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Minimax and Admissible Estimation of the Mean of a Multivariate Normal Distribution.

by

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1. Introduction

1.1. Background and Definitions

Let $\underset{\sim}{X}=(X_1,\cdots\cdots,X_p)'$ be a p variate normal random vector with unknown mean vector $\underset{\sim}{\theta}=(\theta_1,\cdots\cdots,\theta_p)'$ and covariance matrix I_p, $p\times p$ identity matrix, where a' denotes the transpose of a vector a in $\boldsymbol{R}^p$. Consider the problem of estimating $\underset{\sim}{\theta}$ when the loss incurred in estimating $\underset{\sim}{\theta}$ by $\underset{\sim}{\delta}=(\delta_1,\cdots\cdots,\delta_p)'$ is sum-of-squares error loss

$$L(\underset{\sim}{\theta},\underset{\sim}{\delta})=\sum_{i=1}^{p}(\theta_i-\delta_i)^2.$$

Note that this could equivalently be stated as the problem of trying to simultaneously estimate p normal means from independent problems and that this can be done, without loss of generality, in the case of a single observation because of sufficiency. We can also restrict ourselves to the class of all nonrandomized estimators in finding a good estimator because the loss function is convex in $\underset{\sim}{\delta}$ for each $\underset{\sim}{\theta}\in\boldsymbol{R}^p$ (see p. 35 in Berger [4]). Questions of interest are to obtain admissible or/and minimax estimators where usual definitions of admissibility and minimaxity of an estimator are given below.

Definition 1.1.1. An estimator $\underset{\sim}{\delta}^*$ is said to be admissible in $\boldsymbol{D}$, the set of all possible nonrandomized estimators if there does not exist any other estimator $\underset{\sim}{\delta}$ such that $R(\underset{\sim}{\theta},\underset{\sim}{\delta})\leq R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*)$ for all $\underset{\sim}{\theta}\in\boldsymbol{R}^p$ with strict inequality for at least one $\underset{\sim}{\theta}\in\boldsymbol{R}^p$ where $R(\underset{\sim}{\theta},\underset{\sim}{\delta})=\int_{\boldsymbol{R}^p}L(\underset{\sim}{\theta},\underset{\sim}{\delta}(\underset{\sim}{x}))\ f(\underset{\sim}{x};\underset{\sim}{\theta})\,d\underset{\sim}{x}$ is
the risk function of an estimator $\underset{\sim}{\delta}$, where $f(\underset{\sim}{x};\underset{\sim}{\theta})$ is the pdf $N(\underset{\sim}{\theta},I_p)$, the multivariate normal distribution.

Definition 1.1.2. An estimator $\underset{\sim}{\delta}^*$ is called minimax if it minimizes $\sup_{\underset{\sim}{\theta}\in\boldsymbol{R}^p}R(\underset{\sim}{\theta},\underset{\sim}{\delta})$ among all possible estimators $\underset{\sim}{\delta}$ in $\boldsymbol{D}$, that is,

$$\sup_{\underset{\sim}{\theta} \in R^p} R(\underset{\sim}{\theta}, \underset{\sim}{\delta}^*) = \inf_{\underset{\sim}{\delta} \in D} \sup_{\underset{\sim}{\theta} \in R^p} R(\underset{\sim}{\theta}, \underset{\sim}{\delta}).$$

In the above setting, the introduction of a nonnegative measure $\pi(\underset{\sim}{\theta})$ defined over $\boldsymbol{R}^p$ known as the prior distribution yields a Bayes procedure. The formal definition of a Bayes procedure is given as follow.

Definition 1.1.3. The Bayes procedure is to choose $\underset{\sim}{\delta}^\pi$ such that

$$\int_{\boldsymbol{R}^p} R(\underset{\sim}{\theta}, \underset{\sim}{\delta}^\pi)\, d\pi(\underset{\sim}{\theta}) = \inf_{\underset{\sim}{\delta} \in D} \int_{\boldsymbol{R}^p} R(\underset{\sim}{\theta}, \underset{\sim}{\delta})\, d\pi(\underset{\sim}{\theta}) \tag{1.1.1}$$

where $\underset{\sim}{\delta}^\pi$ is a nonrandomized estimator defined as a function from $\boldsymbol{R}^p$ into $\boldsymbol{R}^p$. Here we assume that $R(\underset{\sim}{\theta}, \underset{\sim}{\delta}) < \infty$ for all $\underset{\sim}{\theta} \in \boldsymbol{R}^p$ and $\underset{\sim}{\delta} \in \boldsymbol{D}$.

The left-hand side of (1.1.1) is known as the Bayes risk and we refer to $\underset{\sim}{\delta}^\pi(\underset{\sim}{X})$ as the Bayes estimator. In (1.1.1) if the prior is proper, i.e., $\pi(\boldsymbol{R}^p) < \infty$, then by employing Fubini's Theorem, the Bayes estimator can be formed by choosing for each $\underset{\sim}{x}$, a decision which minimizes

$$\int_{\boldsymbol{R}^p} L(\underset{\sim}{\theta}, \underset{\sim}{\delta})\, f(\underset{\sim}{x}; \underset{\sim}{\theta})\, d\pi(\underset{\sim}{\theta}) \tag{1.1.2}$$

If the prior is improper, i.e., $\pi(\boldsymbol{R}^p) = \infty$, then it is possible for $\underset{\sim}{\delta}^\pi(\underset{\sim}{X})$ to have infinite Bayes risk and hence, (1.1.1) is meaningless. In that case, we use the above result as the definition of the Bayes estimator and we refer to $\underset{\sim}{\delta}^\pi(\underset{\sim}{X})$ as the generalized Bayes estimator for estimating $\underset{\sim}{\theta}$.

The problem has been of considerable interest since Stein [11] demonstrated that the most standard minimax estimator $\underset{\sim}{\delta}_0(\underset{\sim}{X}) = \underset{\sim}{X}$ is admissible for $p=1$ or 2, but is inadmissible for $p \geq 3$. Indeed, he found minimax estimators, $\underset{\sim}{\delta}^{JS}(\underset{\sim}{X}) = \left(1 - \frac{(p-2)}{\sum_{i=1}^{p} X_i^2}\right)\underset{\sim}{X}$, which significantly improved upon the risk of $\underset{\sim}{\delta}_0$ for $p \geq 3$. This result came as a great suprise to most statisticians, in that the estimator $\underset{\sim}{\delta}_0(\underset{\sim}{X}) = \underset{\sim}{X}$ is so natural and satisfies so many "classical" optimality properties. Ever since Stein's result, considerable effort by a number of authors has gone into finding uniformly better(minimax) estimators for $p \geq 3$ (see [2], [10] and [19]). Also extensions of this result have since been obtained in many directions establishing that, in several dimensions, "standard" estimators are often inadmissible for a wide range of densities and loss functions (see [1], [3], [5], [6], [7], [9], [11], [14], [15], [16], [19] and [21] among them).

1.2. Outline

Assume in what follows that $p \geq 3$. Let $\underset{\sim}{X}$ have a multivariate normal distribution with probability density function

$$f(\underset{\sim}{x}|\underset{\sim}{\theta}) = (2\pi)^{-\frac{p}{2}} e^{-\frac{1}{2}|\underset{\sim}{x}-\underset{\sim}{\theta}|^2}, \quad \underset{\sim}{x} \in \boldsymbol{R}^p, \quad \underset{\sim}{\theta} \in \boldsymbol{R}^p \tag{1.2.1}$$

where $|y|$ denotes the Euclidean norm of a vector y in $\boldsymbol{R}^p$. Let the conditional(proper) prior distribution of $\underset{\sim}{\theta}$ given t be $N_p\left(\underset{\sim}{\theta}, \frac{t}{1-t} I_p\right)$ with pdf

$$\pi(\underset{\sim}{\theta}|t) \propto \left(\frac{1-t}{t}\right)^{\frac{p}{2}} e^{-\frac{1-t}{2t}|\underset{\sim}{\theta}|^2}, \quad \underset{\sim}{\theta} \in \boldsymbol{R}^p, \quad 0 < t < 1, \tag{1.2.2}$$

and let t have the marginal (prior) distribution with density

$$h(t) \propto t^{\frac{p}{2}} (1-t)^{\alpha(p-2)-\frac{p}{2}}, \quad 0 < t < 1, \tag{1.2.3}$$

where α is a known constant in $\boldsymbol{R}^1$. Note that if $\alpha > \frac{1}{2}$, then $h(t)$ is proper, i.e., $\int_0^1 h(t)\, dt < \infty$, and if $\alpha \leq \frac{1}{2}$, then $h(t)$ is improper, i.e., $\int_0^1 h(t)\, dt = \infty$, and hence the marginal prior density of $\underset{\sim}{\theta}$, $\pi(\underset{\sim}{\theta}) = \int_0^1 \pi(\underset{\sim}{\theta}|t)\, h(t)\, dt$, will not be proper for $\alpha \leq \frac{1}{2}$. In general, if $h(t)$ is such that the marginal density $m(\underset{\sim}{x})$ of $\underset{\sim}{X}$ is finite for all $\underset{\sim}{x}$, then the posterior density of $\underset{\sim}{\theta}$,

$$\pi(\underset{\sim}{\theta}|\underset{\sim}{x}) = f(\underset{\sim}{x}|\underset{\sim}{\theta})\, \pi(\underset{\sim}{\theta}) / m(\underset{\sim}{x})$$

is a proper density.

