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Remark on Nonresonance Below the First Eigenvalue for a Semilinear Elliptic Problem

by

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1. Introduction

Let V be a bounded domain in $\boldsymbol{R}^n$ and $f(x,s)$ a real-valued Caratheodory continuous function defined on $V\times\boldsymbol{R}$ with the primitive

$$F(x,s)=\int_0^s f(x,t)\ dt.$$

Let λ_1 be the first eigenvalue of the Laplace operator on V, and let $v(x)$ be the corresponding eigenfunction. We consider the following three conditions, in which p and q are chosen so that $p=2n/(n+2)$ and $q=n/2$ if $n>2$, $p>1$ and $q>1$ if $n=2$, and $p=q=1$ if $n=1$.

(1) There exist $a(x)\in L^q(V)$ and $b(x)\in L^p(V)$ such that

$$|f(x,s)|\leq a(x)|s|+b(x)$$

for all $s\in\boldsymbol{R}$ and $a.e.\ x\in V$.

(2) $\limsup\limits_{|s|\to\infty}\dfrac{F(x,s)}{s^2}\leq\dfrac{1}{2}\lambda_1$ for $a.e.\ x\in V$.

(3) $\limsup\limits_{|s|\to\infty}\displaystyle\int_V\frac{F(x,sv(x))}{s^2}dx<\frac{1}{2}\lambda_1\int_V v(x)^2\,dx.$

These conditions imply the existence of a weak solution to the semilinear elliptic Dirichlet problem

$$(1.1)\qquad \begin{cases} -\triangle u=f(x,u)+h & in\ V,\\ u=0 & \text{on } \partial V,\end{cases}$$

for any $h\in H^{-1}(V)$, the dual space of the Sobolev space $H_0^1(V)$. By a weak solution we mean a function $u\in H_0^1(V)$ satisfying that

$$\int_V Du\cdot Dw=\int_V f(x,u)w+\langle h,w\rangle$$

for all $w \in H_0^1(V)$. Here by Du we denote the gradient of u and by the bracket $<,>$ we denote the dual pairing between $H^{-1}(V)$ and $H_0^1(V)$.

To prove the preceeding result, we consider the corresponding energy integral

$$J(u) = \frac{1}{2}\int_V |Du|^2 - \int_V F(x,u) + <h,u>$$

defined for $u \in H_0^1(V)$. Under the conditions (1), (2) and (3), we show that the functional $J(u)$ has a minimum in $H_0^1(V)$. This minimum is a weak solution to (1.1).

As an example that satisfies (1), (2) and (3), we have

$$F(x,s) = F(s) = \frac{1}{2}\lambda_1 s^2 (\frac{1}{2} + \frac{1}{2} \sin(\log|s|))$$

modified to be continuously differentiable near $s=0$. The main difficulty is to justify (3). For this we use the function

$$m(t) = \text{measure of the set } \{x \in V \mid v(x) > t\}.$$

By using the smoothness of $m(t)$ we can prove (3) for this example.

Several authors have studied various nonresonance conditions for semilinear elliptic problems.

A classical result due to Hammerstein [2] says that if $f(x,s)$ is continuous, satisfies a linear growth condition and if for some number $\mu < \lambda_1/2$,

$$\limsup_{|s| \to \infty} \frac{F(x,s)}{s^2} \leq \mu$$

uniformly for $x \in V$, then (1.1) has a solution for any h. This result was extended by Mawhin-Ward-Willem [6], who assume $f(x,s)$ grows at most as $|s|^r$ for some $r < (n+2)/(n-2)$ ($r < \infty$ if $n=1$ or $n=2$) and that for some function $\alpha(x) \in L^\infty(V)$ with $\alpha(x) \leq \lambda_1/2$ $a.e.$ in V and $\alpha(x) < \lambda_1/2$ on a subset of positive measure, and

$$\limsup_{|s| \to \infty} \frac{F(x,s)}{s^2} \leq \alpha(x)$$

uniformly for $a.e. x$ in V. Recently, Figueiredo and Gossez [1] found a nonresonance condition that contains the case when

$$\limsup_{|s|\to\infty} \frac{F(x,s)}{s^2} = \lambda_1/2$$

for $a.e.\, x \in V$. They use the notion of positive density of a set at infinity.

