ANOTHER CLASS OF MINIMAL HP SPACES

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1. Introduction

In an earlier paper [9], we introduced a class of minimal $HP$-spaces called $\alpha$-minimal $HP$-spaces and studied their properties in detail. In this paper we introduce another class of minimal $HP$-spaces called $\beta$-minimal $HP$-spaces and study their properties.

By an $HP$-space we mean a Hausdorff space in which each $G_\delta$-set is open. An $HP$-space $(X, \tau)$ is called minimal $HP$ if and only if $(X, \tau^1)$ is an $HP$-space with $\tau^1 \leq \tau$ implies $\tau = \tau^1$. If $\mathcal{C}$ is a family of subsets of a topological space $X$, then $\mathcal{C}$ is called an almost cover of $X$ if and only if $X = \bigcup\{ \text{cl}U | U \in \mathcal{C} \}$. $\mathcal{C}$ is said to have an almost subcover $\mathcal{C}'$ of $X$ if and only if $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{C}'$ is an almost cover of $X$ in its own right. An $HP$-space is called $HP$-closed if and only if every open cover of $X$ admits a countable almost subcover. The properties and allied results of minimal $HP$ and $HP$-closed spaces are discussed in detail in [7].

An $HP$-space is called $\alpha$-minimal $HP$ if and only if for every closed subset $A(\subseteq X)$ and every open cover $\mathcal{C}$ of $A$ in $X$ (i.e. members of $\mathcal{C}$ are open in $X$) there exists an almost subcover $\mathcal{C}'$ of $A$ in $X$ (i.e. there exists a countable subfamily $\mathcal{C}' \subseteq \mathcal{C}$ such that $A \subseteq \bigcup\{ \text{cl}_X U | U \in \mathcal{C}' \}$). It was proved in [9], $(X, \tau)$ is Linedlöf $HP \Rightarrow (X, \tau)$ is $\alpha$-minimal $HP \Rightarrow (X, \tau)$ is minimal $HP$; but these implications can not be reversed.

It turns out in the present investigation that $(X, \tau)$ is $\alpha$-minimal $HP \Rightarrow (X, \tau)$ is $\beta$-minimal $HP \Rightarrow (X, \tau)$ is minimal $HP$; but these implications can not be reversed. $\beta$-minimal $HP$-spaces are characterized by the elegant property that each of their $HP$ quotients is minimal.

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The second author acknowledges a grant from the University of Auckland Research Fund.
Further an \( HP \)-space is \( \beta \)-minimal \( HP \) if and only if every \( \theta \)-continuous function on \( X \) into any \( HP \)-space is closed; or equivalently \( X \) is \( HP \)-closed and \( \rim - \beta \). (see Definition 2.5 below)

In their paper [2] Dickman and Zame introduce a class of spaces called "functionally compact spaces". Our definition (Definition 2.1) of \( \beta \)-minimal \( HP \)-spaces runs parallel to their definition of functionally compact spaces. Indeed we give an example (Example 3.3) to show that these two classes of spaces are distinct.

We write "nbd" for neighborhood. Also \( \ad(F) \) means the adherence of the filter \( F \) which is \( \cap \{ \cl U | U \in F \} \).

2. \( \beta \)-Minimal \( HP \)-spaces

We say that a filter (base) is a \( \sigma \)-filter (base) if the filter (generated by the filter base) is closed under countable intersection. If the members of a \( \sigma \)-filter (base) are all open then we call the same as open \( \sigma \)-filter (base).

Definition 2.1. An \( HP \)-space \( (X, \tau) \) is called \( \beta \)-minimal \( HP \) if and only if whenever \( F \) is an open \( \sigma \)-filter base on \( X \) such that \( \cap \{ F | F \in F \} = \cap \{ \cl F | F \in F \} = A \), then \( F \) is a base for the nbd of \( A \).

The concept of \( \theta \)-closed subsets of a topological space is due to Velicko [11]. Goss and Viglino [4] and Lim and Tan [5] use the terms regular closed and \( r \)-closed subsets respectively.

Definition 2.2. Let \( X \) be a topological space and \( A \subset X \). An element \( x \in X \) is said to be in \( \theta \)-closure of \( A \) if and only if every closed nbd of \( x \) interests \( A \). \( cl_\theta A = \{ x | x \in X \text{ and every closed nbd of } x \text{ meets } A \} \). \( A \) is said to be \( \theta \)-closed if and only if \( A = cl_\theta A \). Further we call a subset \( B(\subset X) \ \theta \)-open if \( X - B \) is \( \theta \)-closed.

