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Some Remarks of the Depthes and Flatness

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1. Introduction

The center of the theory of commutative algebras is, the auther maintain, the investigation of noetherian ring's properties([14]).

The conception of localization had been formed by W. Krull, the great mathematician during 1920 to 1930 period, and noetherian ring have been studied more exquisitely by the localization([4], [5], [8]).

About 1960, new conceptions of regular sequences and depthes were formed by admitting the technique of homological algebras into the theory of commutative algebras. Accordingly, many mathematicians have concentrated on the research of these conceptions ([2], [3], [6], [15]).

In the first half of 1960, the conceptions of flatness were regarded as of great importance in commutative algebras and therefore it was attempted to study on the theory([1], [12], [13], [16]).

This dissertation was made by assembling the problems which were found newly in theorem 3.1 and 4.5, and some problems which were proved during the seminar of commutative algebras under my academic advicing professors, Ki-an Lee and Kwang-ho Soh.

The contents of this paper is as follows;

In section 2, we deal with the explanation of the conceptions that will be used in section 4 chiefly, for example, the definitions of the $\mathfrak{a}$-adic topology and the Koszul complex and proofs of lemma 2.1, 2.2, 2.3 which will be used in section 4.

In section 3, four properties of depthes are found and proved, one of them which is found newly is proved in theorem 3.1 and its contents is as follows;

For a noetherian ring A and a perfect ideal $\mathfrak{a}$ of A with grade$(A/\mathfrak{a})=n$,

(i) Proj.dim$(\mathfrak{a})=n-1$, $\mathrm{Ext}_A^n(A/\mathfrak{a}, A)\cong A$.

(ii) ${}^{\forall}P \in \mathrm{Ass}(A/\mathfrak{a})$, $\mathrm{grade}(P) = n = \mathrm{grade}(P/\mathfrak{a})$.

The section 4 deals with the proofs of four properties of flatness, one of them that is found newly is proved in theorem 4.5 and its contents is as follows;

For a noetherian local ring homomorphism

φ: $(A, \mathfrak{M}) \longrightarrow (B, \mathfrak{N})$ with a regular ring A and $\dim(A) = \mathrm{ht}(\mathfrak{M}B)$,

we have the following;

(i) B is regular and the image of a regular system of parameters of A under φ is the subsequence of a regular system of parameters of B if and only if $B/\mathfrak{M}B$ is regular and $\dim(B) = \dim(A) + \dim(B/\mathfrak{M}B)$.

(ii) If conditions in (i) hold, then B is a flat A-module.

2. Preliminaries

Throughout this paper by a *ring* we mean a commutative ring with 1 without any statements.

Let A be a ring, $\mathfrak{a}$ an ideal of A and M an A-module. We shall put $\mathfrak{a}^n M = M_n$ for $n = 0, 1, \cdots\cdots$, where $\mathfrak{a}^0 M = M$. Then we have

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots\cdots \supset M_n \supset \cdots\cdots$$

We define a topology on M by taking $\{x + M_n \mid n = 0, 1, \cdots\cdots\}$ as a fundamental system of neighborhoods of x for each $x \in M$. This topology is called the $\mathfrak{a}$*-adic topology* of M or the *linear topology with respect to* $\mathfrak{a}$.

If $\bigcap_{n=0}^{\infty} M_n = 0$, then the $\mathfrak{a}$-adic topology is Hausdorff and it is said to be *separated*. In the $\mathfrak{a}$-adic topology of M, each M_n is open and closed. Since $M_m \supset M_n$ for each $m < n$, we have the natural A-linear surjection

$$\varphi_{m,n}:\ M/M_n \longrightarrow M/M_m$$

and thus it is obvious that $\{M/M_n, \varphi_{m,n}\}$ is an inverse limit system.

We shall put $\hat{M} = \varprojlim M/M_n$, which is called the *completion* of M. If $M \cong \hat{M}$, then M is said to be *complete*. The topology of $\hat{M}$ is defined as follows.

For each $n = 0, 1, \cdots\cdots$, we give the quotient topology on M/M_n and the product topology on $\prod_{n=0}^{\infty} M/M_n$. Since $\hat{M}$ is a subset of $\prod_{n=0}^{\infty} M/M_n$, the topology of $\hat{M}$ is the relative topology in $\prod_{n=0}^{\infty} M/M_n$.

In the above situation, we have two natural maps:

$$\begin{array}{lll} \varphi: M & \longrightarrow & \hat{M}, \\ \quad \Cup & & \Cup \\ \quad a & \longmapsto & (\cdots, a+M_n \cdots) \end{array} \qquad \begin{array}{lll} \rho_n: \hat{M} & \longrightarrow & M/M_n \\ \quad \Cup & & \Cup \\ (\cdots, a_n, \cdots) & \longmapsto & a_n. \end{array}$$

It is well-known that i) $\varphi(M)$ is dense in $\hat{M}$ and ii) if we put $M_n^* = \mathrm{Ker}\ \rho_n$, then the topology of $\hat{M}$ is the linear topology generated by $\{M_n^* \mid n=0, 1, \cdots\cdots\}$ ([9], [10]).

Let $\{L_n\}$, $\{M_n\}$ and $\{N_n\}$ be three inverse limit systems of A-modules where A is a ring. If the diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{n+1} & \longrightarrow & M_{n+1} & \longrightarrow & N_{n+1} & \longrightarrow & 0 \\ & & \varphi_{n,n+1} \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_n & \longrightarrow & M_n & \longrightarrow & N_n & \longrightarrow & 0 \end{array}$$

is commutative then we shall say that we have an exact sequence of inverse limit systems. In this case, if all $\varphi_{n,n+1}$ are surjective, then

$$0 \longrightarrow \varprojlim L_n \longrightarrow \varprojlim M_n \longrightarrow \varprojlim N_n \longrightarrow 0 \quad (※)_1 \text{ is exact.}$$

Let A be a ring, $\mathfrak{a}$ an ideal of A, M an A-module and N a submodule of M. Then we can prove the following by using $(※)_1$.

$$0 \longrightarrow \hat{N} \longrightarrow \hat{M} \longrightarrow (M/N)\hat{} \cong \hat{M}/\hat{N} \longrightarrow 0 \qquad (※)_2$$

is exact where ^ means the completion of the $\mathfrak{a}$-adic topology. For a noetherian ring A, we have the Artin-Rees Lemma as follows.

Property 1°. Let A be a noetherian ring, $\mathfrak{a}$ an ideal of A and M a finitely generated A-module. Then there exists a positive integer c such that $\mathfrak{a}^n M \cap N = \mathfrak{a}^{n-c}(\mathfrak{a}^c M \cap N)$ for all $n > c$ where N is a submodule of M.

With the above notations, since $\mathfrak{a}^{n-c}(\mathfrak{a}^c M \cap N) \subseteq \mathfrak{a}^{n-c} N$ for all $n > c$, we have $\mathfrak{a}^n N \subseteq \mathfrak{a}^n M \cap N \subseteq \mathfrak{a}^{n-c} N$.