In Chapter 2, we derive a class of Bayes estimators with respect to two stage priors

(1.2.2) and (1.2.3) using the confluent hypergeometric function. In Chapter 3, we give a sufficient condition in order for Bayes estimators with respect to a prior to be minimax, and exhibit, using this condition, a subclass of minimax estimators within the class of Bayes estimators derived in Chapter 2. We also show that for two stage priors (1.2.2) and (1.2.3) there exist proper Bayes minimax estimators if $p \geq 5$ which coincides with Strawderman's ([19], [20]) result.

In Appendix A we, applying Brown's ([8]) result, derive a sub-class of admissible estimators within the class of Bayes estimators. Also, we give a class of proper Bayes minimax admissible estimators using two stage priors (1.2.2) and (1.2.3).

2. Two Stage Bayes Estimators

For two stage priors (1.2.2) and (1.2.3), the joint (generalized) prior distribution of $\underset{\sim}{\theta}$ and t has the density

$$\begin{aligned}\pi(\underset{\sim}{\theta}, t) &= \pi(\underset{\sim}{\theta} \mid t)\, h(t) \\ &\propto (1-t)^{a(p-2)}\, e^{-\frac{1-t}{2t}|\underset{\sim}{\theta}|^2}, \quad \underset{\sim}{\theta} \in R^p, \ 0<t<1 \end{aligned} \tag{2.1}$$

and, by (1.1) and (2.1), the joint distribution of $\underset{\sim}{X}, \underset{\sim}{\theta}$, and t has the density

$$\begin{aligned}\pi(\underset{\sim}{x}, \underset{\sim}{\theta}, t) &= \pi(\underset{\sim}{\theta}, t)\, f(\underset{\sim}{x} \mid \underset{\sim}{\theta}) \\ &\propto (1-t)^{a(p-2)}\, e^{-\frac{1}{2t}|\underset{\sim}{\theta} - t\underset{\sim}{x}|^2 - \frac{1}{2}(1-t)|\underset{\sim}{x}|^2}, \\ & \qquad \underset{\sim}{x} \in R^p, \ \underset{\sim}{\theta} \in R^p, \ 0<t<1, \end{aligned} \tag{2.2}$$

and, by (2.2), the marginal distribution of $\underset{\sim}{X}$ has from (2.2), the density

$$\begin{aligned} m(\underset{\sim}{x}) &= \int_0^1 \int_{R^p} f(\underset{\sim}{x}, \underset{\sim}{\theta}, t)\, d\underset{\sim}{\theta}\, dt \\ &\propto \int_0^1 (1-t)^{a(p-2)}\, e^{-\frac{1}{2}(1-t)|\underset{\sim}{x}|^2}\, t^{\frac{p}{2}}\, dt, \quad \underset{\sim}{x} \in R^p \end{aligned} \tag{2.3}$$

which is finite for all $\underset{\sim}{X}$, and which can be written as confluent hypergeometric function.

Hence, by (2.2) and (2.3), the (proper) posterior joint distribution of $\underset{\sim}{\theta}$ and t given $\underset{\sim}{X}$ has the density

$$
\begin{aligned}
\pi(\underset{\sim}{\theta}, t|\underset{\sim}{x}) &= m^{-1}(\underset{\sim}{x})\, f(\underset{\sim}{x}|\underset{\sim}{\theta})\, f(\underset{\sim}{\theta}, t) \\
&\propto m^{-1}(\underset{\sim}{x})\, (1-t)^{a(p-2)}\, e^{-\frac{1}{2t}|\underset{\sim}{\theta}-t\underset{\sim}{x}|^2-\frac{1}{2}(1-t)|\underset{\sim}{x}|^2} \qquad (2.4)\\
&\underset{\sim}{\theta}\in \boldsymbol{R}^p,\ 0<t<1,\ \underset{\sim}{x}\in \boldsymbol{R}^p
\end{aligned}
$$

where $m(\underset{\sim}{x})$is as in (2.3) and the marginal posterior density of $\underset{\sim}{\theta}$ is, from (2.4),

$$
\begin{aligned}
\pi(\underset{\sim}{\theta}|\underset{\sim}{x}) &= f(\underset{\sim}{x}|\underset{\sim}{\theta})\, \pi(\underset{\sim}{\theta})/m(\underset{\sim}{x}) \\
&= \int_0^1 \pi(\underset{\sim}{\theta}, t|\underset{\sim}{x})\, dt \\
&\propto m^{-1}(\underset{\sim}{x}) \int_0^1 (1-t)^{a(p-2)}\, e^{-\frac{1}{2t}|\underset{\sim}{\theta}-t\underset{\sim}{x}|^2-\frac{1}{2}(1-t)|\underset{\sim}{x}|^2}\, dt \qquad (2.5)
\end{aligned}
$$

Remark 2.1. Note that since $m(\underset{\sim}{x})$ is finite for all $\underset{\sim}{x}\in\boldsymbol{R}^p$, $\pi(\underset{\sim}{\theta}, t|\underset{\sim}{x})$ is a proper density. Moreover, the marginal posterior density $\pi(\underset{\sim}{\theta}|\underset{\sim}{x})$ of $\underset{\sim}{\theta}$ given x is also a proper density.

Consider next the Bayes (possibly generalized) estimator (the posterior mean of $\underset{\sim}{\theta}$ based on the joint posterior distribution of $\underset{\sim}{\theta}$ and t with the density (2.4)) of $\underset{\sim}{\theta}$ with respect to the joint prior (possibly improper) distribution with the density (2.1). The Bayes (possibly generalized) estimator $\underset{\sim}{\delta}^{\pi}(\underset{\sim}{X})$ of $\underset{\sim}{\theta}$

(which minimizes $\int_0^1\int_{R^p} |\underset{\sim}{\theta}-\underset{\sim}{\delta}(\underset{\sim}{x})|^2\, \pi(\underset{\sim}{x},\underset{\sim}{\theta},t)\, d\underset{\sim}{\theta}\, dt$ wrt $\underset{\sim}{\delta}$ for each $\underset{\sim}{x}$, i.e.,
$\int_0^1\int_{R^p} |\underset{\sim}{\theta}-\underset{\sim}{\delta}(\underset{\sim}{x})|^2\, \pi(\underset{\sim}{\theta},t|\underset{\sim}{x})\, d\underset{\sim}{\theta}\, dt$ wrt $\underset{\sim}{\delta}$ for each $\underset{\sim}{x}$)

is as follow

$$
\begin{aligned}
\underset{\sim}{\delta}^{\pi}(\underset{\sim}{x}) &= \int_0^1\int_{R^p} \underset{\sim}{\theta}\, \pi(\underset{\sim}{\theta}, t|\underset{\sim}{x})\, d\underset{\sim}{\theta}\, dt \\
&= \frac{\int_0^1 (1-t)^{a(p-2)}\, e^{-\frac{1}{2}(1-t)|\underset{\sim}{x}|^2} \left\{\int_{R^p} \underset{\sim}{\theta}\, e^{-\frac{1}{2t}|\underset{\sim}{\theta}-t\underset{\sim}{x}|^2}\, d\underset{\sim}{\theta}\right\} dt}{\int_0^1 (1-t)^{a(p-2)}\, e^{-\frac{1}{2}(1-t)|\underset{\sim}{x}|^2} \left\{\int_{R^p} e^{-\frac{1}{2t}|\underset{\sim}{\theta}-t\underset{\sim}{x}|^2}\, d\underset{\sim}{\theta}\right\} dt}
\end{aligned}
$$