It is easily seen that \( A \subset \cl A \subset cl_\theta A \). If \( A \) is \( \theta \)-closed then \( A \) is closed. Moreover \( A \) is \( \theta \)-closed if and only if each point \( x \in X - A \) has a closed nbd disjoint from \( A \). Indeed if \( A \) is a closed subset of \( X \) such that for each point \( x \notin A \) there exist disjoint open sets \( U_x \) and \( V_x \) such \( x \in U_x \) \( A \subset V_x \), then \( A \) is \( \theta \)-closed. One can refer to Dickman and Porter [3] for further details about \( \theta \)-closed subsets.

The concept of weakly seminormal spaces was introduced by Lominac [6]. The terms \( \theta \)-seminormal and regular seminormal were used by

**Definition 2.3.** A Hausdorff space \( X \) is called \( \theta \)-seminormal if for every \( \theta \)-closed subset \( A \subset X \) and every open set \( G \) containing \( A \), there is a regular open set \( R \) such that \( A \subset R \subset G \).

The concept of an almost continuous function used in this paper is defined by M. K. Singal and A. R. Sungal [10]. The definition of \( \theta \)-continuous functions is found in [3].

**Definition 2.4.** (a) A function \( f : X \to Y \) is almost continuous if for each \( x \in X \) and open nbd \( V \) of \( f(x) \), there exists an open nbd \( U \) of \( x \) such that \( f(U) \subset intcl V \) (or, equivalently, the inverse of a regular open subset of \( Y \) is open in \( X \).

(b) A function \( f : X \to Y \) is \( \theta \)-continuous if for each \( x \in X \) and each open nbd \( V \) of \( f(x) \), there exists an open nbd \( U \) of \( x \) such that \( f(clU) \subset clV \).

It is clear that continuity implies almost continuity which in turn implies \( \theta \)-continuity. If \( Y \) is almost regular (semiregular) and \( f \) is \( \theta \)-continuous (almost continuous) then \( f \) is almost continuous (continuous).

**Definition 2.5.** An \( HP \)-space \( (X, \tau) \) is rim-\( \beta \) if and only if there exists a nbd system \( \{V\} \) of open sets for each point of \( X \) with the property that given a closed subset \( Q \subset clV - V \) and an open cover \( \mathcal{U} \) of \( Q \) such that \( X - \mathcal{U}\{U|U \in \mathcal{U}\} \) is \( \theta \)-closed, there exists a countable subset \( \mathcal{U}' \subset \mathcal{U} \) such that \( Q \subset \bigcup\{clU|U \in \mathcal{U}'\} \).

**Theorem 2.6.** An \( HP \)-space \( (X, \tau) \) is minimal \( HP \) if and only if for every point \( x \in X \) and every open \( \sigma \)-filter base \( \mathcal{F} \) on \( X \) such that \( \{x\} = \bigcap\{U|U \in \mathcal{F}\} = \bigcap\{clU|U \in \mathcal{F}\} \), \( \mathcal{F} \) is a base for the nbds of \( x \).

**Proof.** Let \( (X, \tau) \) be minimal \( HP \), \( \mathcal{F} \) be an open \( \sigma \)-filter base on \( X \) and \( x \in X \) such that \( \{x\} = \bigcap\{U|U \in \mathcal{F}\} = \bigcap\{clU|U \in \mathcal{F}\} \). Let \( V_0 \) be an open nbd of \( x \). Since \( x \) is the unique point of adherence of \( \mathcal{F} \), by Theorem 2.6 (iii)[7], \( \mathcal{F} \) converges to \( x \). Thus there exists \( U_0 \in \mathcal{F} \) such that \( U_0 \subset V_0 \) so that \( \mathcal{F} \) is a base for the nbds of \( x \).

Conversely let \( G \) be an open \( \sigma \)-filter base on \( X \) with an unique point of adherence, say, \( x \). Let \( V_0 \) be an open nbd of \( x \). Let \( \mathcal{U} \) be the set of all open nbds of \( x \). Let \( \mathcal{F} = \{G \cup U|G \in \mathcal{G} \text{ and } U \in \mathcal{U}\} \). Then \( \mathcal{F} \) is an open
σ-filter base on $X$ such that

$$\{x\} = \cap\{F|F \in \mathcal{F}\} = \cap\{clF|F \in \mathcal{F}\}.$$ 

Hence by the hypothesis of the converse, there exists $F_0 \in \mathcal{F}$ such that $F_0 \subseteq V_0$ so that $F_0 = G_0 \cup U_0 \subseteq V_0$. Hence $G$ converges to $x$. Thus by Theorem 2.6 (iii) of [7], $X$ is minimal $HP$.