This implies that the linear topology of N generated by $\{\mathfrak{a}^n M \cap N \mid n=1, 2, \cdots\cdots\}$ coincides with the linear topology of N generated by $\{\mathfrak{a}^n N \mid n=1, 2, \cdots\}$.

Property 2° ([10]) Let A be a noetherian ring, $\mathfrak{a}$ an ideal of A and M a finitely generated A-module. Then $M\otimes_A\hat{A}\cong\hat{M}$, where $\hat{M}=\varprojlim M/\mathfrak{a}^nM$ and $\hat{A}=\varprojlim A/\mathfrak{a}^n$.

Lemma 2.1. Let A be a noetherian ring and $\mathfrak{a}$ an ideal of A and $\hat{A}$ the completion of A with the $\mathfrak{a}$-adic topology. Then

(i) $\hat{A}$ is a flat A-module.

(ii) $\widehat{(\mathfrak{a}^n)}=(\hat{\mathfrak{a}})^n$.

(iii) $\mathfrak{a}^n/\mathfrak{a}^{n+1}\cong\hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1}$.

(iv) $\hat{\mathfrak{a}}$ is contained in the Jacobson radical of $\hat{A}$.

Proof (i) It suffices to prove that for any ideal I of A, $I\otimes_A\hat{A}\longrightarrow\hat{A}$ is injective. By property 2°, $I\otimes_A\hat{A}=I\hat{A}=\hat{I}$. By the above description and $(※)_2$, $\hat{I}\longrightarrow\hat{A}$ is injective.

(ii) By (i) or property 2°, $\widehat{(\mathfrak{a}^n)}=\mathfrak{a}^n\hat{A}=(\hat{A}\mathfrak{a})^n=(\hat{\mathfrak{a}})^n$.

(iii) For the exact sequence of A-modules

$$0\longrightarrow\mathfrak{a}^n\longrightarrow A\longrightarrow A/\mathfrak{a}^n\longrightarrow 0$$

and inverse limit systems

$$\{\mathfrak{a}^n/\mathfrak{a}^m\,|\,m=1,2,\cdots\},\ \{A/\mathfrak{a}^m\,|\,m=1,2,\cdots\}\text{ and }\{(A/\mathfrak{a}^m)/(\mathfrak{a}^n/\mathfrak{a}^m)\}\ (m>n),$$

it follows from $(※)_1$ and $(※)_2$ that $A/\mathfrak{a}^n\cong\hat{A}/\hat{\mathfrak{a}}^n$, and from which (iii) follows by taking quotients.

(iv) By the above description, it is clear that $\hat{\hat{A}}=\hat{A}$, i.e., $\hat{A}$ is complete for the $\hat{\mathfrak{a}}$-adic topology.

We shall use the fact that for the Jacobson radical rad(A) of A, $x\in\text{rad}(A)$ if and only if for any $y\in A$, $1-xy$ is a unit in A.

For each $x\in\hat{\mathfrak{a}}$, suppose

$$(1-x)^{-1}=1+x+x^2+\cdots\cdots\qquad \text{i.e.,}\quad \begin{aligned} y_1&=1+x,\\ y_2&=1+x+x^2,\\ &\cdots\cdots\cdots\\ y_n&=1+x+x^2+\cdots\cdots+x^n,\\ &\cdots\cdots\cdots. \end{aligned}$$

Then $\{y_n\}$ is a Cauchy sequence in $\hat{A}$. Since $\hat{A}$ is complete, there exists an element

y in $\hat{A}$ such that $\lim_{n\to\infty} y_n = y$.

Thus $\hat{\mathfrak{a}} \subset \mathrm{rad}(A)$. Q.E.D.

Lemma 2.2. Let $(A, \mathfrak{M})$ be a noetherian local ring. Then the completion $\hat{A}$ of A with the $\mathfrak{M}$-adic topology is also a noetherian local ring with its maximal ideal $\hat{\mathfrak{M}} = \mathfrak{M}\hat{A}$. Furthermore, $\hat{A}$ is a faithfully flat A-module.

Proof By lemma 2.1 (i), $\hat{A}$ is A-flat, and by lemma 2.1 (iii),

$\hat{A}/\hat{\mathfrak{M}} = \hat{A}/\mathfrak{M}\hat{A} \cong A/\mathfrak{M}$ which is a field.

Therefore, $\mathfrak{M}\hat{A}$ is a maximal ideal of $\hat{A}$ and $\hat{A} \neq \hat{\mathfrak{M}}\hat{A} = \mathfrak{M}\hat{A}$.

Thus $\hat{A}$ is a faithfully flat A-module. Moreover, by lemma 2.1 (iv), $\hat{\mathfrak{M}}$ is the Jacobson radical of $\hat{A}$ and so is the unique maximal ideal. Hence, $\hat{A}$ is a local ring. Let us prove that $\hat{A}$ is noetherian.

Define $G(A) = G_{\mathfrak{M}}(A) = \bigoplus_{n=0}^{\infty} \mathfrak{M}^n/\mathfrak{M}^{n+1} \quad (\mathfrak{M}^0 = A)$

and for an A-module M $\quad G(M) = G_{\mathfrak{M}}(M) = \bigoplus_{n=0}^{\infty} \mathfrak{M}^n M/\mathfrak{M}^{n+1} M \quad (\mathfrak{M}^0 M = M)$.

Then for $\mathfrak{M} = (x_1, \cdots\cdots, x_n)$ and

$$\begin{array}{ccc} \mathfrak{M} & \longrightarrow & \mathfrak{M}/\mathfrak{M}^2 \\ \cup\!\!\!\!\!\cup & & \cup\!\!\!\!\!\cup \\ x_i & \longmapsto & \bar{x}_i, \end{array}$$

it follows that $G(A) = (A/\mathfrak{M})[\bar{x}_1, \cdots\cdots, \bar{x}_n]$.

Therefore, since A is noetherian so is $G(A)$. By lemma 2.1 (iii), it is obvious that $G_{\mathfrak{M}}(A)$ and $G_{\hat{\mathfrak{M}}}(\hat{A})$ are isomorphic as graded rings.

In order to prove our assertion, we shall use the following:

"For a ring A, an ideal $\mathfrak{a}$ of A and an A-module M, we assume that $A = \hat{A}$ (the completion of A with the $\mathfrak{a}$-adic topology) and $\bigcap_n \mathfrak{a}^n M = 0$. If $G(M)$ is a noetherian $G(A)$-module, then M is a noetherian A-module."

We have already proved that $\hat{A}$ is complete for the $\hat{\mathfrak{M}}$-adic topology and $G_{\hat{\mathfrak{M}}}(\hat{A})$ is noetherian. Therefore if we take $\hat{A}$ as A and $\hat{A}$ as M in the above statement, then it follows that $\hat{A}$ is noetherian. Q.E.D.