$$
=\frac{\int_0^1 (1-t)^{\alpha(p-2)}\, e^{-\frac{1}{2}(1-t)|\underset{\sim}{x}|^2}\, t\underset{\sim}{x}\cdot t^{\frac{p}{2}}\, dt}{\int_0^1 (1-t)^{\alpha(p-2)}\, e^{-\frac{1}{2}(1-t)|\underset{\sim}{x}|^2}\cdot t^{\frac{p}{2}}\, dt}
$$

$$
=\left\{\frac{\int_0^1 t^{\frac{p}{2}+1}\,(1-t)^{\alpha(p-2)}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt}{\int_0^1 t^{\frac{p}{2}}\,(1-t)^{\alpha(p-2)}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt}\right\}\underset{\sim}{x}
$$

$$
=\left[1-\frac{\int_0^1 t^{\frac{p}{2}}\,(1-t)^{\alpha(p-2)+1}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt}{\int_0^1 t^{\frac{p}{2}}\,(1-t)^{\alpha(p-2)}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt}\right]\underset{\sim}{x} \tag{2.6}
$$

But, using integration by parts to the numerator of (2.6), we have

$$
\begin{aligned}
&\int_0^1 t^{\frac{p}{2}}\,(1-t)^{\alpha(p-2)+1}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt \\
&=t^{\frac{p}{2}}\,(1-t)^{\alpha(p-2)+1}\,\frac{2}{|\underset{\sim}{x}|^2}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\Big|_{t=0}^{t=1} \\
&\quad-\frac{2}{|\underset{\sim}{x}|^2}\int_0^1\left[\frac{p}{2}\, t^{\frac{p}{2}-1}\,(1-t)^{\alpha(p-2)+1}-t^{\frac{p}{2}}\,[\alpha(p-2)+1]\,(1-t)^{\alpha(p-2)}\right] e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt. \\
&=-\frac{2}{|\underset{\sim}{x}|^2}\left\{\frac{p}{2}\int_0^1 t^{\frac{p}{2}-1}\,(1-t)^{\alpha(p-2)+1}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt\right. \\
&\quad\left.-[\alpha(p-2)+1]\int_0^1 t^{\frac{p}{2}}\,(1-t)^{\alpha(p-2)}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt\right\} \\
&=\frac{2[\alpha(p-2)+1]}{|\underset{\sim}{x}|^2}\int_0^1 t^{\frac{p}{2}}\,(1-t)^{\alpha(p-2)}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt-\frac{p}{|\underset{\sim}{x}|^2}\int_0^1 t^{\frac{p}{2}}(1-t)^{\alpha(p-2)}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt.
\end{aligned}
$$

Hence,

$$
\delta^*(\underset{\sim}{x})=\left[1-\frac{2[\alpha(p-2)+1)}{|\underset{\sim}{x}|^2}+\frac{p}{|\underset{\sim}{x}|^2}\,\frac{\int_0^1 t^{\frac{p}{2}-1}\,(1-t)^{\alpha(p-2)+1}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt}{\int_0^1 t^{\frac{p}{2}}\,(1-t)^{\alpha(p-2)}\, e^{\frac{1}{2}t|\underset{\sim}{x}|^2}\, dt}\right]\underset{\sim}{x}. \tag{2.7}
$$

Now, we will use the confluent hypergeometric function in order to get another form of (2.7). The confluent hypergeometric function $M(s;r;y)$ with arguments s, r and y is defined by

$$
\begin{aligned}
M(s;r;y)&=1+\frac{s}{r}\cdot\frac{y}{1!}+\frac{s(s+1)}{r(r+1)}\cdot\frac{y^2}{2!}+\cdots \\
&=\sum_{j=0}^{\infty}\frac{s(s+1)\cdots(s+j-1)}{r(r+1)\cdots(r+j-1)}\cdot\frac{y^j}{j!}\quad(r>0) \\
&=\frac{\Gamma(r)}{\Gamma(s)\,\Gamma(r-s)}\int_0^1 t^{s-1}\,(1-t)^{r-s-1}\, e^{ty}\, dt\ (s>0)
\end{aligned} \tag{2.8}
$$

Which is the moment generating function of Beta distribution with parameters $s>0$ and $r>0$. (see Johnson and Kotz [12] p. 23~24)

First, put $S=\frac{p}{2}$, $r=\alpha(p-2)+2+s$, and $y=\frac{1}{2}|\underset{\sim}{x}|^2$ in (2.8). Then, for $\alpha>-\frac{2}{p-2}$,

$$\int_0^1 t^{\frac{p}{2}-1}(1-t)^{\alpha(p-2)+1}e^{\frac{1}{2}t|\underset{\sim}{x}|^2}dt$$

$$=\frac{\Gamma(\frac{p}{2})\,\Gamma(\alpha(p-2+2}{\Gamma(\alpha(p-2)+2+\frac{p}{2}}M(\frac{p}{2}: \alpha(p-2)+2+\frac{p}{2};\ \frac{1}{2}|\underset{\sim}{x}|^2). \tag{2.9}$$

Next, put $S=\frac{p}{2}+1$, $r=\alpha(p-2)+1+s$, and $y-\frac{1}{2}|\underset{\sim}{x}|^2$ in (2.8) Then, for $\alpha>-\frac{1}{p-2}$,

$$\int_0^1 t^{\frac{p}{2}}(1-t)^{\alpha(p-2)}e^{\frac{1}{2}t|\underset{\sim}{x}|^2}dt$$

$$=\frac{\Gamma(\frac{p}{2}+1)\,\Gamma(\alpha(p-2)+1)}{\Gamma(\alpha(p-2)+\frac{p}{2}+2)}M(\frac{p}{2}+1;\ \alpha(p-2)+\frac{p}{2}+2;\ \frac{1}{2}|\underset{\sim}{x}|^2). \tag{2.10}$$

Note that both (2.9) and (2.10) hold when $\alpha>-\frac{1}{p-2}$. Hence, using (2.9) and (2.10) for $\alpha>-\frac{1}{p-2}$, (2.7) reduces to

$$\underset{\sim}{\delta}^*(\underset{\sim}{x})=\left[1-\frac{2[\alpha(p-2)+1]}{|\underset{\sim}{x}|^2}+\frac{p}{|\underset{\sim}{x}|^2}\cdot\frac{\frac{\Gamma(\frac{p}{2})\Gamma(\alpha(p-2)+2)}{\Gamma(\alpha(p-2)+2+\frac{p}{2})}M(\frac{p}{2};\ \alpha(p-2)+2+\frac{p}{2};\ \frac{1}{2}|\underset{\sim}{x}|^2}{\frac{\Gamma(\frac{p}{2}+1)\,\Gamma(\alpha(p-2)+1)}{\Gamma(\alpha(p-2)+\frac{p}{2}+2)}M(\frac{p}{2}+1;\ \alpha(p-2)+\frac{p}{2}+2;\ \frac{1}{2}|\underset{\sim}{x}|^2)}\right]\underset{\sim}{x}$$

$$=\left[1-\frac{2[\alpha(p-2)+1]}{|\underset{\sim}{x}|^2}+\frac{p}{|\underset{\sim}{x}|^2}\cdot\frac{\alpha(p-2)+1}{\frac{p}{2}}\cdot\frac{M(\frac{p}{2};\ \alpha(p-2)+\frac{p}{2}+2;\ \frac{1}{2}|\underset{\sim}{x}|^2}{M(\frac{p}{2}+1;\ \alpha(p-2)+\frac{p}{2}+2;\ \frac{1}{2}|\underset{\sim}{x}|^2}\right]\underset{\sim}{x}$$

$$=\left[1-\frac{2[\alpha(p-2)+1]}{|\underset{\sim}{x}|^2}\left\{1-\frac{M(\frac{p}{2};\ \alpha(p-2)+\frac{p}{2}+2;\ \frac{1}{2}|\underset{\sim}{x}|^2}{M(\frac{p}{2}+1;\ \alpha(p-2)+\frac{p}{2}+2;\frac{1}{2}|\underset{\sim}{x}|^2}\right\}\right]\underset{\sim}{x}$$

$$=\left[1-\frac{r(|\underset{\sim}{x}|^2)}{|\underset{\sim}{x}|^2}\right]\underset{\sim}{x} \qquad 2.11)$$

Where $r(|\underset{\sim}{x}|^2)=2[\alpha(p-2)+1]\left\{1-\frac{M(\frac{p}{2};\ \alpha(p-2)+\frac{p}{2}+2;\ \frac{1}{2}|\underset{\sim}{x}|^2}{M(\frac{p}{2}+1;\ \alpha(p-2)+\frac{p}{2}+2;\frac{1}{2}|\underset{\sim}{x}|^2)}\right\}.$

Remark 2.2 $\underset{\sim}{\delta}^*(\underset{\sim}{x})$ is a spherically symetric estimator about O, since

$\underset{\sim}{\delta}^*(p\underset{\sim}{x})=P\underset{\sim}{\delta}^*(\underset{\sim}{x})$

for any orthogonal transformation P relative to the origin O.

