※)$_2$ is a finite free resolution over M. //

(iii) Since A is a noetherian ring, for each prime ideal $\mathscr{P}$ of A, A_p is a noetherian local ring.

Recall that for a noetherian local ring A.

A is a regular local ring$\Longleftrightarrow$gl. dim $A=\dim A\Longleftrightarrow$gl. $\dim A<\infty$ (※※※)$_2$

where gl. $\dim A$ is the global dimension of A ([10],[14] and [15]). Then, it suffices to prove proj. $\dim(A_{\mathscr{P}}/\mathscr{P}A_{\mathscr{P}})=$gl. $\dim A_{\mathscr{P}}<\infty$.

Note that

$$A_{\mathscr{P}}/\mathscr{P}A_{\mathscr{P}}=A/\mathscr{P}\otimes_A A_{\mathscr{P}}$$

and $A/\mathscr{P}$ is a finitely generated A-module with generator $1+\mathscr{P}$. Thus by our hypothesis there exists a finite free resolution over $A/\mathscr{P}$ such that

$$0\longrightarrow F_n\longrightarrow F_{n-1}\longrightarrow\cdots\cdots\longrightarrow F_0\longrightarrow A/\mathscr{P}\longrightarrow 0$$

where $F_i(0\leqq i\leqq n)$ is a finitely generated free A-module.

Hence

$$0\longrightarrow F_n\otimes_A A_{\mathscr{P}}\longrightarrow\cdots\cdots\longrightarrow F_0\otimes_A A_{\mathscr{P}}\longrightarrow A/\mathscr{P}\otimes_A A_{\mathscr{P}}=A_{\mathscr{P}}/\mathscr{P}A_{\mathscr{P}}\longrightarrow 0$$

is a free resolution over $A_{\mathscr{P}}$-module $A_{\mathscr{P}}/\mathscr{P}A_{\mathscr{P}}$. This implies that

$$\text{gl. }\dim A_{\mathscr{P}}=\text{proj. }\dim(A_{\mathscr{P}}/\mathscr{P}A_{\mathscr{P}})<\infty.$$

By (※※※)$_2$ above, $A_{\mathscr{P}}$ is a regular local ring. //

Let A be a ring. For an A-module M is said to be an A-module of *finite presentation* if there are finitely generated A-modules A^m and A^n such that

$$A^m\longrightarrow A^n\longrightarrow M\longrightarrow 0$$

is an exact sequence.

Lemma 4.2. Let A be a ring, and let M be an A-module of finite presentation. M is a projective A-module if and only if M_m is a free A_m-module for all maximal ideals m.

Proof. Recall that

"for a local ring (A, m) every projective A-module is a free A-module" $(※※※)_3$ ([10]).

Assume M is a projective A-module. Then M is a direct summand of a free A-module because there exists an epimorphism $A^n \longrightarrow M$. Take a maximal ideal m of A. Then A_m is a local ring and M_m is a projective A_m-module. By $(※※※)_3$ above M_m is a free A_m-module.

Conversely, take an exact sequence of A-modules

$$N_1 \longrightarrow N_2 \longrightarrow 0$$

and we have to prove that

$$\mathrm{Hom}_A(M, N_1) \longrightarrow \mathrm{Hom}_A(M, N_2)$$

is an epimorphism, because then M is a projective A-module. We put

$$C = \mathrm{Hom}_A(M, N_2) / \mathrm{Im}\ \mathrm{Hom}_A(M, N_1)$$

and recall that for a ring A, an A-module M and $x \in M$ if for every maximal ideals m of A, $x=0$ in M_m then $x=0$ $(※※※※)_3$. We assume that for every maximal ideals m of A, M_m is a free A_m-module. Then

$$\begin{aligned} C_m &= \mathrm{Coker}(\mathrm{Hom}_{A_m}(M_m, (N_1)_m) \longrightarrow \mathrm{Hom}_{A_m}(N_m, (N_2)_m)) \\ &= \mathrm{Coker}(((N_1)_m)n \longrightarrow ((N_2)_m)n) \\ &= 0 \end{aligned}$$

because $(N_1)m \longrightarrow (N_2)m \longrightarrow 0$ is exact. Therefore by $(※※※※)_3$ above $C=0$ and thus

$$\mathrm{Hom}_A(M, N_1) \longrightarrow \mathrm{Hom}_A(M, N_2)$$

is surjective. ///

We want to prove the following Proposition 4.3 by using Lemma 3.1, Proposition 3.2, Lemma 3.3, Theorem 4.1 and Lemma 4.2.

Proposition 4.3. A regular local ring is a unique factorization domain.