Let A be a ring, M an A-module and $a_1, \cdots\cdots, a_n$ a sequence of elements of A.

Then $a_1, \cdots\cdots, a_n$ is said to be a *M-regular sequence* or simply *M-sequence* if it satisfies the following conditions ([11], [14], [16]).

(i) For each $1 \leq i \leq n$ a_i is not a zero-divisor on $M/(a_1, \cdots\cdots, a_{i-1})M$.

(ii) $M \neq (a_1, \cdots\cdots, a_n)M$.

If $(a_1, \cdots\cdots, a_n) \subset \mathfrak{a}$ for an ideal $\mathfrak{a}$ of A, then $a_1, \cdots\cdots, a_n$ is a *M-sequence in* $\mathfrak{a}$. Moreover, if there is no element $a\in\mathfrak{a}$ such that $a_1, \cdots\cdots, a_n, a$ is M-regular, then $a_1, \cdots\cdots, a_n$ is said to be *a maximal* M-sequence in $\mathfrak{a}$.

The length of a maximal M-regular sequence in $\mathfrak{a}$ is called the $\mathfrak{a}$-*depth* of M and it is denoted by $\mathrm{depth}_{\mathfrak{a}}(M)$ (or $\mathrm{depth}(\mathfrak{a}, M)$).

When $(A, \mathfrak{M})$ is a local ring $\mathrm{depth}_{\mathfrak{M}}(M)$ is written as $\mathrm{depth}(M)$ or $\mathrm{depth}_A(M)$ and call it simply the *depth* of M.

Let A be a ring and $\mathfrak{a}$ a proper ideal of A. $\mathrm{ht}(\mathfrak{a})$ means the ***height*** of $\mathfrak{a}$ and $\dim(A)$ denotes the *Krull dimension* of A.

In general, the following holds ([9]).

Property 3° (i) For each $P\in\mathrm{Spec}(A)$ (the *spectrum* of A), $\mathrm{ht}(P)=\dim(A_P)$ where A_P is the localization of A by P.

(ii) For each ideal $\mathfrak{a}$ of A, $\dim(A/\mathfrak{a})+\mathrm{ht}(\mathfrak{a})\leq\dim(A)$.

(iii) For a noetherian local ring $(A, \mathfrak{M})$ and a finitely generated A-module M, depth $(M)\leq\dim(M)$. where $\dim(M)=\dim(A/\mathrm{Ann}(M))$.

(iv) Let φ: $A\longrightarrow B$ be a noetherian ring homomorphism, $\boldsymbol{P}\in\mathrm{Spec}(B)$ and $P=\boldsymbol{P}\cap A$. Then $\mathrm{ht}(\boldsymbol{P})\leq\mathrm{ht}(P)+\dim(B_{\boldsymbol{P}}/PB_{\boldsymbol{P}})$. In particular, if φ is a flat homomorphism (i.e., B is A-flat ([1], [12])), the equality holds.

Let A be a ring. A complex M_* of A-modules is a sequence

$$M_* : \cdots\cdots\longrightarrow M_n\xrightarrow{d_M} M_{n-1}\xrightarrow{d_M}\cdots\cdots\xrightarrow{d_M} M_0\xrightarrow{d_M} 0$$

of A-modules with A-linear maps d_M such that $d_M{}^2=0$. For two complexes L_* and M_* of A-modules, we define $(L_*\otimes_A M_*)_n=\bigoplus_{p+q=n} L_p\otimes_A M_q$
and for $x\otimes y\in L_p\otimes M_q$, $d(x\otimes y)=d_L(x)\otimes y+(-1)^{p-1}x\otimes d_M(y)$.
Then

$$L_*\otimes M_* : \cdots\cdots\longrightarrow(L_*\otimes_A M_*)_n\xrightarrow{d}\cdots\cdots\xrightarrow{d}(L_*\otimes M_*)_0\longrightarrow 0$$

is also a complex of A-modules. For $x_1, \cdots\cdots, x_n\in A$ we define a complex K_* of A-modules as follows;

$$K_* : \cdots\cdots\longrightarrow 0\longrightarrow K_n\xrightarrow{d} K_{n-1}\xrightarrow{d}\cdots\cdots\longrightarrow K_1\xrightarrow{d} K_0\longrightarrow 0$$

where $K_0=A$, for $1\leq p\leq n$, $K_p=\oplus\{Ae_{i_1\cdots i_p}\,|\,1\leq i_1<\cdots\cdots<i_p\leq n\}$ which is a free A-

module with rank$\binom{n}{p}$, and for $p > n$ $K_p = 0$.

The differentiation $d : K_p \longrightarrow K_{p-1}$ is defined by

$$d(ae_{i_1, \dots, i_p}) = a\left(\sum_{r=1}^{p} (-1)^{r-1} x_{i_r} e_{i_1, \dots, i_{r-1}, i_{r+1}, \dots, i_p}\right) \text{ where } a \in A.$$

This complex K_* is sometimes denoted by $K_*(x_1, \cdots\cdots, x_n)$ or $K_*(\underline{x})$ and is called the *Koszul complex* for $x_1, \cdots\cdots, x_n$.

For a complex C_* of A-modules, we shall put $C_*(\underline{x}) = C_* \otimes K_*(\underline{x})$. When $n=1$,

$$K_*(x) : \cdots\cdots\cdots \longrightarrow 0 \longrightarrow Ae \xrightarrow{x} A \longrightarrow 0.$$

and $$K_*(x_1, \cdots\cdots, x_n) = K_*(x_1) \otimes_A \cdots\cdots \otimes_A K_*(x_n).$$

For a Koszul complex $K_*(x_1, \cdots\cdots, x_n)$ and an A-module M we define the complex $K_*(\underline{x}, M)$ of A-modules as follows;

$$K_*(\underline{x}, M) : \longrightarrow K_n \otimes_A M \longrightarrow \cdots\cdots \longrightarrow K_0 \otimes_A M \cdots\cdots 0.$$

Moreover, for the homology groups $H_*(K_*(\underline{x}, M))$ of $K_*(\underline{x}, M)$, we shall put

$$H_*(K_*(\underline{x}, M)) = H_*(\underline{x}, M)$$

Then it is clear that $H_0(\underline{x}, M) \cong M / \sum_{i=1}^{n} x_i M$ and

$$H_n(\underline{x}, M) \cong \{\xi \in M \mid x_1 \xi = \cdots\cdots = x_n \xi = 0\}.$$

For $x \in A$ and a complex C_* of A-modules,

$$(C_*(x))_p = C_p \otimes_A A \oplus C_{p-1} \otimes_A A \cong C_p \oplus C_{p-1}$$

and thus we have an exact sequence of complexes

$$0 \longrightarrow C_* \longrightarrow C_*(x) \longrightarrow C'_* \longrightarrow 0 \quad \text{where } C'_{p+1} = C_p.$$

For the differentiation d of C_*, the differentiation d' of $C_*(x)$ is defined by

$$d'(\xi, \eta) = (d\xi + (-1)^{p-1} x\eta,\ d\eta) \text{ for each } (\xi, \eta) \in C_p \oplus C_{p-1}.$$

Therefore $$d'(\xi, \eta) = 0 \Longrightarrow d\eta = 0 \text{ and } d\xi = (-1)^{p-1} x\eta$$
$$\Longrightarrow x \cdot (\xi, \eta) = (x\xi, x\eta) = d'(0, (-1)^p \xi) \in d' C_{p+1}(x)$$

and thus for all $p = 0, 1, 2, \cdots\cdots$, $x \cdot H_p(C_*(x)) = 0$.