3. Minimaxity and Admissibility of Two Stage Bayes Estimators

3.1. Minimaxity

We will consider a general situation as follows:

Suppose that the conditional (proper) prior density of $\underset{\sim}{\theta}$ given t is given by (1.2.2) in chapter 1. Let t have the marginal prior (possibly improper) density $h(t)$ such that the resulting marginal density $m(\underset{\sim}{x})$ of $\underset{\sim}{X}$ is finite for all $\underset{\sim}{x}\in R^p$ (and bence the posterior density $\pi(\underset{\sim}{\theta}|\underset{\sim}{x})$ of $\underset{\sim}{\theta}$ given $\underset{\sim}{X}=\underset{\sim}{x}$ is a proper density).

The following Lemma gives a sufficient condition on $h(t)$ under which the Bayes (possibly generalized) estimator has risk less than P, the risk of $\underset{\sim}{X}$, and hence is minimax.

Lemma 3.1.1. If the marginal (prior) density $h(t)$ of t is differentiable in t, if $\lim_{t\to 0^+} h(t)<\infty$, and if $l(t)=\frac{h'(t)}{h(t)}(t-1)$ is well defined for $0<t<1$, then the Bayes (possibly generalized) estimator $\hat{\underset{\sim}{\theta}}(\underset{\sim}{X})=E(\underset{\sim}{\theta}|\underset{\sim}{X})$ has risk $R(\underset{\sim}{\theta},\hat{\underset{\sim}{\theta}})$ less than P if

(i) $l(t)$ is nondecreasing in t, and

(ii) $l(1^-)=\lim_{t\to 1^-} l(t)\leq \frac{p}{2}-3$.

Proof. The proof of this lemma is given in Appendix.

Remark 3.1.1. The above lemma can be derived, with some modification, from Theorem 1 in Faith [10] (see also Corollary 1 following Theorem 1 in his paper). Although the proof of this lemma can also follows, with some modification, the same lines with the Proof of Theorem 1 in Faith, we give the proof in Appendix for the purpose of completeness.

Now, applying Lemma 3.1.1 to our problem of Chapter 1 in which

$$h(t) \propto t^{\frac{p}{2}} (1-t)^{\alpha(p-2)-\frac{p}{2}},\ 0<t<1,\ \text{(see (1.2.3))}$$

yields the following theorem:

Theorem 3.1.1. In the setting of Chapter 1, the Bayes (Possibly generalized) estimator $\underset{\sim}{\delta}^{\pi}(\underset{\sim}{X})$ in (2.11) has risk less than P, the risk of $\underset{\sim}{X}$, (and hence is minimax) if $-\frac{1}{p-2}<\alpha\leq 1-\frac{1}{p-2}$.

Proof. With $h(t) \propto t^{\frac{p}{2}}(1-t)^{\alpha(p-2)-\frac{p}{2}}$, $0<t<1$, we have $h'(t)=h(t)\left[\frac{p}{2t}-\frac{\alpha(p-2)-\frac{p}{2}}{1-t}\right]$ so that $l(t)=\frac{h'(t)}{h(t)}(t-1)=\alpha(p-2)-\frac{p}{2}-\frac{1-t}{t}\cdot\frac{p}{2}$, $0<t<1$, which is well defined and is obviously nondecreasing in t, $0<t<1$. Also note that $\lim_{t\to 0^+} h(t)=0<\infty$. Now, consider condition (ii) of Lemma 3.1.1 Since $l(1^-)=\alpha(p-2)-\frac{p}{2}$, the Bayes (possibly generalized) estimator $\underset{\sim}{\delta}^{\pi}(\underset{\sim}{X})$ in (2.11) has risk less than p if $-\frac{1}{p-2}<\alpha\leq 1-\frac{1}{p-2}$. Note that $\underset{\sim}{\delta}^{\pi}(\underset{\sim}{X})$ in (2.11) is defined only for $\alpha>-\frac{1}{p-2}$.

Remark 3.1.2. Since the marginal density $m(\underset{\sim}{x})$ of $\underset{\sim}{X}$ is finite for all $\underset{\sim}{x}\in \boldsymbol{R}^p$ which was shown in Chapter 1, the Bayes (Possibly generalized) estimator $\hat{\underset{\sim}{\theta}}(\underset{\sim}{X})=\underset{\sim}{\delta}^{\pi}(\underset{\sim}{x})$ is well definded and is minimax for $-\frac{1}{p-2}<\alpha\leq 1-\frac{1}{p-2}$. Recall that Bayes estimators

$\underset{\sim}{\delta}^{\pi}(\underset{\sim}{X})$ in (2.11) is proper Bayes for $\alpha > \frac{1}{2}$, is generalized Bayes for $\alpha \leq \frac{1}{2}$. Hence, for $p=3$ or 4, there exists no (spherically symmetric) proper Bayes minimax estimators of $\underset{\sim}{\theta}$ in our problem. More generally, conditions (i) and (ii) in Lemma 3.1.1 are not met by any proper prior $h(t)$ when $p=3$ or $p=4$, which is consistent with Strawderman's ([19], [20]) result. The proof of this statement proceeds as follows:

When $p=3$ or $p=4$, from (ii) of Lamma 3.1.1, $\lim_{t \to 1^-} l(t) = l(1^-) \leq -1$. Since $l(t)$ is nondecreasing in t by (i) of Lemma 3.1.1, we have

$$l(t) = \frac{h'(t)}{h(t)}(t-1) \leq -1 \text{ for all } 0<t<1,$$

which reduces to

$$\frac{h'(t)}{h(t)} \geq \frac{1}{1-t}, \quad 0<t<1. \tag{3.1.1}$$

Integrating both sides of (3.1.1) over the interval $[t_1, t_2]$ when $0<t_1<t_2<1$, we get, for fixed t_1,

$$h(t_2) \geq \frac{1-t_1}{1-t_2} h(t_1) = \frac{c}{1-t_2}, \text{ say,} \tag{3.1.2}$$

but the right hand side of (3.1.2) is not integrable between t_1 and 1, and hence $h(t_2)$ is not integrable between t_1 and 1 of $0<t_1<1$.

3.2. Admissibility

To get criterions of admissibility and inadmissibility of $\underset{\sim}{\delta}^{\pi}(\underset{\sim}{X})$ in (2.11) is based on Brown's ([8]) result and on the asymptotic expression of the confluent hypergeometric function (see Alam [1]).