As an immediate corollary we have

**Corollary 2.7.** If $X$ is $\beta$-minimal $HP$, then $X$ is minimal $HP$.

Since minimal $HP$-spaces are $HP$-closed, we obtain

**Corollary 2.8.** If $X$ is $\beta$-minimal $HP$, then $X$ is $HP$-closed.

We have defined the concepts of $HP$-closed sets and sets $HP$-closed relative to $X$ in [8].

**Definition 2.9** [8]. Let $(X, \tau)$ be an $HP$-space. Let $A \subseteq X$.

(i) $A$ is said to be an $HP$-closed set if $(A, \tau/A)$ is $HP$-closed in its own right.

(ii) $A$ is said to be $HP$-closed relative to $X$ if every open cover $U$ of $A$ in $X$ admits a countable almost subcover of $A$ in $X$.

It was also proved in [8] that:

**Theorem 2.10.** Let $(X, \tau)$ be an $HP$-space. Let $A \subseteq X$.

(i) If $A$ is $HP$-closed relative to $X$, then $A$ is closed in $X$.

(ii) If $A$ is an $HP$-closed set, then $A$ is $HP$-closed relative to $X$.

(iii) If $A$ is regular closed and $HP$-closed relative to $X$, then $A$ is an $HP$-closed set.

(iv) If $A$ is regular closed and $X$ is $HP$-closed then $A$ is $HP$-closed relative to $X$.

The straightforward verification of the following lemma is omitted.

**Lemma 2.1.** Let $(X, \tau)$ be an $HP$-space. Let $A \subseteq X$. Then $A$ is $HP$-closed relative to $X$ if and only if every cover $U$ of $A$ consisting of sets regular open in $X$ admits a countable almost subcover of $A$ in $X$.

**Proposition 2.12.** Let $(X, \tau)$ and $(Y, \sigma)$ be $HP$-spaces. Let $f : X \rightarrow Y$ be a function. If $A$ is $HP$-closed relative to $X$, then $f(A)$ is $HP$-closed relative to $Y$ if $f$ is $\theta$-continuous.

**Proof.** Let $\mathcal{V}$ be a cover of $f(A)$ consisting of sets open in $Y$. For each $a \in A$ there exists an open nbd $V_a$ of $f(a)$ such that $V_a \in \mathcal{V}$. Since $f$ is
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There exists a countable subset $B \subset A$ such that $A \subset \bigcup \{clU_a | a \in B\}$. It follows, therefore, $f(A) \subset \bigcup \{clV_a | a \in B\}$.

We observe that Lemma 2.11 and Proposition 2.12 are proved for the whole space in [1].

**Theorem 2.13.** Let $(X, \tau)$ be an $HP$-space. Then the following are equivalent:

(i) $X$ is $\beta$-minimal $HP$.

(ii) Every $\sigma$-filter base $\mathcal{F}$ consisting of open or regular open sets such that $\cap \{clU | U \in \mathcal{F}\} = \cap \{U | U \in \mathcal{F}\} = A$ converges to its adherence.

(iii) Every $\sigma$-filter base $\mathcal{F}$ consisting of open or regular open subsets of $X$ with $\theta$-closed adherence converges to its adherence.

(iv) $X$ is minimal $HP$ and $\theta$-seminormal.

(v) $X$ is $HP$-closed and $\theta$-seminormal.

(vi) Given any $\theta$-closed subset $C$ of $X$, an open cover $\mathcal{C}$ of $X - C$ consisting of open or regular open subsets of $X$ and an open nbd $U$ of $\mathcal{C}$, there exists a countable subset $\mathcal{D} \subset \mathcal{C}$ such that $X = U \cup (\bigcup \{clV | V \in \mathcal{D}\})$.

(vii) Given any closed subset $C$ of $X$ and any open cover $\mathcal{C}$ of $C$ such that $X - \bigcup \mathcal{C}$ is $\theta$-closed, then there exists a countable subset $\mathcal{D} \subset \mathcal{C}$ such that $c \subset \bigcup \{clV | V \in \mathcal{D}\}$.

(viii) If $C$ is any closed subset of $X$ and $\mathcal{G}$ is a family of open subsets of $X$ such that $\mathcal{G} \cup \{C\}$ has the countable intersection property and $\cap \{clG | G \in \mathcal{G}\}$ is $\theta$-closed, then $C \cap (\cap \{clG | G \in \mathcal{G}\}) \neq \emptyset$.