Moreover, for each $\eta \in C'_p = C_{p-1}$ with $d\eta = 0$ in $(C_*(x))_p = C_p \oplus C_{p-1}$ we have $d'(0,\eta) = ((-1)^{p-1}x\eta, 0)$. Hence, from the above exact sequence of complexes and by the above statements, we have

$$\left.\begin{array}{l}\cdots\cdots \to H_p(C_*) \longrightarrow H_p(C_*(x)) \longrightarrow H_{p-1}(C_*) \xrightarrow{(-1)^{p-1}x} H_{p-1}(C_*) \longrightarrow \cdots, \text{(exact)} \\ \text{and for all } p=0,1,2,\cdots\cdots\cdots\cdots, \quad x\cdot H_p(C_*(x))=0\end{array}\right\}. \quad (※)_3$$

Lemma 2.3. Let A be a ring and M an A-module.

(i) Let $(\underline{x})=(x_1,\cdots,x_n)$ be an ideal of A. Then for all $p=0,1,\cdots\cdots$, $(\underline{x})H_p(\underline{x},M)=0$.

(ii) If $x_1,\cdots\cdots,x_n$ is a M-sequence, then $H_p(\underline{x},M)=0$ $(p>0)$ and $H_0(\underline{x},M)\cong M/\sum_{i=1}^{n}x_iM$.

Proof (i) Suppose $(\underline{x}')=(x_1,\cdots,x_{n-1})$ and $K_*(x_1,\cdots,x_{n-1})\otimes M=K_*(\underline{x}',M)$. Then $K_*(\underline{x}',M)\ (x_n)\cong K_*(\underline{x},M)$. In this case, by $(※)_3$ for all $p=0,1,2,\cdots\cdots$,

$$x_n\cdot H_p(K_*(\underline{x}',M)\ (x_n))=x_n\cdot H_p(\underline{x},M)=0.$$

Similarly, for $1\leq i<n$ we have $x_i\cdot H_p(\underline{x},M)=0$. $p=0,1,\cdots\cdots$.
Therefore we have $(\underline{x})\cdot H_p(\underline{x},M)=0$ for all $p=0,1,2,\cdots\cdots$.

(ii) We shall prove our assertion by induction on n.

(a) $n=1$; $H_1(x,M)=\{\xi\in M \mid x\xi=0\}=0$ (x is a M-regular element) and thus our assertion is true.

(b) We assume that for all $1,2,\cdots\cdots,n-1$, (ii) is true. Then by $(※)_3$, we have the exact sequence

$$H_p(x_1,\cdots,x_{n-1},M)=0 \longrightarrow H_p(x_1,\cdots,x_n,M) \longrightarrow H_{p-1}(x_1,\cdots,x_{n-1},M)=0 \text{ for } p>1.$$

Therefore $H_p(\underline{x},M)=0$ for $p>1$. We put $M_i=M/(x_1,\cdots\cdots,x_i)M$. Then when $p=1$,

$$H_1(x_1\cdots x_{n-1},M)=0 \longrightarrow H_1(\underline{x},M) \longrightarrow H_0(x_1,\cdots,x_{n-1},M)=M_{n-1} \xrightarrow{\pm x_n} M_{n-1} \longrightarrow \cdots \text{ is exact.}$$

Since x_n is a M_{n-1}-regular element, $H_1(\underline{x},M)$ must be zero. Q.E.D.

3. Some properties of depthes

Let A be a noetherian ring and let M be a finitely generated A-module. The grade of M is defined by

$$\text{grade}(M)=\inf\{i \mid \text{Ext}_A^i(M,A)\neq 0\}.$$

In fact, for $\mathfrak{a}=\mathrm{Ann}(M)$ $\quad \mathrm{supp}(M)=V(\mathfrak{a})=\{P\in\mathrm{Spec}(A)\mid P\supset\mathfrak{a}\}$ and

$$\mathrm{grade}(M)=\mathrm{depth}(\mathfrak{a}, A).$$

Moreover, if $n=\mathrm{grade}(M)$ then $\mathrm{Ext}_A^n(M, A)\neq 0$ and thus

$\mathrm{grade}(M)\leq \mathrm{Proj.\,dim}(M)$ where $\mathrm{Proj.\,dim}(M)$ is the *projective dimension* of M.

Let $\mathfrak{a}$ be an ideal of a noetherian ring A. We put $\mathrm{grade}(\mathfrak{a})=\mathrm{grade}(A/\mathfrak{a})$.

If $a_1, \cdots\cdots, a_r$ is a maximal A-sequence in $\mathfrak{a}$, then we have

$r=\mathrm{depth}(\mathfrak{a}, A)=\mathrm{grade}(\mathfrak{a})=\mathrm{ht}(a_1, \cdots, a_r)\leq \mathrm{ht}(\mathfrak{a})$.

In particular, if $\mathrm{grade}(\mathfrak{a})=\mathrm{Proj.\,dim}(A/\mathfrak{a})$ then $\mathfrak{a}$ is called a *perfect ideal.*

Theorem 3.1. Let A be a noetherian ring and let $\mathfrak{a}$ be a perfect ideal with grade $(A/\mathfrak{a})=\mathrm{Proj.\,dim}(A/\mathfrak{a})=n$.

Then (i) $\mathrm{Ext}_A^0(A/\mathfrak{a}, A)\cong A$ and $\mathrm{Proj.\,dim}(\mathfrak{a})=n-1$.

(ii) For each $P\in\mathrm{Ass}(A/\mathfrak{a})$ (= the set of all associated prime ideals of $A/\mathfrak{a}$), $\mathrm{grade}(\mathfrak{a})=\mathrm{grade}(P)=\mathrm{grade}(P/\mathfrak{a})=n$.

Proof (i) For the exact sequence

$$0\longrightarrow\mathfrak{a}\longrightarrow A\longrightarrow A/\mathfrak{a}\longrightarrow 0,$$

since $\mathrm{Proj.\,dim}(A/\mathfrak{a})=n$, it is obvious that $\mathrm{Proj.\,dim}(\mathfrak{a})=n-1$.