Lemma 3.2.1 (Brown [8]):

(i) A generalized Bayes estimator $\underset{\sim}{\delta}^{\pi}(\underset{\sim}{X})$ with respect to some prlor π is admissible if $\underset{\sim}{\delta}^{\pi}(\underset{\sim}{X})$ has uniformly bounded risk $R(\underset{\sim}{\theta}, \underset{\sim}{\delta}^{\pi})$ on the closed convex hull K_{π} of the support of π and

$$\underset{\sim}{x}\cdot\underset{\sim}{\delta}^{\pi}(\underset{\sim}{x})\leq\underset{\sim}{x}\cdot\left(1-\frac{p-2}{|\underset{\sim}{x}|^2}\right)\underset{\sim}{x} \text{ for } |\underset{\sim}{x}|\geq M;$$

for some constant $M<\infty$,

(ii) A generalized Bayes estimator $\underset{\sim}{\delta}^{\pi}(\underset{\sim}{x})$ with respect to some prior π is inadmissible if for some constants $c<p-2$ and $M<\infty$,

$$\underset{\sim}{x}\cdot\underset{\sim}{\delta}^{\pi}(\underset{\sim}{x})\geq\underset{\sim}{x}\cdot\left(1-\frac{c}{|\underset{\sim}{x}|^2}\right)\underset{\sim}{x} \text{ for } |\underset{\sim}{x}|\geq M.$$

Lemma 3.2.2. (Alam [1]). For large y, $M(s;r;y)=\frac{\Gamma(r)}{\Gamma(s)}e^{y}y^{s-r}\{1+(1-s)(r-s)y^{-1}+O(y^{-2})\}$. In order to apply Lemma 3.2.1 and 3.2.2 to our problem, we need to show that $\underset{\sim}{\delta}^{\pi}(\underset{\sim}{x})$ has uniformly bounded risk (on R^p). But, as was seen in Brown [4], this is equivalent to show that

$$\underset{\sim}{\gamma}_{\pi}(\underset{\sim}{x})=\underset{\sim}{\delta}^{\pi}(\underset{\sim}{x})-\underset{\sim}{x}=-\frac{r(|\underset{\sim}{x}|^2)}{|\underset{\sim}{x}|^2}\underset{\sim}{x} \text{ is uniformly bounded on } R^p.$$

To do this, we need to know about the behavior of $r(|\underset{\sim}{x}|^2)$, and we proceed this with the following lemma:

Lemma 3.2.3. Let $g(y)=\left(\sum_{i=0}^{\infty}b_iy^i\right)\Big/\left(\sum_{i=0}^{\infty}a_iy^i\right)$ where a_i, b_i are nonnegative, and $\sum a_iy^i$ converge for all $y>0$.

If the sequence $\{b_i/a_i\}$ is monotone nondecreasing (nonincreasing), then $g(y)$ is monotone nondecreasing (nonincreasing) in y.

This lemma can be shown by differentiating $g(y)$ (see Lehamann [13], problem 4(i), page 312)

Now applying Lemma 3.2.3 to

$$\frac{M\left(\frac{p}{2};\frac{p}{2}+\alpha(p-2)+2;\frac{1}{2}|\underset{\sim}{x}|^2\right)}{M\left(\frac{p}{2}+1;\frac{p}{2}+\alpha(p-2)+2;\frac{1}{2}|\underset{\sim}{x}|^2\right)}=\frac{\sum_{j=0}^{\infty}b_jy^j}{\sum_{j=0}^{\infty}a_jy^j}$$

where $(c)_j=c(c+1)\cdots\cdots(c+j-1)$ with $(c)_0=1$,

$$b_j=\frac{\left(\frac{p}{2}\right)_j}{\left(\frac{p}{2}+\alpha(p-2)+2\right)_j}\cdot\frac{1}{j!},\quad a_j=\frac{\left(\frac{p}{2}+1\right)_j}{\left(\frac{p}{2}+\alpha(p-2)+2\right)_j}\cdot\frac{1}{j!}\ \text{ and } y=\frac{1}{2}|\underset{\sim}{x}|^2.$$

Then $\frac{b_j}{a_j}=\frac{\left(\frac{p}{2}\right)_j}{\left(\frac{p}{2}+1\right)_j}=\frac{p}{p+2j}$, which is clearly nonincreasing in j and hence

$\dfrac{M\left(\frac{p}{2};\frac{p}{2}+\alpha(p-2)+2;\frac{1}{2}|\underset{\sim}{x}|^2\right)}{M\left(\frac{p}{2}+1;\frac{p}{2}+\alpha(p-2)+2;\frac{1}{2}|\underset{\sim}{x}|^2\right)}$ is nonincreasing in $\frac{1}{2}|\underset{\sim}{x}|^2$, i.e., in $|\underset{\sim}{x}|^2$, and hence $r(|\underset{\sim}{x}|)^2$ is nondecreasing in $|\underset{\sim}{x}|^2$.

In addition to this, consider the behavior of $r(|\underset{\sim}{x}|)^2$ at $+\infty$. To do this, use Lemma 3.2.2. Then for large $|\underset{\sim}{x}|^2$,

$$(|\underset{\sim}{x}|^2)=2[\alpha(p-2)+1]\left\{1-\frac{M\left(\frac{p}{2};\frac{p}{2}+\alpha(p-2)+2;\frac{1}{2}|\underset{\sim}{x}|^2\right)}{M\left(\frac{p}{2}+1;\frac{p}{2}+\alpha(p-2)+2;\frac{1}{2}|\underset{\sim}{x}|^2\right)}\right\}$$

$$=2[\alpha(p-2)+1]\left\{1-\frac{\frac{\Gamma\left(\frac{p}{2}+\alpha(p-2)+2\right)}{\Gamma\left(\frac{p}{2}\right)}e^{\frac{1}{2}|\underset{\sim}{x}|^2}\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{\alpha-(p-2)-2}}{\frac{\Gamma\left(\frac{p}{2}+\alpha(p-2)+2\right)}{\Gamma\left(\frac{p}{2}+1\right)}e^{\frac{1}{2}|\underset{\sim}{x}|^2}\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{-\alpha(p-2)-2}}\right.$$

$$\left.\frac{\left\{1+\left(1-\frac{p}{2}\right)(\alpha(p-2)+2)\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{-1}+O\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{-2}\right\}}{\left\{1-\frac{p}{2}(\alpha(p-2)+1)\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{-1}+O\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{-2}\right\}}\right\}$$

$$=2[\alpha(p-2)+1]\left\{1-\frac{p}{2}\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{-1}\right.$$

$$\left.\left[\frac{1+\left(1-\frac{p}{2}\right)(\alpha(p-2)+2)\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{-1}+O\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{-2}}{1-\frac{p}{2}(\alpha(p-2)+1)\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{-1}+\frac{1}{2}(|\underset{\sim}{x}|^2)^{-1}+O\left(\frac{1}{2}|\underset{\sim}{x}|^2\right)^{-2}}\right]\right\}$$

$$=2[\alpha(p-2)+1]\left\{1-p(|\underset{\sim}{x}|^2)^{-1}\left[1+\frac{2[\alpha(p-2)+2]-p}{|\underset{\sim}{x}|^2}+O(|\underset{\sim}{x}|^2)^{-2}\right]\right\}$$

$$=2[\alpha(p-2)+1]\left\{1-p(|\underset{\sim}{x}|^2)^{-1}+O(|\underset{\sim}{x}|^2)^{-2}\right\}$$

$$=2[\alpha(p-2)+1]\left\{1-p(|\underset{\sim}{x}|^2)^{-1}+(|\underset{\sim}{x}|^2)^{-2}[(|\underset{\sim}{x}|^2)^2O(|\underset{\sim}{x}|^2)^{-2}]\right\}$$

$$=2[\alpha(p-2)+1]\left\{1-p(|\underset{\sim}{x}|^2)^{-1}+o(|\underset{\sim}{x}|^2)^{-1}\right\},$$

and hence $\lim_{|\underset{\sim}{x}|^2\to\infty} r(|\underset{\sim}{x}|^2)=2[\alpha(p-2)+1]$.

So far, we have shown that

① $r(|\underset{\sim}{x}|^2)$ is nondecreasing in $|\underset{\sim}{x}|^2$, and

② $\lim_{|\underset{\sim}{x}|^2\to\infty} r(|\underset{\sim}{x}|^2)=2[\alpha(p-2)+1]$.

Therefore $r_x(\underset{\sim}{x})=\underset{\sim}{\delta}^*(\underset{\sim}{x})-\underset{\sim}{x}=-\frac{r(|\underset{\sim}{x}|^2)}{|\underset{\sim}{x}|^2}\underset{\sim}{x}$ is uniformly bounded with probability 1 on R^p.

Next check the second condition of Lemma 3.2.1, using (i).