**Proof.** (i)⇒(ii) by Definition 2.1.

(ii)⇒(iii). Let $\mathcal{F}$ be as stated in (iii) with $adh(\mathcal{F}) = A$. Since $A$ is $\theta$-closed, if $x \in X - A$, there exists an open nbd $V_x$ of $x$ such that $clV_x \cap A = \emptyset$. Clearly $A \subset X - clV_x \subset cl(X - clV_x) \subset X - V_x$. If $\mathcal{G}' = \{X - clV_x | x \in X - A\}$ then $\cap \mathcal{G}' = A$ and $\mathcal{G}'$ has countable intersection property. $\mathcal{G}'$ thus generates an open $\sigma$-filter base, $\mathcal{G}$, the set of all countable intersections coming from $\mathcal{G}'$. Thus if $G \in \mathcal{G}$, then $G = \cap \{X - clV_x | x \in B\}$ for a suitable countable subset $B \subset X - A$. Indeed $\cap \{G | G \in \mathcal{G}\} = A = \cap \{clG | G \in \mathcal{G}\}$. Let $\mathcal{E} = \{F \cup G | F \in \mathcal{F}$ and $G \in \mathcal{G}\}$. Let $U$ be an open nbd of $A$. $\mathcal{E}$ is an open $\sigma$-filter base such that $\cap \{E | E \in \mathcal{E}\} = A = \cap \{clE | E \in \mathcal{E}\}$. Thus $\mathcal{E}$ converges to its adherence. In other words there exists $E_0 \in \mathcal{E}$ such that $A \subset E_0 \subset U$ and $E_0 = F_0 \cup G_0$ for some $F_0 \in \mathcal{F}$ and $G_0 \in \mathcal{G}$. Hence there exists
Let \( F_0 \in \mathcal{F} \) such that \( A \subset F_0 \subset U \).

(iii) \( \Rightarrow \) (i). Let \( \mathcal{F} \) be an open \( \sigma \)-filter base such that \( \cap \{F \mid F \in \mathcal{F} \} = A = \cap \{clF \mid F \in \mathcal{F} \} \). Let \( x \notin A \). Then there exists \( F_0 \in \mathcal{F} \) such that \( x \notin clF_0 \). Indeed \( x \in U_x = X - clF_0 \subset X - intclF_0 \subset X - F_0 \subset X - A \) so that \( clU_x \cap A = \emptyset \). Thus \( A \) is \( \theta \)-closed so that, by the hypothesis in (iii), \( \mathcal{F} \) converges to \( A \). Hence \( X \) is \( \beta \)-minimal HP.

(iii) \( \Rightarrow \) (iv). Suppose \( \mathcal{F} \) is an open \( \sigma \)-filter base with unique adherent point \( x \). Since \( X \) is Hausdorff, \( \{x\} \) is \( \theta \)-closed. Thus by (iii), \( \mathcal{F} \) converges to \( x \); and hence by Theorem 2.6 (iii) of [7], \( X \) is minimal HP. Let \( A \) be \( \theta \)-closed and let \( U \) be an open nbd of \( A \). For each point \( x \notin A \) there exists an open nbd \( V_x \) of \( x \) such that \( clV_x \cap A = \emptyset \). Thus \( \mathcal{G}' = \{X - clV_x \mid x \in X - A\} \) generates an open \( \sigma \)-filter base \( \mathcal{G} \) such that \( \cap \{G \mid G \in \mathcal{G} \} = A = \cap \{clG \mid G \in \mathcal{G} \} \). Hence there exists an open set \( G = \cap \{X - clV_x \mid x \in B\} \) (where \( B \) is a suitable countable subset of \( X - A \)) such that \( A \subset G \subset U \). \( G = X - \cup \{clV_x \mid x \in B\} = X - cl(\cup \{V_x \mid x \in B\}) \) is regular open. Hence \( X \) is \( \theta \)-seminormal.

(iv) \( \Rightarrow \) (v). If \( X \) is minimal HP, then \( X \) is HP-closed so that the result follows.

(v) \( \Rightarrow \) (vi). Let \( C, \mathcal{C} \) and \( U \) be all given as in (vi). Since \( X \) is \( \theta \)-seminormal there is a regular open nbd \( V_0 \) of \( C \) such that \( C \subset V_0 \subset U \). Thus \( X = (X - V_0) \cup U \). Since \( X - V_0 \) is a regular closed subset of \( X \) which is HP-closed and \( C \) is an open cover of \( X - V_0 \), there exists a countable subset \( \mathcal{D} \subset \mathcal{C} \) (by Theorem 2.10 (iv)) such that \( X - V_0 \subset \cup \{clV \mid V \in \mathcal{D} \} \) so that \( X = U \cup (\cup \{clV \mid V \in \mathcal{D} \}) \).