Furthermore, $\mathrm{Ext}_A^0(\mathfrak{a}, A)=\mathrm{Ext}_A^1(A/\mathfrak{a}, A)=0$ implies that

$$A\cong\mathrm{Ext}_A^0(A, A)\cong\mathrm{Ext}_A^0(A/\mathfrak{a}, A).$$

(ii) We shall put $P=\mathrm{Ann}(x+\mathfrak{a})\in\mathrm{Ass}(A/\mathfrak{a})$. Then we have the short exact sequence $\quad 0\longrightarrow A/P\xrightarrow{x} A/\mathfrak{a}\longrightarrow P/\mathfrak{a}\longrightarrow 0 \qquad (※※)_1$

Since $P/\mathfrak{a}$ is a submodule of $A/\mathfrak{a}$, the exact sequence $(※※)_1$ is split, i.e.,

$A/\mathfrak{a}\cong A/P \oplus P/\mathfrak{a}$. Therefore we have

$\mathrm{Ext}_A^i(A/\mathfrak{a}, A)\cong\mathrm{Ext}_A^i(A/P, A) \oplus \mathrm{Ext}_A^i(P/\mathfrak{a}, A)$ for all $i=0, 1, \cdots$.

Since by our hypothesis $\mathrm{Ext}_A^i(A/\mathfrak{a}, A)=0$ if $i<n$, we have also

$\mathrm{Ext}_A^i(A/P, A)=0=\mathrm{Ext}_A^i(P/\mathfrak{a}, A)$ if $i<n$.

From $\mathrm{Ext}_A^n(A/\mathfrak{a}, A)\neq 0$, we get $\mathrm{Ext}_A^n(A/P, A)\neq 0\neq\mathrm{Ext}_A^n(P/\mathfrak{a}, A)$ because that

① if $\mathrm{Ext}_A^n(A/P, A)=0$ and $\mathrm{Ext}_A^n(P/\mathfrak{a}, A)\neq 0$,

then it contradicts with $\mathrm{grade}(A/P)<\infty$, and

② if $\mathrm{Ext}_A^n(A/P, A)\neq 0$ and $\mathrm{Ext}_A^n(P/\mathfrak{a}, A)=0$, then it contradicts with grade$(P/\mathfrak{m})<\infty$.

Thus grade$(A/P)-n=$grade$(P/\mathfrak{m})$. Q.E.D.

Proposition 3.2. Let $f: A\longrightarrow B$ be a flat ring homomorphism and let M be an A-module. Then for a M-sequence $a_1,\cdots\cdots,a_r\in A$,
if $(M/(a_1,\cdots\cdots,a_r)M)\otimes_A B\neq 0$
then $f(a_1),\cdots\cdots,f(a_r)$ is a $M\otimes_A B$-sequence in B.

Proof Since a_1 is a M-regular element of A,

$$0\longrightarrow M\xrightarrow{a_1}M\longrightarrow M/a_1M\longrightarrow 0 \quad \text{is exact.}$$

By our hypothesis, B is a flat A-module and thus

$$0\longrightarrow M\otimes_A B\xrightarrow{a_1\otimes 1}M\otimes_A B\longrightarrow (M/a_1M)\otimes_A B\longrightarrow 0 \quad \text{is exact.}$$

By $(M/a_1M)\otimes_A B=M\otimes_A B/_{a_1M\otimes_A B}=M\otimes_A B/_{M\otimes_A f(a_1)B}$,

the above exact sequence can be written as

$$0\longrightarrow M\otimes_A B\xrightarrow{1\otimes f(a_1)}M\otimes_A B\longrightarrow M\otimes_A B/_{M\otimes_A f(a_1)B}\longrightarrow 0.$$

Therefore $1\otimes f(a_1)$ $(\in A\otimes_A B=B)=f(a_1)$ is $M\otimes_A B$-regular.

Similarly, since a_2 is M/a_1M-regular, we have the exact sequence

$$\begin{array}{llll}
0\longrightarrow M/a_1M & \xrightarrow{a_2} M/a_1M & \longrightarrow M/(a_1M+a_2M)\longrightarrow 0 & \\
 & \Downarrow & & \\
0\longrightarrow (M/a_1M)\otimes_A B & \xrightarrow{a_2\otimes 1} (M/a_1M)\otimes_A B & & \text{(exact)} \\
\qquad \| & \qquad \| & & \\
\quad M\otimes_A B/_{a_1M\otimes_A B} & \quad M\otimes_A B/_{a_1M\otimes_A B} & & \\
\qquad \| & \qquad \| & & \\
0\longrightarrow M\otimes_A B/_{M\otimes_A f(a_1)B} & \xrightarrow{1\otimes f(a_2)} M\otimes_A B/_{M\otimes_A f(a_1)B} & & \text{(exact)}
\end{array}$$

Thus $1\otimes_A f(a_2)$ $(\in A\otimes_A B=B)=f(a_2)$ is $M\otimes_A B/_{M\otimes_A f(a_1)B}$-regular.
Repeating this way, we arrive at the following situation;

$$\begin{array}{ll}
0\longrightarrow M/\sum_{i=1}^{r-1}a_iM\xrightarrow{a_r}M/\sum_{i=1}^{r-1}a_iM & \text{(exact)} \\
\qquad\qquad \Downarrow & \\
0\longrightarrow M\otimes_A B/_{M\otimes_A\sum_{i=1}^{r-1}f(a_i)B}\xrightarrow{1\otimes f(a_r)}M\otimes_A B/_{M\otimes_A\sum_{i=1}^{r-1}f(a_i)B} & \text{(exact).}
\end{array}$$

Therefore $1\otimes f(a_1),\cdots\cdots,1\otimes f(a_r)$, i.e., $f(a_1),\cdots\cdots,f(a_r)$ is $M\otimes_A B$-sequence because $(M/(a_1,\cdots\cdots,a_r)M)\otimes_A B=M\otimes_A B/_{M\otimes_A(f(a_1),\cdots\cdots,f(a_r))}\neq 0$. Q.E.D.

Let A be a ring and M an A-module. If $x_1, \cdots\cdots, x_i = ab, \cdots\cdots, x_n$ is a M-sequence in A, then $x_1, \cdots\cdots, x_{i-1}, a, x_{i+1}, \cdots\cdots, x_n$ and $x_1, \cdots\cdots, x_{i-1}, b, x_{i+1}, \cdots\cdots, x_n$ are M-sequences in A. (※※)$_2$

(See some properties of modules of generalized fractions by D.S. Kim (Chonbuk National Univ.) to appear).

If A is a noetherian ring and M is a finitely generated A-module, then for a M-sequence $x_1, \cdots\cdots, x_n$ in A,

$$x_i, x_1, \cdots\cdots, x_{i-1}, x_{i+1}, \cdots\cdots, x_n \text{ is a } M\text{-sequence in } A \quad (※※)_3$$

where $(x_1, \cdots\cdots, x_n) \subset \mathrm{rad}(A)$ ([7]).