Proof. Let(A, m) be a regular local ring. We shall prove our assertion by using induction on $\dim A$.

(i) When $\dim A=0$: Then $m=(0)$ and A is a field. Thus A is a unique factorization domain.

(ii) When $\dim A=1$: Since A is a regular local ring, $\dim A=1$ implies that there exists a regular element $a\in A$ such that $m=aA$.

Thus A is a principal ideal domain. By Lemma 3.1 A is a unique factorization domain.

(iii) When $\dim A>1$. Since(A, m) is a regular local ring for each $x\in m-m^2$ there exists a regular system of parameters $\{x, x_2, \cdots, x_n\}$, where $n=\dim A$([10]). Moreover xA is a prime ideal of A([10],[17]). Put $\Omega=\{x\}$ in Proposition 3.2. For $S=\{x, x^2, x^3, \cdots\}$ we shall put $A_S=A_x$. By proposition 3.2 it suffices to prove that A_x is a unique factorization domain.

Take a prime ideal $\mathfrak{P}$ of A_x with $ht(\mathfrak{P})=1$. Put

$$\mathscr{P}=\mathfrak{P}\cap A$$

then $\mathfrak{P}=\mathscr{P}A_x$.

Note that A_x is a noetherian local ring. By Lemma 2.3 $\mathfrak{P}$ is finitely generated and thus $\mathscr{P}$ is a finitely generated A-module. By (※※※)$_2$ above, proj. dim $\mathscr{P}$ is finite. Thus $\mathscr{P}$ has a finite free resolution, and an A_x-module $\mathfrak{P}$ has a finite free resolution. Take a prime ideal Q of A_x. Then $m\supsetneqq Q\cap A$. Thus $ht(m)\supsetneqq ht(Q)$. Since

$$(A_x)_Q=A_{Q\cap A}$$

and $\dim(A_{Q\cap A})=ht(Q\cap A)<ht(m)=\dim A$,

by our induction hypothesis $A_{Q\cap A}$ is a unique factorization domain. Note that in a local ring R every finitely generated R-module N is a projective if and only if N is R-free. Thus $\mathfrak{P}_Q$ is a free $(A_x)_Q$-module. We want to prove that $\mathfrak{P}$ is a principal ideal because that if this is done then A_x is a unique factorization ring by Lemma 3.1. By Lemma 4.2 $\mathfrak{P}$ is a projective A_x-module which has a finite free resolution. By (i) of Theorem 4.1 $\mathfrak{P}$ is stably free and thus $\mathfrak{P}$ is a principal ideal by Lemma 3.3.//

References

1. H. cartan and S. Eilenberg: Homological Algebra, Princeton Univ. 1973.
2. J. Eagon and M. Hochster: R-sequences and indeterminates, Quart. *J. Math. Oxford* **25** 1974, 61~71.
3. J. Dieudonne: On regular sequences, *Nagoya Math. J.* **27-1**, 1966, 355~356.
4. M. Hochster: Topics in the Homological theory of Modules over commutative rings, number 24 Conference Board of the Math. Sciences, A.M.S. 1975.
5. M. Hochster and J.L. Roberto: Action of reductive groups on regular rings and Cohen-Macaulay rings, *Bull Amer. Math. Soc.* **80**, 1974, 281~284.
6. I. Kaplansky: Projective modules, *Ann. of Math.* **68** 1958, 372~377.
7. I. Kaplansky: Commutative Rings, The Univ. of Chicago Press 1974.
8. E. Kunz: Characterization of regular local rings of characteristic p, *Amer. J.* **91** 1969, 772~784.
9. S. Lichtenbaum: On the vanishing of Tor in regular local rings, *Illinois J. Math.* **10**, 1966, 220~226, MR 32 # 4130.
10. H. Matsumura: Commutative Algebra, The Benjamin/Cummings Publishing Company, INC. 1980.
11. S. McAdam: Saturated chains of primes in Noetherian rings, Indiana *J. Math.* **23**, 1974, 719~728.
12. M. Nagata: Local Rings, Interscience Publishers 1962.
13. W. Vasconcelos: Ideals generated by R-sequences, *J. of Alg.* **6**, 1967, 309~316.
14. J. Nishimura: Note on Krull dimension, *J. Math. Kyoto Univ.* **15**, 1975, 397~400.
15. C. Peskine and L. Szpiro: Dimension projective finie et cohomologie locale, Inst. Haute Etudes Sci. *Publ. Math. no.* **42**, Paris 1973, 323~395.
16. P. Samuel: On unique factorization domains, Ill. J.5 1961, 1~17.
17. P. Samuel and O. Zariski: *Commutative Algebra*, Vol. I, II. Van Nostrand 1960.