Nonresonance between any two consecutive eigenvalues of semilinear elliptic operators is studied by Landesman and Lazer [3], Lazer and Leach [4], Mawhin and Ward [7], and Metzen [9].

Our result contains the classical one and that of [1]. Concerning with the result of [6], ours is a partial generalization in the special case when $f(x,s)$ grows linearly in the variable s.

2. Preliminaries

In this section we give some preliminary lemmas which will be used in the proof of our main results. At first, we start with the following lemma which appears in [5]. We give here a slightly modified proof.

Lemma 2.1. Let X be a finite measure space, u_j and u functions in $L^r(X)$, $1<r<\infty$, such that

$$\{u_j\} \text{ is a bounded sequence and } u_j \longrightarrow u \ a.e. \text{ in } X$$

Then $u_j \longrightarrow u$ in L^r weakly.

Proof. Let $r'=r/(r-1)$. For any natural number N, let

$$E_N = \{x \in X \mid |u_j(x)-u(x)|<1 \text{ for all } j \geq N\}.$$

The measurable set E_N increases with N and measure(E_N) converges to measure(X) as $N \longrightarrow \infty$. Let $\varepsilon>0$ be given, and let w be a function in $L^{r'}$. Let c_N denote the characteristic function of the set E_N. Then the sequence $\{wc_N\}$ converges to w in $L^{r'}$ by the Lebesgue's theorem. Hence there exists N such that

$$\|w - wc_N\|_{L^{r'}} < \varepsilon.$$

Again, by the Lebesgue's theorem, $\{wc_N(u_j-u)\}$ converges to 0 in L^1 as $j \longrightarrow \infty$. By Hölder's inequality, we have

$$\limsup_{j\to\infty} \|w(u_j-u)\|_{L^1}\leq \varepsilon C$$

since $\{u_j\}$ is bounded. Since this holds for any $\varepsilon>0$, we conclude that $\{u_j\}$ converges weakly in $L^r(X)$.

Remark 2.2. Note that $\{|u_j-u|\}$ also converges to 0 weakly in $L^r(X)$.

On account of the following lemma, we were able to remove the uniform convergence from the condition (2).

Lemma 2.3. Assume (1) and (2). Then

$$\limsup_{|s|\to\infty} \int_V \frac{F(x,sw(x))}{s^2}dx\leq \frac{1}{2}\int_V w(x)^2 dx$$

for all w *in* $H_0^1(V)$.

Proof. By (1), we have

$$\frac{F(x,sw(x))}{s^2}\leq \frac{1}{2}a(x)|w(x)|^2+\frac{b(x)|w(x)|}{s}$$

for all $s\in \boldsymbol{R}$ and $a.e. x\in V$. By the Sobolev imbedding theorem and the Hölder's inequality, the right hand side of the above inequality belongs to $L^1(V)$ with any $w\in H_0^1(V)$. Now by applying the Fatou's lemma to the difference of both sides, we obtain

$$\limsup_{|s|\to\infty} \int_V \frac{F(x,sw(x))}{s^2}dx\leq \int_V \limsup_{|s|\to\infty} \frac{F(x,sw(x))}{s^2}\, dx.$$

By (2), the conclusion follows immediately.

Forthcoming two lemmas are essential to the proof of our main results.

Lemma 2.4. Assume (1). Then J is wealky sequentially lower semicontinuous in $H_0^1(V)$.

Proof. Assume that $n>2$ (the idea of proof is the same in the case when $n=2$ or $n=1$).

Let $\{u_j\}$ be a sequence in $H_0^1(V)$ which converges weakly to some u. Without any loss of generality we may assume that $\{J(u_j)\}$ is convergent. Since $\int|Du|^2\leq\liminf_{j\to\infty}$

$\int |Du_j|^2$ and $<h,u>=\lim<h,u_j>$, to complete the proof, it is sufficient to show that

$$\lim_{j\to\infty}\int F(x,u_j)=\int F(x,u)$$

By (1), we have

$$|F(x,u_j)-F(x,u)|\leq a(x)w_j+b(x)v_j$$

with $w_j=(|u|+u_j)u_j/2$ and $v_j=|u_j-u|$.