(vi) \( \Rightarrow \) (vii). Let \( C, \mathcal{C} \) be as given in (vii). Let \( X - \cup \mathcal{C} = A \). \( A \) is \( \theta \)-closed, \( X - A = \cup \mathcal{C} \) and \( X - C \) is an open nbd of \( A \). Hence there exists a countable subset \( \mathcal{D} \subset \mathcal{C} \) such that \( X = (X - C) \cup (\cup \{clV \mid V \in \mathcal{D} \}) \). Thus \( C \subset \cup \{clV \mid V \in \mathcal{D} \} \).

(vii) \( \Rightarrow \) (viii). Let \( C \) and \( \mathcal{G} \) be as given in (viii). Assume \( C \cap (\cap \{clG \mid G \in \mathcal{G} \}) = \emptyset \). Then \( C \subset \cup \{X - clG \mid G \in \mathcal{G} \} = X - \cap \{clG \mid G \in \mathcal{G} \} \). Thus \( \mathcal{C} = \{X - clG \mid G \in \mathcal{G} \} \) is an open cover of \( C \) such that \( X - \cup \mathcal{C} \) is \( \theta \)-closed. Hence, by (vii), there exists a countable subset \( \mathcal{G}' \subset \mathcal{G} \) and a corresponding countable subset \( \mathcal{D} \subset \mathcal{C} \) (\( \mathcal{D} = \{X - clG \mid G \in \mathcal{G}' \} \)) such that \( C \subset \cup \{clV \mid V \in \mathcal{D} \} = \cup \{cl(X - clG) \mid G \in \mathcal{G}' \} \subset X - \cap \{G \mid G \in \mathcal{G}' \} \). Hence \( C \subset (\cap \{G \mid G \in \mathcal{G}' \}) = \emptyset \) so that \( \mathcal{G} \cup \{C\} \) violates the countable intersection property, a contradiction.

(viii) \( \Rightarrow \) (iii). Let \( \mathcal{F} \) be any open \( \sigma \)-filter base such that \( \text{adh}(\mathcal{F}) \) is
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$\theta$-closed. Let $A = \text{adh}(\mathcal{F})$. Let $U$ be any open set such that $U \supset A$. Then $X - U$ is a closed set such that $(X - U) \cap (\bigcap \{clF | F \in \mathcal{F}\}) = \emptyset$ so that $\mathcal{F} \cup \{X - U\}$ violates the countable intersection property. Hence there exists a countable subset $\mathcal{F}' \subset \mathcal{F}$ such that $(X - U) \cap (\bigcap \{F | F \in \mathcal{F}'\}) = \emptyset$. Since $\mathcal{F}$ is an open $\sigma$-filter base, there exists $G \in \mathcal{F}$ such that $G \subset \bigcap \{F | F \in \mathcal{F}'\}$ and hence $A \subset G \subset U$. Thus $\mathcal{F}$ converges to $A$.

**Theorem 2.14.** The following are equivalent for an $HP$-space $X$.

(i) $X$ is $\beta$-minimal $HP$.

(ii) $X$ is $HP$-closed and $rim - \beta$.

Proof. That (i) $\Rightarrow$ (ii) is easily seen.

(ii) $\Rightarrow$ (i). To prove that $X$ is $\beta$-minimal $HP$ we use the criterion (vii) of Theorem 2.13 above. Let $C$ be a closed subset of $X$ and $C$ be an open cover of $C$ such that $X - UC$ is $\theta$-closed. Since $X - C$ is open for each $x \in X - C$ there exists a basic open nbd $V_x$ such that $x \in V_x \subset X - C$ satisfying the condition of $rim - \beta$. Thus $C \cup \{V_x | x \in X - C\}$ is an open cover of $X$. Since $X$ is $HP$-closed there exist countable subsets $C'(\subset C)$ and $B(\subset X - C)$ such that $X = (\bigcup \{clU | U \in C'\}) \cup (\bigcup \{clV_x | x \in B\})$. Now $clV_x - P \subset C$ for each $x \in X - C$ where we take $X - C = P$. Indeed $C - \bigcup \{clU | U \in C'\} \subset \bigcup \{clV_x - P | x \in B\}$. Since $clV_x - P$ is a closed subset of $clV_x - V_x$ and $C$ is an open cover of $clV_x - P$ such that $X - UC$ is $\theta$-closed, by the condition of $rim - \beta$ at $x$, there exists a countable subset $C^x \subset C$ such that $clV_x - P \subset \bigcup \{clV_x | V_x \in C^x \}$. Thus $C \subset (\bigcup \{clU | U \in C'\}) \cup (\bigcup \{clV_x | V_x \in C^x \}$ and $x \in B\})$. If we write $\mathcal{D} = C' \cup (\bigcup \{C^x | x \in B\})$, then $\mathcal{D}$ is a countable subset of $C$ and $C \subset \bigcup \{clU | U \in \mathcal{D}\}$. Hence $B$ is $\beta$-minimal $HP$.