Proposition 3.3. Let $(A, \mathfrak{M})$ and $(B, \mathfrak{N})$ be noetherian local rings, $\varphi: (A, \mathfrak{M}) \hookrightarrow (B, \mathfrak{N})$ a ring monomorphism with $\varphi(\mathfrak{M}) \subset \mathfrak{N}$. We assume that $\mathfrak{N} \cap A = \mathfrak{M}$ and $\mathfrak{M}B$ is $\mathfrak{N}$-primary. Then for each finitely generated B-module M, $\mathrm{depth}_B(M) = \mathrm{depth}_A(M)$.

Proof Since $\mathfrak{M}B$ is $\mathfrak{N}$-primary, in the topological space Spec(B), we have $V(\mathfrak{N}) = V(\mathfrak{M}B) = \{\mathfrak{N}\}$. This implies that

$$\mathrm{depth}_B(M) = \mathrm{depth}(\mathfrak{N}, M) = \mathrm{depth}(\mathfrak{M}B, M) \quad ([10]).$$

Thus it suffices to prove that $\mathrm{depth}(\mathfrak{M}B, M) = \mathrm{depth}(\mathfrak{M}, M) = \mathrm{depth}_A(M)$.

Let $m_1b_1, \cdots\cdots, m_rb_r$ ($m_i \in \mathfrak{M}$, $b_i \in B$ for $i = 1, 2, \cdots\cdots, r$) be a maximal M-sequence in $\mathfrak{M}B$.

Then by (※※)$_2$, above $m_1, \cdots\cdots, m_r$ is also a maximal M-sequence in $\mathfrak{M}B$.

Since $m_i \in \mathfrak{M}$ for $i = 1, 2, \cdots\cdots, r$, $m_1, \cdots\cdots, m_r$ is a maximal M-sequence in $\mathfrak{M}$, because that if $\mathrm{depth}_A(M) > \mathrm{depth}(\mathfrak{M}B, M)$ and $x_1, \cdots\cdots, x_l$ ($l > r$) is a maximal M-sequence in $\mathfrak{M}$, then $\mathfrak{M} \subset \mathfrak{M}B$ implies that $m_1b_1, \cdots\cdots, m_rb_r$ is not a maximal M-sequence in $\mathfrak{M}B$ and that $\mathfrak{M}M \subset \mathfrak{M}BM \neq M$.

Hence we have $\quad \mathrm{depth}(\mathfrak{M}B, M) = \mathrm{depth}(\mathfrak{M}, M)$,

that is, $\quad \mathrm{depth}_B(M) = \mathrm{depth}_A(M)$. $\quad$ Q.E.D.

Proposition 3.4. Let $(A, \mathfrak{M})$ be a noetherian local ring. Then for all $P \in \mathrm{Spec}(A)$

(i) $\mathrm{depth}(A) \leq \mathrm{depth}(P, A) + \dim(A/P)$

(ii) $\dim(A) - \mathrm{depth}(A) = \mathrm{Codepth}(A) \geq \dim(A_P) - \mathrm{depth}(A_P) = \mathrm{Codepth}(A_P)$.

Proof (i) We shall prove (i) by induction on depth(A).

Let depth(A) $=0$.

Then depth(P, A) $=0$ and dim(A/P) ≥ 0 ($A/P \neq 0$). Thus in this case

$$\text{depth}(A) \leq \text{depth}(P, A) + \dim(A/P).$$

Let depth(A) $= n$ (≥ 1).

We assume that for all noetherian local ring A with depth(A) $\leq n-1$ our assertion holds. We have two cases such that

$$\text{depth}(P, A) = 0 \quad \text{and} \quad \text{depth}(P, A) \geq 1.$$

In case depth(P, A) $=0$. Take a maximal A-sequence $x_1, \cdots\cdots, x_n$ in $\mathfrak{M}$. Then for all $i=1, 2, \cdots\cdots, n$, $x_i \notin P$ because that since $x_i, x_1, \cdots\cdots, x_{i-1}, x_{i+1}, \cdots\cdots, x_n$ is an A-sequence by (※※)$_3$ if $x_i \in P$ then depth(P, A) $\neq 0$.

Thus for $\begin{array}{ccc} \mathfrak{M} & \longrightarrow & \mathfrak{M}/P \\ \Cup & & \Cup \\ x_i & \longmapsto & \bar{x}_i \end{array}$

it follows that $\bar{x}_1, \cdots\cdots, \bar{x}_n$ is an A/P sequence in $\mathfrak{M}/P$.

This implies that $n \leq \text{depth}(A/P) \leq \dim(A/P)$.

Hence we have $\text{depth}(A) \leq \text{depth}(P, A) + \dim(A/P)$.

In case depth(P, A) ≥ 1.

Let $x_1, \cdots\cdots, x_n$ be a maximal A-sequence in $\mathfrak{M}$.

At first, we assume that for all $i=1, 2, \cdots\cdots, n$ $x_i \notin P$.

Then by the above reason, $\bar{x}_1, \cdots\cdots, \bar{x}_n$ is an A/P-sequence in $\mathfrak{M}/P$, and

$$n \leq \text{depth}(A/P) \leq \dim(A/P).$$

Since depth(P, A) ≥ 1, we have depth(A) $<$ depth(P, A) $+$ dim(A/P).

Next, we assume that there exists x_i ($1 \leq i \leq n$) with $x_i \in P$.

Then by (※※)$_3$ $x_i, x_1, \cdots\cdots, x_{i-1}, x_{i+1}, \cdots\cdots, x_n$ is a maximal A-sequence in $\mathfrak{M}$.

In this case $\text{depth}_{\mathfrak{M}/Ax_i}(A/Ax_i) = n-1$, $\text{depth}(P/Ax_i, A/Ax_i) \leq r-1$

where depth(P, A) $= r$ ($1 \leq r \leq n$).

By our inductive hypothesis

$$\text{depth}_{\mathfrak{M}/Ax_i}(A/Ax_i) \leq \text{depth}(P/Ax_i, A/Ax_i) + \dim(A/P)$$

because that $(A/Ax_i)/(P/Ax_i) \cong A/P$.

Thus $\quad \text{depth}\,(A/Ax_i)+1\leq\text{depth}(P/Ax_i,\ A/Ax_i)+1+\dim(A/P)$

that is, $\quad \text{depth}\,(A)\leq\text{depth}(P, A)+\dim(A/P)$.

(ii) It is well-known that $\quad \text{depth}(A)\leq\text{depth}(A_P)$

$\dim(A_P)=\text{ht}(P)$

$\dim(A/P)+\text{ht}(P)\leq\dim(A)$ ([9]).

Hence, $\quad \dim(A/P)+\text{ht}(P)-\text{depth}(A_P)\leq\dim(A)-\text{depth}(A)$

and thus $\quad \dim(A_P)-\text{depth}(A_P)+\dim(A/P)\leq\dim(A)-\text{depth}(A)$.

Therefore, $\text{Codepth}(A_P)\leq\text{Codepth}(A)$. Q.E.D.

4. Some properties of flatness

Let A be a ring, I an ideal of A and M an A-module. If for every finitely generated ideal $\mathfrak{a}$, $\mathfrak{a}\otimes_A M$ is separated with the I-adic topology then M is said to be *idealwise separated* for I.