By the Sobolev imbedding theorem, $\{u_j\}$ is bounded in $L^s(V)$ with $s=2n/(n-2)$ and converges strongly in $L^1(V)$. By passing to a subsequence, we may assume that it converges $a.e.$ to u in V. Then $\{w_j\}$ is bounded in $L^r(V)$ with $r=n/(n-2)$ and converges $a.e.$ to 0 in V. Not that $r'=n/2$. Since $a(x)\in L^{r'}(V)$, by Lemma 2.1, $\{a(x)w_j\}$ converges to 0 in $L^1(V)$. By the similar reason, $\{b(x)v_j\}$ converges to 0 in $L^1(V)$. Thus $\{|F(x,u_j)-F(x,u)|\}$ converges to 0 in $L^1(V)$. This completes the proof.

Lemma 2,5, Assume (1), (2) and (3). Then J is coersive in $H_0^1(V)$, *i.e.*, $J(u)\longrightarrow\infty$ as $\|u\|\longrightarrow\infty$.

Proof. Suppose on the contrary that J is not coersive in $H_0^1(V)$. Then we can choose a sequence $\{u_j\}$ such that

$$\{J(u_j)\}\text{ is bounded in } H_0^1(V),\text{ and } \|u_j\|\longrightarrow\infty.$$

Put $t_j=\|u_j\|$ and $v_j=u_j/t_j$. Note that

$$\frac{1}{2}=\frac{1}{2}|Dv_j|^2\leq\limsup_{j\to\infty}\frac{F(x,t_jv_j)}{t_j^2}.$$

Note also that

$$\frac{F(x,t_jv_j)-F(x,t_jv)}{t_j^2}\leq a(x)\ (|v|+|v_j-v|)\ (v_j-v)+b(x)|v_j-v|/t_j$$

Since $\|v_j\|=1$, by passing to a subsequence and on account of the Sobolev imbedding theorem, we may assume that $\{v_j\}$ converges weakly to some v in $H_0^1(V)$, $\{v_j\}$ is bounded in $L^r(V)$ with $r=2n/(n-2)$ and v_j converges $a.e.$ to v in V. By the same reasoning as in the proof of Lemma 2.3, we deduce that

$$\lim_{j\to\infty} \frac{F(x,t_jv_j)-F(x,t_jv)}{t_j^2}=0.$$

From this and the first inequality in this proof, it follows that

$$\liminf_{j\to\infty} \frac{1}{2}\int |Dv_j|^2 \leq \limsup_{j\to\infty}\int \frac{F(x,t_jv)}{t_j^2}$$

By Lemma 2.3, we obtain

$$\int |Dv|^2 \leq \lambda_1 \int v^2$$

since $\|v\| \leq \liminf_{j\to\infty} \|v_j\|$. Note also that $v \neq 0$. Thus v is the eigenfunction corresponding to the first eigenvalue of the Laplace operator on V. The condition (3) leads us to a contradiction.

3. Main Results

Theorem 3.1. Assume (1), (2) and (3). Then there exists $u \in H_0^1(V)$ such that $J(u) \leq J(w)$ for all $w \in H_0^1(V)$.

Proof. Let $\{u_j\}$ be a minimizing sequence of J in $H_0^1(V)$. Then $J(u)$ is bounded above, and hence $\{u_j\}$ is bounded in $H_0^1(V)$ by Lemma 2.5. By passing to a subsequence, we may assume that it converges to some u in $H_0^1(V)$. By Lemma 2.4, we have

$$J(u) \leq \liminf_{j\to\infty} J(u_j).$$

Hence $J(u)$ is the minimum value of J in $H_0^1(V)$.

Theorem 3.2. Assume (1), (2) and (3). Then the problem (1.1) has a weak solution in $H_0^1(V)$.