**Theorem 2.15.** Let $(X, \tau)$ be an $HP$-space. Then the following are equivalent:

(i) $X$ is $\beta$-minimal $HP$.

(ii) Every $\theta$-continuous function of $X$ into any $HP$-space is closed

(iii) Every almost continuous function of $X$ into any $HP$-space is closed

(iv) Every continuous function of $X$ into any $HP$-space is closed

Proof. (i) $\Rightarrow$ (ii). Let $f : X \to Y$ be $\theta$-continuous and $Y$ be an $HP$-space. To prove that $f$ is closed, assume $C$ is closed in $X$ and $y \in cl_Y f(C)$. By Theorem 2.13 (v), $X$ is $HP$-closed. $f(X)$ is a $\theta$-continuous image of $X$ so that $f(X)$ is $HP$-closed relative to $Y$ (by proposition 2.12) and
hence closed in $Y$. Thus $y \in \text{cl}_Y f(C) \subset f(X)$. Suppose $y \notin f(C)$. \(f^{-1}(y) \subset X - C\). Further if $x \notin f^{-1}(y)$, then there exists an open nbd $V$ of $f(x)$ such that $y \notin \text{cl}V$. Since $f$ is $\theta$-continuous there exists an open nbd $U$ of $x$ such that $f(\text{cl}U) \subset \text{cl}V$. Clearly $(\text{cl}U) \cap f^{-1}(y) = \emptyset$ so that $f^{-1}(y)$ is $\theta$-closed. By Theorem 2.13(v), since $X$ is $\theta$-seminormal, there exists a regular open set $R$ such that $f^{-1}(y) \subset R \subset X - C$. $C \subset X - R$ and $X - R$ is a regular closed subset of $X$ which is $HP$-closed. Hence $X - R$ is $HP$-closed relative to $X$ (by Theorem 2.10(iv)) so that $f(X - R)$ is also $HP$-closed relative to $Y$ (by Theorem 2.12) and hence closed in $Y$. This leads to $f(C) \subset \text{cl}_Y f(C) \subset f(X - R)$ whereas $y \notin f(X - R)$, a contradiction.

That $(ii) \implies (iii) \implies (iv)$ is obvious.

$(iv) \implies (i)$. Let $F$ be an open $\sigma$-filter base on $X$ such that $\cap \{F | F \in \mathcal{F}\} = \cap \{\text{cl}F | F \in \mathcal{F}\} = A$ and that there exists an open set $U \supset A$ with $(X - U) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Consider the equivalence relation $R = (A \times A) \cup \Delta \subset X \times X$ where $\Delta$ is the diagonal of $X \times X$. Let us form the quotient set $Y = X/R$. In $Y$ we introduce the topology generated by the collection of all open subsets of $X - A$ and those subsets $V \subset Y$ such that $p^{-1}(V) \in \mathcal{F}$ where $p : X \rightarrow Y$ is the projection. $Y$ is an $HP$-space and $p$ is continuous. But $p(X - U)$ is not closed in $Y$, though $X - U$ is closed in $X$.

As a consequence of Theorem 2.7 of [9], we get the following.

**Corollary 2.16.** If $X$ is $\alpha$-minimal $HP$, then $X$ is $\beta$-minimal $HP$.

**Theorem 2.17.** Let $(X, \tau)$ be an $HP$-space. Then the following are equivalent:

(i) $X$ is $\beta$-minimal $HP$.

(ii) Every continuous map of $X$ onto an $HP$-space is a quotient map.

(iii) Every continuous $HP$-image of $X$ is minimal $HP$.

(iv) Every Hausdorff $P$-quotient of $X$ is minimal $HP$.

(v) Every closed continuous $HP$-image of $X$ is minimal $HP$.

**Proof.** That $(i) \implies (ii)$ is clear.

$(ii) \implies (iii)$. Suppose $(ii)$ is true and $f$ is a continuous image of $X$ onto an $HP$-space $Y$. If $Y$ is not minimal $HP$ then we can retopologize $Y$ with a strictly coarser topology and let us call the resulting space $Y^*$. Then clearly the same map $f : X \rightarrow Y^*$ is a continuous function onto the $HP$-space $Y^*$, but obviously not a quotient map, a contradiction.
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That (iii)$\Rightarrow$(iv)$\Rightarrow$(v) is obvious.