Lemma 3.2.2. Then for large $|\underset{\sim}{x}|^2$,

$$\begin{aligned}\underset{\sim}{x}\cdot\delta^*(\underset{\sim}{x})&=\underset{\sim}{x}\cdot\left[1-\frac{r(|\underset{\sim}{x}|^2)}{|\underset{\sim}{x}|^2}\right]\underset{\sim}{x}\\&=\underset{\sim}{x}\cdot\underset{\sim}{x}-r(|\underset{\sim}{x}|^2)\\&=\underset{\sim}{x}\cdot\underset{\sim}{x}-2[\alpha(p-2)+1]\{1-p(|\underset{\sim}{x}|^2)^{-1}+o(|\underset{\sim}{x}|^2)^{-1}\}.\end{aligned}$$

By Lemma 3.2.1 (i), $\underset{\sim}{\delta}^*(\underset{\sim}{x})$ is admissible if there exist some constant $M<\infty$ such that $\underset{\sim}{x}\cdot\underset{\sim}{x}-2[\alpha(p-2)+1]\ \{1-p(|\underset{\sim}{x}|^2)^{-1}+o(|\underset{\sim}{x}|^2)^{-1}\}\leq\underset{\sim}{x}\cdot\underset{\sim}{x}-(p-2)$ for $|\underset{\sim}{x}|^2\geq M$, i.e., $2[\alpha(p-2)+1]\ \{1-p(|\underset{\sim}{x}|^2)^{-1}+o(|\underset{\sim}{x}|^2)^{-1}\}\geq p-2$ for $|\underset{\sim}{x}|^2\geq M$ (3.2.1). Letting $|\underset{\sim}{x}|^2\to\infty$ in (3.2.1), we have $2[\alpha(p-2)+1]\geq p-2$.

But when $2[\alpha(p-2)+1]=p-2$, there existant any constant $M<\infty$ such that (3.2.1) holds for $|\underset{\sim}{x}|^2\geq M$. Hence $2[\alpha(p-2)+1]>p-2$, i.e., $\alpha>\frac{1}{2}-\frac{1}{p-2}$.

Similarly, by Lemma 3.2.1 (ii), $\underset{\sim}{\delta}^*(\underset{\sim}{x})$ is inadmissible if for some constants $c<p-2$ and $M<\infty$,

$$2[\alpha(p-2)+1]\ \{1-p(|\underset{\sim}{x}|^2)^{-1}+o(|\underset{\sim}{x}|^2)^{-1}\}\leq c \text{ for } |\underset{\sim}{x}|^2\geq M.$$

Letting $|\underset{\sim}{x}|^2\to\infty$, we have $2[\alpha(p-2)+1]\leq c$ for some $c<p-2$, i.e., $2[\alpha(p-2)+1]<p-2$, i.e., $\alpha<\frac{1}{2}-\frac{1}{p-2}$,

Recall that $\underset{\sim}{\delta}^*(\underset{\sim}{x})$ in (2.11) is defined only for $\alpha>-\frac{1}{p-2}$.

Hence, we have the following theorem for the admissibility of $\underset{\sim}{\delta}^*(\underset{\sim}{x})$.

Theorem 3.2.1. Let $\underset{\sim}{\delta}^*(\underset{\sim}{X})$ be as in (2.11). If $\alpha > \frac{1}{2} - \frac{1}{p-2}$, then $\underset{\sim}{\delta}^*(\underset{\sim}{x})$ is admissible, and if $-\frac{1}{p-2} < \alpha < \frac{1}{2} - \frac{1}{p-2}$, then $\underset{\sim}{\delta}^*(\underset{\sim}{x})$ is inadmissible.

Remark 3.2.1. We cannot conclude admissibility and inadmissibility of $\underset{\sim}{\delta}^*(\underset{\sim}{X})$ in case of $\alpha = \frac{1}{2} - \frac{1}{p-2}$, since the estimator satisfies neither conditions (i) nor (ii) in Lemma A.1.

Combining Theorem 3.1.1 with Theorem 3.2.1 gives the following theorem.

Theorem 3.2.2. For the class of prior densities $h(t) \propto t^{\frac{p}{2}}(1-t)^{\alpha(p-2)-\frac{p}{2}}$, $0<t<1$, with $\frac{1}{2} < \alpha \leq 1 - \frac{1}{p-2}$, $p \geq 5$, a class of proper Bayes minimax admissible estimators is given by

$$\underset{\sim}{\delta}^*(\underset{\sim}{x}) = \left[1 - \frac{r(|\underset{\sim}{x}|^2)}{|\underset{\sim}{x}|^2}\right]\underset{\sim}{x},$$

where $$r(|\underset{\sim}{x}|^2) = 2[\alpha(p-2)+1]\left\{1 - \frac{M\left(\frac{p}{2};\alpha(p-2)+\frac{p}{2}+2;\frac{1}{2}|\underset{\sim}{x}|^2\right)}{M\left(\frac{p}{2}+1;\alpha(p-2)+\frac{p}{2}+2;\frac{1}{2}|\underset{\sim}{x}|^2\right)}\right\}.$$

Appendix

In this appendix we will prove Lemma 3.1.1.

Proof of Lemma 3.1.1: Let $\lambda = 1-t$. This is strictly monotone decreasing function of t. The random parameter λ has the density

$$d(\lambda) = h(1-\lambda), \quad o<\lambda<1 \quad (d(1)=d(1^-)=h(0^+)<\infty)$$

Differentiate this to obtain the function

$$e(\lambda) = \frac{d'(\lambda)}{d(\lambda)}\lambda = -\frac{h'(1-\lambda)}{h(1-\lambda)}\lambda$$
$$= l(t(\lambda)), \text{ where } t(\lambda) = 1-\lambda$$

Observe the conditions (i) and (ii) are equivalent to

(i)′ $e(\lambda)$ is nonincreasing in λ, and

(ii)′ $e(0^+) \leq \frac{p}{2} - 3$.

At this point, we need the following two lemma: One provides a convenient expression for the generalized Bayes estimator of $\underset{\sim}{\theta}$, and the other (which is due to Strawderman [19]) gives sufficient conditions in order for an estimator $\hat{\underset{\sim}{\theta}}$ of $\underset{\sim}{\theta}$ to be minimax.

Lemma A.1. Under the above specification of the prior density for $\underset{\sim}{\theta}$, the generalized Bayes estimator of $\underset{\sim}{\theta}$ is given by

$$E(\underset{\sim}{\theta}|\underset{\sim}{X}) = [E(t|\underset{\sim}{X})]\underset{\sim}{X} = [1 - E(1-t|\underset{\sim}{X})]\underset{\sim}{X} = [1 - E(1-t|Y)]\underset{\sim}{X}$$
$$= [E(t|Y)]\underset{\sim}{X} \text{ where } Y = |\underset{\sim}{X}|^2 = \sum_{i=1}^{p} X_i^2.$$

Proof. Note that $E(\underset{\sim}{\theta}|\underset{\sim}{x}) = E\{E[\underset{\sim}{\theta}|t, \underset{\sim}{x}]\underset{\sim}{x}\}$ Now the joint density of $\underset{\sim}{\theta}$, t, and $\underset{\sim}{X}$ is

$$f(\underset{\sim}{x}, \underset{\sim}{\theta}, t) = (2\pi)^{-\frac{p}{2}} e^{-\frac{1}{2}|\underset{\sim}{x}-\underset{\sim}{\theta}|^2} (2\pi)^{-\frac{p}{2}} \left(\frac{1-t}{t}\right)^{\frac{p}{2}} e^{-\frac{1-t}{2t}|\underset{\sim}{\theta}|^2} h(t)$$
$$= (2\pi)^{-p} \left(\frac{1-t}{t}\right)^{\frac{p}{2}} e^{-\frac{1}{2}|\underset{\sim}{x}-\underset{\sim}{\theta}|^2 - \frac{1-t}{2t}|\underset{\sim}{\theta}|^2} h(t),$$
$$\underset{\sim}{x} \in \boldsymbol{R}^p, \ \underset{\sim}{\theta} \in \boldsymbol{R}^p, \ 0 < t < 1.$$

and the Joint density of $\underset{\sim}{x}$ and t is

$$f(\underset{\sim}{x}, t) = \int_{R^p} (2\pi)^{-p} \left(\frac{1-t}{t}\right)^{\frac{p}{2}} e^{-\frac{1}{2}|\underset{\sim}{x}-\underset{\sim}{\theta}|^2 - \frac{1-t}{2t}|\underset{\sim}{\theta}|^2} h(t) d\underset{\sim}{\theta}$$
$$= \int_{R^p} (2\pi)^{-p} \left(\frac{1-t}{t}\right)^{\frac{p}{2}} h(t) e^{-\frac{1}{2t}|\underset{\sim}{\theta} - t\underset{\sim}{x}|^2 - \frac{1}{2}(1-t)|\underset{\sim}{x}|^2} d\underset{\sim}{\theta}$$
$$= (2\pi)^{-\frac{p}{2}} (1-t)^{\frac{p}{2}} h(t) e^{-\frac{1}{2}(1-t)|\underset{\sim}{x}|^2}, \underset{\sim}{x} \in \boldsymbol{R}^p, \ 0 < t < 1,$$

and hence the conditional density of $\underset{\sim}{\theta}$ given $\underset{\sim}{X} = \underset{\sim}{x}$ and t is

$$g(\underset{\sim}{\theta}|t, \underset{\sim}{x}) = (2\pi)^{-\frac{p}{2}} t^{-\frac{p}{2}} e^{-\frac{1}{2t}|\underset{\sim}{\theta} - t\underset{\sim}{x}|^2}, \ \underset{\sim}{\theta} \in \boldsymbol{R}^p$$