E **Example** 4.1. Let B be a noetherian ring, φ: $A\longrightarrow B$ a ring homomorphism, M a finitely generated B-module and I an ideal of A. Then if $\varphi(I)B$ $(=IB)\subset\text{rad}(B)$ then M is idealwise separated for I.

Proof)Let $\mathfrak{a}$ be a finitely generated ideal of A. Since M is a finitely generated B-module so is $\mathfrak{a}\otimes_A M$. By using property 2° in §2, the following theorem can be proved.

"Let A be a noetherian ring and $I\subset\text{rad}(A)$ an ideal of A. Then every finitely generated A-module is separated with the I-adic topology ([10])."

With the above theorem it follows that $\mathfrak{a}\otimes_A M$ is separated with the IB-adic topology ($IB\subset\text{rad}(B)$).

Therefore
$$\begin{aligned}\{0\}&=\bigcap_{n\geq 0}(IB)^{n+1}(\mathfrak{a}\otimes_A M)\\&=\bigcap_{n\geq 0}I^{n+1}B(\mathfrak{a}\otimes_A M)\\&=\bigcap_{n\geq 0}I^{n+1}(\mathfrak{a}\otimes_A BM)\\&=\bigcap_{n\geq 0}I^{n+1}(\mathfrak{a}\otimes_A M)\end{aligned}$$

and as an A-module M is idealwise separated for I. Q.E.D.

Let A be a ring, $\mathfrak{a}$ an ideal of A and M an A-module. We shall put as the following

$$A_n = A/\mathfrak{a}^{n+1}, \qquad M_n = M/\mathfrak{a}^{n+1}M \qquad (n \geq 0)$$
$$\mathrm{gr}(A) = \bigoplus_{n\geq 0} \mathfrak{a}^n/\mathfrak{a}^{n+1}, \qquad \mathrm{gr}(M) = \bigoplus_{n\geq 0} \mathfrak{a}^n M/\mathfrak{a}^{n+1}M.$$

Then there is the canonical mapping

$$\begin{array}{ccc} \gamma_n: (\mathfrak{a}^n/\mathfrak{a}^{n+1}) \otimes_{A_0} M_0 & \longrightarrow & \mathfrak{a}^n M/\mathfrak{a}^{n+1}M \\ \Cup & & \Cup \\ (a^n + \mathfrak{a}^{n+1}) \otimes (m + \mathfrak{a}M) & \longmapsto & (a^n m + \mathfrak{a}^{n+1}M) \end{array}$$

and thus we have the $\mathrm{gr}(A)$-module homomorphism γ: $\mathrm{gr}(A) \otimes_{A_0} M_0 \to \mathrm{gr}(M)$. The following lemma is very useful to study flatness ([10]).

Lemma 4.2. With the above notations, if

(a) $\mathfrak{a}$ is a nilpotent ideal

or (b) A is noetherian and M is idealwise separated for $\mathfrak{a}$, then the followings are equivalent.

(i) M is A-flat.

(ii) For all A_0-module N, $\mathrm{Tor}_1^A(N, M) = 0$.

(iii) M_0 is A_0-flat and $\mathfrak{a} \otimes_A M \cong \mathfrak{a}M$.

(iv) M_0 is A_0-flat and $\mathrm{Tor}_1^A(A_0, M) = 0$.

(v) M_0 is A_0-flat and for all $n \geq 0$, γ_n is an isomorphism.

(vi) M_0 is A_0-flat and γ is an isomorphism.

(vii) For all $n \geq 0$, M_n is A_n-flat.

This lemma is effective for a noetherian local ring. Let $(A, \mathfrak{M})$ be a noetherian local ring, and let an A-module M be idealwise separated for $\mathfrak{M}$. In fact, in lemma 4.2, since $A_0 = A/\mathfrak{M}$ is a field, M_0 is automatically A_0-flat.

Since the maximal ideal $\mathfrak{M}/\mathfrak{M}^{n+1}$ is a nilpotent ideal in the noetherian local ring $(A_n = A/\mathfrak{M}^{n+1}, \mathfrak{M}/\mathfrak{M}^{n+1})$, M_n is A_n-flat if and only if M_n is A_n-free.

Proposition 4.3. Let A be a noetherian ring, B an A-algebra and a noetherian ring, M a finitely generated B-module. If there exists an element $x \in A$ such that it is A-regular and M-regular with $xB \subset \mathrm{rad}(B)$ then, if M/xM is A/xA-flat then M is A-flat.

Proof Since B is noetherian, M is a finitely generated B-module and $xB \subset \mathrm{rad}(B)$, M is separated with the xB-adic topology of M from the theorem in the proof of example 4.1.

That is, $\bigcap_{\nu>0}(xB)^{\nu}M=\{0\}$.
From $(xA)^{\nu}M\subset(xB)^{\nu}M$, it follows that $\bigcap_{\nu>0}(xA)^{\nu}M=0$.
That is, M is idealwise separated for xA.

By our hypothesis, since A is noetherian and M_0 ($=M/xM$) is A_0 ($=A/xA$)-flat it suffices to prove that

$$\mathrm{Tor}_1^A(A_0, M)=\mathrm{Tor}_1^A(A/xA, M)=0 \quad \text{by lemma 4.2. (iv).}$$

Since x is A-regular, the Koszul complex

$$0\longrightarrow Ae_x\longrightarrow A\longrightarrow A/xA\longrightarrow 0$$

is a free resolution over A/xA. Furthermore, since x is also M-regular and in $K_*(x, A)\otimes_A M=K_*(x, M)$, we have

$$H_i(K_*(x, M))=\mathrm{Tor}_i^A(A/xA, M)=0 \text{ for all } i\geq 1 \text{ by lemma 2.3 (ii).}$$

Therefore, M is a flat A-module. Q.E.D.

Proposition 4.4. Let A and B be noetherian rings, $A\longrightarrow B$ a flat ring homomorphism, $\mathfrak{a}$ an ideal of A and J an ideal of B such that $\mathfrak{a}B\subset J$. Then $\hat{B}$ is a flat $\hat{A}$-module where $\hat{A}=\varprojlim A/\mathfrak{a}^n$ and $\hat{B}=\varprojlim B/J^n$.

Proof By lemma 2.1 (i), it is clear that $\hat{A}$ is a flat A-module and $\hat{B}$ is a flat B-module. By our hypothesis, since B is a flat A-module, it follows that $\hat{B}$ is a flat A-module by the transitivity with respect to flatness. As in the proof of lemma 2.1 (i), it suffices to prove that for any ideal I of A

$$\hat{I}\otimes_{\hat{A}}\hat{B}\longrightarrow\hat{B} \text{ is injective where } \hat{I}=\varprojlim I/\mathfrak{a}^n.$$

As in the proof of lemma 2.2, $\hat{A}$ is a noetherian ring.
Hence $\hat{I}=I\otimes_A\hat{A}$ by property 2° in §2. Thus we have the following;

$$\hat{I}\otimes_{\hat{A}}\hat{B}=I\otimes_A\hat{A}\otimes_{\hat{A}}\hat{B}\cong I\otimes_A\hat{B}\longrightarrow\hat{B}$$

is injective because that $\hat{B}$ is an A-flat module.