(v)$\Rightarrow$(i). If $X$ is not $\beta$-minimal $HP$ for some open $\sigma$-filter base $\mathcal{F}$ on $X$, $\cap\{F|F \in \mathcal{F}\}=\cap\{clF|F \in \mathcal{F}\}=A$, while $\mathcal{F}$ is not a nbd base at $A$. If $Y=X/R$ where $R$ is the equivalence relation defined by $R=(A \times A) \cup \Delta \subset X \times X$, the resulting quotient space is not minimal $HP$, but then the quotient map of $X$ onto $Y$ is closed, a contradiction.

3. Examples

As remarked already we note that $X$ is $\alpha$-minimal $HP \Rightarrow X$ is $\beta$-minimal $HP \Rightarrow X$ is minimal $HP$. The first implication is due to the Corollary 2.16 and the second implication is due to Corollary 2.7. But these implications can not be reversed.

Example 3.1. Let $A=[1,\Omega], B=[0,\Omega]=A \cup \{0\}$. Let $X=B \times B$.

Let $\{A_\alpha|\alpha \in A\}$ be an uncountable partition of $A$ such that each $A_\alpha$ is uncountable. Define

$$P_\xi=\{((\xi,\eta)|\eta \geq 0\} \cup (A_\xi \times A)$$

$$Q_\xi=\{(\eta,\xi)|\eta \geq 0\} \cup (A \times A_\xi)$$

Now let us define a topology $\tau$ on $X$ as follows. If $p \in A \times A$ then $\{p\} \in \tau$.

Let us write $A_\alpha=[1,\Omega_\alpha]$ for each $\alpha \in A$. Let

$$V(\alpha,0)=\{(\alpha,0)\} \cup (\{\alpha\} \times [\beta,\Omega]) \cup ([\gamma_\alpha,\Omega_\alpha] \times [\beta,\Omega]) \quad (1)$$

As $\beta$ and $\gamma_\alpha$ varies over $A$ and $A_\alpha$ we get a system of basic $\tau$-open nbds of $(\alpha,0)$. Let

$$V(0,\alpha)=\{(0,\alpha)\} \cup ([\beta,\Omega] \times \{\alpha\})$$

$$\cup (\{\beta,\Omega[\times [\gamma_\alpha,\Omega_\alpha[\times [\beta,\Omega]\cup \{p_\xi|1 \leq \xi < \theta\}) \quad (2)$$

where $\beta,\theta \in A$ and $\gamma_\alpha \in A_\alpha$. As $\beta$ and $\theta$ vary over $A$ and $\gamma_\alpha$ over $A_\alpha$ we get a system of basic $\tau$-open nbds of $(\alpha,0)$. Let

$$V(0,0)=\{(0,0)\} \cup (A \times A)$$

$$-\{(\cup\{P_\xi|1 \leq \xi < \theta\}) \cup (\cup\{Q_\xi|1 \leq \xi < \theta'\})) \quad (3)$$

where $\theta,\theta' \in A$. As $\theta$ and $\theta'$ vary over $A$ we get a system of basic $\tau$-open nbds of $(0,0)$. 
In this way we get an $HP$-space $(X, \tau)$. To prove that $(X, \tau)$ is $\beta$-minimal $HP$, let us show that $X$ is $HP$-closed and $rim - \beta$.

Let $C$ be an open cover of $X$. We can take the members of $C$ to be basic open sets. Then there exists an open nbd $U$ of $(0, 0)$ such that $U \in C$ and $U$ is of the form (3). Moreover

$$X - clU = (\cup \{P_\xi | 1 \leq \xi < \theta\}) \cup (\cup \{Q_\xi | 1 \leq \xi < \theta'\})$$

(4)

By a careful analysis of points of the union of sets occurring in the right side of (4), it is easy to prove that this union of sets is covered by countably many open sets coming from $C$. Thus $X$ is $HP$-closed.