Hence $E[\underset{\sim}{\theta}|t, \underset{\sim}{x}] = \int_{R^p} \underset{\sim}{\theta} g(\underset{\sim}{\theta}|t, \underset{\sim}{x}) d\underset{\sim}{\theta}$

$$= t\underset{\sim}{x}$$

and So $E(\underset{\sim}{\theta}|\underset{\sim}{x}) = E(t\underset{\sim}{x}|\underset{\sim}{x}) = [E(t|\underset{\sim}{x})]\underset{\sim}{x} = [1-E(1-t|\underset{\sim}{x})]\underset{\sim}{x}$.

Now the conditional density of t given $\underset{\sim}{X}=\underset{\sim}{x}$ is

$$f(t|\underset{\sim}{x}) = \frac{(2\pi)^{-\frac{p}{2}}(1-t)^{\frac{p}{2}}h(t)e^{-\frac{1}{2}(1-t)|\underset{\sim}{x}|^2}}{\int_0^1 (2\pi)^{-\frac{p}{2}}(1-t)^{\frac{p}{2}}h(t)e^{-\frac{1}{2}(1-t)|\underset{\sim}{x}|^2}\,dt}$$

which depends on $\underset{\sim}{x}$ only through $y=|\underset{\sim}{x}|^2$. Hence

$$[E(t|\underset{\sim}{x})]\underset{\sim}{x} = [E(t|y)]\underset{\sim}{x} = [1-E(1-t|y)]\underset{\sim}{x}.$$

Lemma A.2. (Strawderman [19]). Let $\hat{\underset{\sim}{\theta}} = \left[1-\frac{(p-2)S(|\underset{\sim}{x}|^2)}{|\underset{\sim}{x}|^2}\right]\underset{\sim}{x}$, where $p \geq 3$, and $S(\cdot)$ is any nondecreasing function such that $0 \leq S(\cdot) \leq 2$, $S(\cdot) \not\equiv 0$, and $S(\cdot) \not\equiv 2$. Then the estimator $\hat{\underset{\sim}{\theta}}$ has risk $R(\underset{\sim}{\theta}, \hat{\underset{\sim}{\theta}}) = E_{\underset{\sim}{\theta}}[|\underset{\sim}{\theta}-\hat{\underset{\sim}{\theta}}|^2] > P$ for all $\underset{\sim}{\theta} \in R^p$.

From Lemma A.1 and Lemma 4.2,

$$\begin{aligned} S(|\underset{\sim}{x}|^2) &= \frac{|\underset{\sim}{x}|^2}{p-2}E(1-t|\underset{\sim}{X}=\underset{\sim}{x}) \\ &= \frac{|\underset{\sim}{x}|^2}{p-2}E(\lambda|\underset{\sim}{X}=\underset{\sim}{x}) \\ &= \frac{|\underset{\sim}{x}|^2}{p-2}\frac{\int_0^1 \lambda d(\lambda)\, f(\underset{\sim}{x}|\lambda)d\lambda}{\int_0^1 d(\lambda)\, f(\underset{\sim}{x}|\lambda)d\lambda} \end{aligned}$$

Now

$$\begin{aligned} f(\underset{\sim}{x},\lambda) &= \int_{R^p} f(\underset{\sim}{x},\underset{\sim}{\theta},\lambda)d\underset{\sim}{\theta} \\ &= \int_{R^p} (2\pi)^{-\frac{p}{2}} e^{-\frac{1}{2}|\underset{\sim}{x}-\underset{\sim}{\theta}|^2} d(\lambda)(2\pi)^{-\frac{p}{2}}\left(\frac{\lambda}{1-\lambda}\right)^{\frac{p}{2}} e^{-\frac{\lambda}{2(1-\lambda)}|\underset{\sim}{\theta}|^2}d\underset{\sim}{\theta} \\ &= (2\pi)^{-\frac{p}{2}}\lambda^{\frac{p}{2}} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}, \quad \underset{\sim}{x}\in R^p, \quad 0<\lambda<1. \end{aligned}$$

Therefore $f(\underset{\sim}{x}|\lambda) = (2\pi)^{-\frac{p}{2}}\lambda^{\frac{p}{2}}e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}$ and hence

$$S(|\underset{\sim}{x}|^2) = \frac{|\underset{\sim}{x}|^2}{p-2}\frac{\int_0^1 \lambda^{\frac{p}{2}+1} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}d\lambda}{\int_0^1 \lambda^{\frac{p}{2}} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}d\lambda}$$

Integrating the numerator by parts gives

$$\int_0^1 \lambda^{\frac{p}{2}+1} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\, d\lambda = \lambda^{\frac{p}{2}+1} d(\lambda)\left(-\frac{2}{|\underset{\sim}{x}|^2}e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\right)_{\lambda=0}^{\lambda=1}$$

$$+\frac{2}{|\underset{\sim}{x}|^2}\int_0^1\left[\left(\frac{p}{2}+1\right)\lambda^{\frac{p}{2}} d(\lambda)+\lambda^{\frac{p}{2}+1} d'(\lambda)\right] e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2} d\lambda$$

$$=-d(1)\frac{2}{|\underset{\sim}{x}|^2}\, e^{-\frac{1}{2}|\underset{\sim}{x}|^2}+\frac{2}{|\underset{\sim}{x}|^2}\left(\frac{p}{2}+1\right)\int_0^1 \lambda^{\frac{p}{2}} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\, d\lambda$$

$$+\frac{2}{|\underset{\sim}{x}|^2}\int_0^1 \lambda^{\frac{p}{2}+1} d'(\lambda)\, e^{\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\, d\lambda \text{ (using } d(1)=d(1^-)<\infty)$$

and hence $S(|\underset{\sim}{x}|^2)=\frac{2}{p-2}\left\{\left(\frac{p}{2}+1\right)+\frac{\int_0^1 \lambda^{\frac{p}{2}+1} d'(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\, d\lambda}{\int_0^1 \lambda^{\frac{p}{2}} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\, d\lambda}-\frac{d(1)\, e^{-\frac{1}{2}|\underset{\sim}{x}|^2}}{\int_0^1 \lambda^{\frac{p}{2}} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\, d\lambda}\right\}$

$$=\frac{2}{p-2}\left\{\left(\frac{p}{2}+1\right)+\frac{\int_0^1 \lambda^{\frac{p}{2}} e(\lambda)\, d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\, d\lambda}{\int_0^1 \lambda^{\frac{p}{2}} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\, d\lambda}-\frac{d(1)}{\int_0^1 \lambda^{\frac{p}{2}} d(\lambda)\, e^{\frac{1}{2}(1-\lambda)|\underset{\sim}{x}|^2}\, d\lambda}\right\}$$

$$=\frac{2}{p-2}\left\{\left(\frac{p}{2}+1\right)+E[e(\lambda)\,|\,\underset{\sim}{X}=\underset{\sim}{x}]-\frac{d(1)}{\int_0^1 \lambda^{\frac{p}{2}} d(\lambda)\, e^{\frac{1}{2}(1-\lambda)|\underset{\sim}{x}|^2}\, d\lambda}\right\}$$

$$=\frac{2}{p-2}\left\{\left(\frac{p}{2}+1\right)+A+B\right\}$$

Claim 1. $S(|\underset{\sim}{x}|^2)$ is nondecreasing.