Proof. In view of the preceeding theorem, it is sufficient to show that J is Gateaux differentiable in $H_0^1(V)$. For this, note that

$$\begin{aligned}(F(x,u+tw)-F(x,u))/t &= \int_u^{u+tw} f(x,s)\,ds \\ &= \int_0^1 f(x,u+stw)\,w\,ds.\end{aligned}$$

This function converges to $f(x,u)w$ *a.e.* in V as $t \longrightarrow 0$, and it is dominated by the integrable function $(a(x)|u+w|+b(x))|w|$ near $t=0$. By the Lebesgue's theorem, it converges in $L^1(V)$ to $f(x,u)w$. Clearly, the linear functional $w \longrightarrow \int_V f(x,u)w$ belongs to $H^{-1}(V)$. This completes the proof.

Concluding this paper, consider the following

Example 3.3. $F(s)=\frac{\lambda_1}{4}s^2(1+\sin(\log|s|))$ for $|s|\geq 1$. For $|s|\leq 1$, we extend $F(s)$ to be smooth. Let $f(s)=F'(s)$, so we have $f(s)=\frac{\lambda_1}{2}s(1+\sin(\log|s|))\pm\frac{\lambda_1}{4}s\cos(\log(|s|)$ for $|s|\geq 1$.

Note that

$$\limsup_{|s|\to\infty}\frac{F(s)}{s^2}=\frac{\lambda_1}{2}$$

$$\limsup_{|s|\to\infty}\frac{f(s)}{s}=\frac{2+\sqrt{5}}{4}\lambda_1$$

Note also that

$$F(s)\leq\frac{\lambda_1}{2}s^2 \text{ and } |f(s)|\leq\frac{2+\sqrt{5}}{4}|s|\lambda_1 \text{ for } |s|>1.$$

Thus $f(x,s)$ satisfies (1) and (2). To justify (3), let $v(x)$ be the eigenfunction corresponding to the first eigenvalue of the Laplace operator on a regular domain. We may assume that $v(x)>0$ in V. Put $m(t)=$measure of the set $\{x|v(x)>t\}$. We may assume that $m'(t)\leq -c<0$ for $0<t<\max\ v(x)=M$ with some $c>0$. This will imply that $F(s)$ satisfies (3). Indeed, note that, for $s>0$,

$$\frac{\lambda_1}{2}\int v^2-\int\frac{F(sv)}{s^2}\geq-\int_{1/s}^{M}t^2(1-\sin(\log st))\ dm(t)$$

$$\geq c\int_{1/s}^{M}t^2(1-\sin(\log st)\ dt.$$

With some calculation, we see that the last term has limit inferior equal to $(\frac{1}{3}-\frac{1}{\sqrt{10}})cM^3$. This completes the justification of (3) for our example.

References

1. De Figueiredo, D. and Gossez, J.-P., Nonresonance below the first eigenvalue for a semilinear elliptic problem, *Math. Ann.* ***281*** (1988), 581～610.
2. Hammerstein, A., Nichtlineare Integralgleichungen nebst Anwendungen, *Acta Math.* ***54*** (1930), 117～176.
3. Landesman, E. M. and Lazer, A. C., Linear eigenvalues and a nonlinear boundary value problem, *Pac. J. Math.* ***33*** (1970), 311～328.
4. Lazer, A.C. and Leach, D. E., On a nonlinear two point boundary value problem, *J. Math. Analysis Applic.* ***26*** (1969), 20～27.
5. Lions, J. L., Quelque methodes de resolution des problemes aux limites non lineaires, Dunod, Paris (1969).
6. Mawhin, J., Ward, J. Jr. and Willem, M., Variational methods and semilinear elliptic problems, *Arch. Rat. Mech. Anal.* ***95*** (1986), 269～277.
7. Mawhin, J. and Ward, J.R., Nonresonace and existence for nonlinear elliptic boundary value problems, *Nonlinear Analysis, TMA* ***6*** (1981), 677～684.
8. Maz'ja, V.G., Sobolev spaces, Springer-Verlag, Berlin Heidelberg (1985).
9. Metzen, G., Nonresonance semilinear operator equations in unbounded domains, Nonlinear Analysis, TMA ***11*** (1987), 1185～1191.