To verify that $(X, \tau)$ is $rim - \beta$, we proceed as follows. If $p \in A \times A$, \{p\} is both open and closed. Hence the condition of $rim - \beta$ is satisfied at $p$ for each $p \in A \times A$. Consider a basic $\tau$-open nbd $V(\alpha, 0)$ of $(\alpha, 0)$ as given in (1). $clV(\alpha, 0) - V(\alpha, 0) = \{0\} \times [\beta, \Omega]$. Let $K$ be any subset of $\{0\} \times [\beta, \Omega]$. Let $(0, \mu) \in K$. A basic open nbd $V(0, \mu)$ of $(0, \mu)$ is given in (2). $clV(0, \mu) = V(0, \mu) \cup ([\beta', \Omega[\times\{0\})$. Let $C$ be any open cover of $K$ such that $X - \cup\{U|U \in C\}$ is $\theta$-closed. Then there exists an open nbd $V(0, \mu)$ of $(0, \mu)$ such that $(0, \mu) \subset V(0, \mu) \subset clV(0, \mu) \subset \cup\{U|U \in C\}$. Thus $\cup\{U|U \in C\}$ contains $A \times \{0\}$, but for countably many elements. The closure of an open nbd, say $U_2(\in C)$ of any element in $(A \times \{0\}) \cap (\cup\{U|U \in C\})$ contains $\{0\} \times A$, but for countably many elements. Thus $K$ gets covered by $clU_1, clU_2$ and countably many open sets coming from $C$. (Here we have written $U_1$ for $V(\mu, 0)$). Thus $rim - \beta$ condition is verified at $(\alpha, 0)$ for every $\alpha \in A$. The argument is precisely the same for $(0, \alpha)$ for every $\alpha \in A$. Now consider a basic open nbd $V(0, 0)$ of $(0, 0)$ given as in (3). $clV(0, 0) - V(0, 0) = (\{0\} \times [\theta', \Omega]) \cup ([\theta, \Omega[\times\{0\})$. Thus an argument similar to the one we gave for the verification of the condition of $rim - \beta$ at $(\alpha, 0)$ works here also.

Furthermore the space is not $\alpha$-minimal $HP$. For, consider the closed set $A \times \{0\}$. $U = \{P_\xi | \xi \in A\}$ is an open cover of $A \times \{0\}$ which does not admit a countable subfamily $U' \subset U$ such that $A \times \{0\} \subset \cup\{clU|U \in U'\}$.  

**Example 3.2.** This example is given in [7, Example 2.8]. $X = \{0\} \cup \{1\} \cup A \cup B \cup C$ where $A = [2_a, \Omega_a[, B = B_1 \times B_1$ with $B_1 = [2_b, \Omega_b[ \text{ and } C = C_1 \times C_1$ with $C_1 = [2_c, \Omega_c[x_a, x_b, x_c]$ denote three points having the same ordinal $x \in [2, \Omega]$ with $x_a \in A$, $x_b \in B_1$ and $x_c \in C_1$. Let the topology $\tau$ on $X$ be as defined in [7, Example 2.8]. Let $Z = [2, \Omega]$ endowed with the discrete topology $\rho$. Let $Y = [2, \Omega]$ be its one-point
Lindelöf extension i.e. if $\sigma$ is this topology on $Y$, then $U$ is a basic $\sigma$-open nbd of $\Omega$ if and only if $Y - U$ is a countable subset of $Z$. Let us define $f : (X, \tau) \rightarrow (Y, \sigma)$ as follows:

$$f(x_a) = x \quad \text{for every} \quad x_a \in A$$

$$f((x_b, y_b)) = x \quad \text{for every} \quad (x_b, y_b) \in B$$

$$f((x_c, y_c)) = x \quad \text{for every} \quad (x_c, y_c) \in C$$

$$f(0) = f(1) = \Omega$$

$f$ is a continuous function. Indeed $f^{-1}(x) = \{x_a\} \cup B(\emptyset, x_b) \cup C(\emptyset, x_c)$ which is open in $X$ for every $x \in [2, \Omega]$. Also

$$f^{-1}(U) = f^{-1}(Y - P), \quad \text{where} \quad p \quad \text{is a countable subset of} \quad Z$$

$$= \{0\} \cup \{1\} \cup V(P_b, 0) \cup V(P_c, 1) \cup (A - P_a)$$

where $P_b$ and $P_c$ are given by the equation

$$x \in P \Leftrightarrow x_b \in P_b \Leftrightarrow x_c \in P_c \Leftrightarrow x_a \in P_a$$

Clearly $f^{-1}(U)$ is open. Thus $f$ is continuous. But $f(A)$ is not closed in $Y$, though $A$ is closed in $X$. Hence $f$ is not closed. Thus $X$ is minimal $HP$, but not $\beta$-minimal $HP$. 

**Example 3.3.** Consider the discrete space $(Z, \rho)$ of Example 3.2. $(Y, \sigma)$ is a Lindelöf $HP$-space. Let $(Z^*, \rho^*)$ be the one-point Hausdorff compactification of $Z$. Clearly $(Y, \sigma)$ is $\beta$-minimal $HP$ but not minimal Hausdorff since $\rho^* \leq \sigma$ taking the set $Z^*$ the same as the set $Y$. Hence $(Y, \sigma)$ is not functionally compact. But $(Z^*, *)$ is functionally compact but not a $P$-space so that it is not $\beta$-minimal $HP$. Thus the classes of $\beta$-minimal $HP$-spaces and functionally compact spaces are distinct.

**References**


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