Therefore $\hat{B}$ is an $\hat{A}$-flat module. Q.E.D.

Theorem 4.5. Let φ: $(A,\mathfrak{M})\longrightarrow(B,\mathfrak{N})$ be a noetherian ring homomorphism. Let us assume that A is regular and $\dim(A)=\mathrm{ht}(\mathfrak{M}B)$. Then

(i) B is regular and the image of a regular system of parameters of A under φ is the subsequence of a regular system of parameters of B if and only if $B/\mathfrak{M}B$ is regular and $\dim(B)=\dim(A)+\dim(B/\mathfrak{M}B)$.

(ii) If conditions in (i) hold, then B is a flat A-module.

Proof (i) Since B is a regular ring it is also a Cohen-Macaulay ring.

Therefore, since $\mathfrak{M}B\subsetneq\mathfrak{N}$ is a proper ideal, we have

$$\begin{aligned}\dim(B)&=\mathrm{ht}(\mathfrak{M}B)+\dim(B/\mathfrak{M}B) \quad ([10])\\ &=\dim(A)+\dim(B/\mathfrak{M}B).\end{aligned}$$

Let $x_1,\cdots\cdots,x_r$ be a regular system of parameters of A such that $\varphi(x_1),\cdots\cdots,\varphi(x_r)$ is a subsequence of a regular system of parameters of B where $\dim(A)=r$.

Since $(\varphi(x_1),\cdots\cdots,\varphi(x_r))=\mathfrak{M}B$, $\mathfrak{M}B$ is a prime ideal of B and B is a regular ring, we have $B/\mathfrak{M}B$ is a regular ring ([9]).

Conversely, let $\Psi: \begin{array}{ccc} B & \longrightarrow & B/\mathfrak{M}B \\ \Downarrow & & \Downarrow \\ b & \longmapsto & \bar{b} \end{array}$

be the canonical projection.

Take a regular system $x_1,\cdots\cdots,x_r$ of parameters of A, and let $\bar{y}_1,\cdots\cdots,\bar{y}_s$ be a regular system of parameters of $B/\mathfrak{M}B$.

Let us put $\Psi^{-1}(\bar{y}_i)\ni y_i$ $(\neq 0)$ for all $i=1,2,\cdots\cdots,s$.

Then it follows that $\mathfrak{N}$ is generated by $\{\varphi(x_1),\cdots\cdots,\varphi(x_r),\ y_1,\cdots\cdots,y_s\}$.

By our hypothesis $\quad \dim(B)=\dim(A)+\dim(B/\mathfrak{M}B)$.

Hence we have $\quad \dim(B)=r+s$.

Thus B is a regular local ring.

Furthermore, in this case, for a regular system $x_1,\cdots\cdots,x_r$ of parameters of A, $\varphi(x_1),\cdots\cdots,\varphi(x_r)$ is a subsequence of a regular system of parameters of B.

(ii) We assume that $B/\mathfrak{M}B$ is regular and $\dim(B)=\dim(A)+\dim(B/\mathfrak{M}B)$. Since A is a regular local ring by our hypothesis, it follows that A is an integral domain ([10]).

Hence we have $\quad \bigcap_{\nu>0}\mathfrak{M}^{\nu}=\{0\}$ $\quad$ ([10]).

This means that as A-module B is idealwise separated for $\mathfrak{M}$. It suffices to prove that

$$B/\mathfrak{M}B \text{ is } A/\mathfrak{M}\text{-flat and } \mathrm{Tor}_1^A(A/\mathfrak{M},B)=0$$

for the flatness of B by lemma 4.2. Since $A/\mathfrak{M}$ is a field, it is obvious that $B/\mathfrak{M}B$

is $A/\mathfrak{M}$-flat. Let $x_1, \cdots\cdots, x_r$ be a regular system of parameters of A. Then the Koszul complex $K_*(x_1, \cdots\cdots, x_r, A)$ is a free resolution over the A-module $A/\mathfrak{M}$.

$$\text{Therefore } \mathrm{Tor}_i^A(A/\mathfrak{M}, B) = H_i(K_*(x_1, \cdots\cdots, x_r, A)\otimes_A B)$$
$$= H_i(K_*(\varphi(x_1), \cdots\cdots, \varphi(x_r), B)) \quad \text{for all } i \geq 0.$$

Since $\varphi(x_1), \cdots\cdots, \varphi(x_r)$ is a B-regular sequence in $\mathfrak{M}B$ as in the proof of (i), $\mathrm{Tor}_i^A(A/\mathfrak{M}, B) = 0$ for $i \geq 1$ by lemma 2.3 (ii).
Therefore, B is a flat A-module by lemma 4.2. Q.E.D.

REFERENCES

1. L.L. Avramov: Flat homomorphisms of complete intersections, Dokl. Akad. Nauk. S.S.S.R 225(1975), soviet *Math. Dokl.* **16**(1975), pp.1413~1417.
2. E. Davis: Ideals of the principal class, R-sequence. *Pacific J.* **20**(1967), pp.197~209.
3. J. Dieudonne: On regular sequences, *Nagoya Math. J.* **27~1**(1969), pp.355~356.
4. P. Eakin: The converse to a well-known theorem on noetherian rings, *Math. Ann.* **177** (1968), pp.278~282.
5. D. Ferrand and M. Raynaud: Fibres formelles dum anneau local noetherian, *Ann. Sci. Ec. Norm. Sup* **3**(1970), pp.295~311.
6. R. Hartshorne: A property of A-sequences, *Bull Soc. Math. France* **94**(1966), pp.61~66.
7. I. Kaplanski: Commutative Rings, The university of Chicago Press (1974).
8. J. Matijevic: Maximal ideal transforms of noetherian rings, *Proc. A.M.S.* **54**(1976), pp.49~52.
9. H. Matsumura: Commutative Algebra, The Benjamin/Cummings Publishing Company. INC. (1980).
10. H. Matsumura: Commutative Rings, Kyositzsia, Japan (1980).
11. M. Nagata: Local Rings, Interscience (1962).
12. M. Nagata: Flatness of an extension of a commutative ring, *J. Math. Kyoto Univ* (1969), pp.439~448.
13. L. Robbiano and G. Valla: On normal flatness and normal torsion-freeness, *J. of Algebra* **43**(1976), pp.552~560.
14. P. Samuel and O. Zariski: Commutative Algebra Ⅰ, Ⅱ, Springer (1975).
15. W. Vasconcelos: Ideals generated by R-sequences, *J. of Alg.* **6**(1967), pp.309~316.
16. W. Vasconcelos: On finitely generated flat Modules, *Traus. A.M.S.* **136**(1969), pp. 505~512.