Proof. $B=-\frac{d(1)}{\int_0^1 \lambda^{\frac{p}{2}} d(\lambda)\, e^{\frac{1}{2}(1-\lambda)|\underset{\sim}{x}|^2}\, d\lambda}$

$\begin{cases} =0 \quad \text{if } d(1)=0 \\ \text{is increasing function of } |\underset{\sim}{x}|^2 \text{ if } d(1)>0 \text{ since } 0<\lambda<1. \end{cases}$

and $A=E(e(\lambda)\,|\,\underset{\sim}{X}=\underset{\sim}{x})=E(e(\lambda)\,|\,Y=y)$ where $Y=|\underset{\sim}{X}|^2$ and $y=|\underset{\sim}{x}|^2$,

Since

$$f(\lambda\,|\,\underset{\sim}{x})=\frac{\lambda^{\frac{p}{2}} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}}{\int_0^1 \lambda^{\frac{p}{2}} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\, d\lambda}$$

depends on $\underset{\sim}{x}$ only through $|\underset{\sim}{x}|^2=y$.

For $0<\lambda_1<\lambda_2<1$, $0<y_1<y_2$,

$$f^*(\lambda_1|y_1)\ f^*(\lambda_2|y_2)-f^*(\lambda_1|y_2)\ f^*(\lambda_2|y_1)$$
$$=\frac{(\lambda_1\lambda_2)^{\frac{p}{2}}\,d(\lambda_1)\,d(\lambda_2)}{\int_0^1\lambda^{\frac{p}{2}}\,d(\lambda)\,e^{-\frac{1}{2}\lambda y_1}\,d\lambda.\int_0^1\lambda^{\frac{p}{2}}\,d(\lambda)\,e^{-\frac{1}{2}\lambda y_2}\,d\lambda}\left[e^{-\frac{1}{2}\lambda_1 y_1-\frac{1}{2}\lambda_2 y_2}-e^{-\frac{1}{2}\lambda_1 y_2-\frac{1}{2}\lambda_2 y_1}\right]$$

Now

$$\frac{e^{-\frac{1}{2}\lambda_1 y_1-\frac{1}{2}\lambda_2 y_2}}{e^{-\frac{1}{2}\lambda_1 y_2-\frac{1}{2}\lambda_2 y_1}}=e^{-\frac{1}{2}\lambda_1(y_1-y_2)-\frac{1}{2}\lambda_2(y_2-y_1)}=e^{\frac{1}{2}(\lambda_1-\lambda_2)(y_2-y_1)}<1$$

when $0<\lambda_1<\lambda_2<1$, $0<y_1<y_2$. Hence $f^*(\lambda_1|y_1)\ f^*(\lambda_2|y_2)-f^*(\lambda_1|y_2)\ f^*(\lambda_2|y_1)<0$ whenever $0<\lambda_1<\lambda_2<1$, $0<y_1<y_2$. This shows that the family of conditional densities $f^*(\lambda|y)$ has monotone likelihood ratio in $-y$. Hence, since $e(\lambda)$ is nonincreasing in λ, using a well-known result (see Lehmann [13], p. 74, Lemma 2). It follows that $E(e(\lambda)|y)$ is nondecreasing in $y>0$.

Hence $S(|\underset{\sim}{x}|^2)$ is nondecreasing in $|\underset{\sim}{x}|^2$.

Claim 2. $S(|\underset{\sim}{x}|^2)\geq 0$ and $S(|\underset{\sim}{x}|^2)\not\equiv 0$.

Proof. $p(S(|\underset{\sim}{x}|^2)>0)>0$ since $p(\lambda)>0)=1$.

Claim 3. $S(|\underset{\sim}{x}|^2)\leq\frac{2}{p-2}\left\{\left(\frac{p}{2}+1\right)+\lim\limits_{\substack{\lambda\to 0\\(\lambda>0)}}e(\lambda)\right\}\leq 2.$

Proof. Since $-\frac{d(1)}{\int_0^1\lambda^{\frac{p}{2}}\,d(\lambda)\,e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2}\,d\lambda}\leq 0\ \forall\underset{\sim}{x}\in\boldsymbol{R}^p$ and $e(\lambda)$ is nonincreasing in λ, $0<\lambda<1$

$$\begin{aligned}S(|\underset{\sim}{x}|^2)&\leq\frac{2}{p-2}\left\{\left(\frac{p}{2}+1\right)+E(e(\lambda)|\underset{\sim}{X}=\underset{\sim}{x})\right\}\\&\leq\frac{2}{p-2}\left\{\left(\frac{p}{2}+1\right)+[\lim_{\substack{\lambda\to 0\\(\lambda>0)}}e(\lambda)]\,E(1|\underset{\sim}{X}=\underset{\sim}{x})\right\}\\&=\frac{2}{p-2}\left\{\left(\frac{p}{2}+1\right)+\lim_{\substack{\lambda\to 0\\(\lambda>0)}}e(\lambda)\right\}\\&\leq\frac{2}{p-2}\left\{\left(\frac{p}{2}+1\right)+\left(\frac{p}{2}-3\right)\right\}\quad\text{by (ii)}'\\&=2\end{aligned}$$

Claim 4. $S(|\underset{\sim}{x}|^2)\not\equiv 2$

Proof. Suppose not, *i.e.*, $S(|\underset{\sim}{x}|^2)\equiv 2$, then

$$-\frac{d(1)}{\int_0^1 \lambda^{\frac{p}{2}} d(\lambda)\, e^{-\frac{1}{2}\lambda|\underset{\sim}{x}|^2} d\lambda}=0, \quad i.e., \quad d(1)=0,$$

and $E(e(\lambda)\,|\,\underset{\sim}{X}=\underset{\sim}{x})\equiv\frac{p}{2}-3.$

This implies that $\lim\limits_{\substack{\lambda\to 1\\(\lambda<1)}} d(\lambda)=0$ and

$$e(\lambda)=\frac{d'(\lambda)}{d(\lambda)}\lambda\equiv\frac{p}{2}-3.$$

From the second equation, $\log d(\lambda)=\left(\frac{p}{2}-3\right)\log\lambda+a$, and hence $d(\lambda)=\left(\frac{p}{2}-3\right)e^a=C\lambda^{\frac{p}{2}-3}$, $C>0$, $0<\lambda<1$, and hence $\lim\limits_{\substack{\lambda\to 1\\(\lambda<1)}} d(\lambda)=C>0$ which contradicts to $\lim\limits_{\substack{\lambda\to 1\\(\lambda<1)}} d(\lambda)=0$. Therefore $S(|\underset{\sim}{x}|^2)\equiv 2$.

Hence, by claim 1, 2, 3 and 4, conditions of Lemma A.2 are satisfied, and so

$$\hat{\underset{\sim}{\theta}}(\underset{\sim}{X})=[1-E(\lambda|\underset{\sim}{X})]\underset{\sim}{X}=[1-E(-t|\underset{\sim}{X})]\underset{\sim}{X}$$

has risk less than that of $\underset{\sim}{X}$ and hence, is minimax. Note that the above proof of lemma 3.1.1 includes the following corollary.

Corollary A.1. If the marginal prior density $d(\lambda)$ of $\lambda=1-t$ is differentiable in t, if $\lim\limits_{\lambda\to 1^-} d(\lambda)<\infty$, and if $e(\lambda)=\frac{d'(\lambda)}{d(\lambda)}\lambda$, $0<\lambda<1$, Satisfies

(i)′ $e(\lambda)$ is nonincreasing in λ, and

(ii)′ $e(0^+)\leq\frac{p}{2}-3$,

then $\hat{\underset{\sim}{\theta}}(\underset{\sim}{X})=E(\underset{\sim}{\theta}|\underset{\sim}{X})$ has risk less than p.

In our problem with

$d(\lambda)=C_1\lambda^{\alpha(p-2)-\frac{p}{2}}(1-\lambda)^{\frac{p}{2}}$, $0<\lambda<1$, $\lim_{\lambda\to 1^-} d(\lambda)=0<\infty$, and, with $\varepsilon=\frac{p}{2}$, $\lim_{\lambda\to 0^+} \lambda^{1+\varepsilon} d(\lambda)=0$, and $d'(\lambda)=d(\lambda)\left(\frac{\alpha(p-2)-\frac{p}{2}}{\lambda}-\frac{\left(\frac{p}{2}\right)}{1-\lambda}\right)$ so that

$$e(\lambda)=\alpha(p-2)-\frac{p}{2}-\frac{\lambda}{1-\lambda}\left(\frac{p}{2}\right),\quad 0<\lambda<1.$$

By Corollary A.1, the Bayes (possibly generalized) estimator has risk less than p(and hence, is minimax) if $-\frac{1}{p-2}<\alpha\leq 1-\frac{1}{p-2}$.

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