The notion of a Dirichlet set has been studied for several decades. Such sets are named in honour of Dirichlet's Theorem [3, p235] which, in modern terminology, simply says that every finite set in $\mathbb{R}$ is a Dirichlet set.

In this paper, we present a Criterion for proving that a symmetric Cantor set is a Dirichlet set.

**Definition 1.** A bounded subset $A$ of $\mathbb{R}$ is called a Dirichlet set (in short, $D$-set) if there exists a sequence $(\alpha_k)_{k=1}^{\infty}$ in $\mathbb{R}$ such that

$$
\lim_{k \to \infty} \alpha_k = \infty \quad \text{and} \quad \lim_{k \to \infty} (\sup_{x \in A} |\sin \alpha_k x|) = 0.
$$

[Define $\sup \emptyset = 0$ for the empty set $\emptyset$, so $\emptyset$ is a $D$-set.]

**Notation 2.** Let $C = (c_n)_{n=1}^{\infty}$ be a fixed sequence of real numbers such that $0 < 2c_n < c_{n-1}$ for $n \geq 1$ and put $r_n = c_{n-1} - c_n$ for $n \geq 1$. Let

$$
F_n = \left\{ \sum_{j=1}^{n} \varepsilon_j r_j \mid \varepsilon_j = 0 \text{ or } 1 \text{ for all } j \right\}.
$$

Then it is clear that $|s - t| > c_n$ for $s \neq t \in F_n$. In particular, $F_n$ has exactly $2^n$ points. Next, put $E_n = \cup_{t \in F_n} [t, t + c_n]$ which, by the above, is a disjoint union of $2^n$ closed intervals of length $c_n$ each. Note that for $t \in F_n$, we have $t \in F_{n+1}$, $t + r_{n+1} \in F_{n+1}$, and $t < t + c_{n+1} < t + r_{n+1} < t + c_n$. This shows that $E_{n+1} \subseteq E_n$ for all $n \geq 1$. The set $E = E_C = \cap_{n=1}^{\infty} E_n$ will be called the symmetric Cantor set on $[0, c_0]$ determined by $C$. It is easy to show that $E = \{ \sum_{i=1}^{\infty} \varepsilon_i r_i \mid \varepsilon_i = 0 \text{ or } 1 \text{ for all } i \}$

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In what follows we write the following classical result of Dirichlet taken from [3].

**Lemma 3.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_k \) be any \( k \) real numbers and let \( Q \) be any positive integer. Then we can find an integer \( q \) with \( 1 \leq q \leq Q^k \) and integers \( p_1, p_2, \ldots, p_k \) such that
\[
|\alpha_j - \frac{p_j}{q}| < \frac{1}{Qq} \leq \frac{1}{q^{1+k}} (j = 1, 2, \ldots, k).
\]
In particular, \( |\sin \pi q \alpha_j| < \frac{\pi}{Q} (j = 1, 2, \ldots, k) \).

**Proposition 4.** Adopt the Notation (2). If \( \lim_n \sum_{k=1}^{\infty} |\sin nr_k| = 0 \) then \( E \) is a \( D \)-set.

**Proof.** Let \( \{n_p\} (\uparrow \infty) \) be a sequence in \( \mathbb{N} \) such that
\[
\sum_{k=1}^{\infty} |\sin n_p r_k| < \eta_p \quad \text{with} \quad \{\eta_p\} \downarrow 0.
\]
For \( x \in E \), we have
\[
|\sin n_p x| \leq \sum_{k=1}^{\infty} |\sin n_p r_k| < \eta_p.
\]
Thus
\[
\sup_{x \in E} |\sin n_p x| < \eta_p.
\]
It follows that
\[
\lim_{p \to \infty} \sup_{x \in E} |\sin n_p x| = 0.
\]

Now we are ready for the main theorem.

**Theorem 5.** Adopt the Notation as before. If \( \lim_{p \to \infty} \frac{1}{p} c_p = 0 \) then \( E_C \) is a \( D \)-set.

**Proof.** Let
\[
c_k^\frac{1}{k} = \frac{1}{k^\psi(k)} \quad \text{with} \quad \lim k^\psi(k) = \infty.
\]
It follows from Lemma (3) that for given \( p \), \( A \in \mathbb{Z}^+ \) and \( t \in R(t \geq 1) \), there exist \( n = n(p) \) such that
\[
A \leq n \leq At^p \quad \text{and} \quad |\sin nr_k| < \frac{\pi}{[t]} \quad \text{for} \quad 1 \leq k \leq p.
\]
Let $A = p$, and $t = p\sqrt{\psi(p)} \geq 2$. Then there exists $n = n(p)$ such that
\[ p \leq n < p(p\sqrt{\psi(p)})^p \quad \text{and} \quad |\sin nr_k| < \frac{\pi}{|p\sqrt{\psi(p)}|} \quad (k = 1, 2, \ldots, p). \]

Thus $\sum_{k=1}^{p} |\sin nr_k| < \frac{2\pi}{\sqrt{\psi(p)}}$ since $p\sqrt{\psi(p)} \geq 2$ and
\[ \sum_{k=p+1}^{\infty} |\sin nr_k| \leq n \sum_{k=p+1}^{\infty} r_k = nc_p < p(p\sqrt{\psi(p)})^p \cdot \left(\frac{1}{p\psi(p)}\right)^p = \frac{p}{(\sqrt{\psi(p)})^p}. \]

Therefore we have
\[ \sum_{k=1}^{\infty} |\sin nr_k| \leq \frac{2\pi}{\sqrt{\psi(p)}} + \frac{p}{(\sqrt{\psi(p)})^p} \to 0 \quad \text{as} \quad p \to \infty. \]

Let us choose a increasing sequence $\{n = n(p)\}$ by letting $A$ and $p \to \infty$. It follows from proposition (4) that $E_C$ is a $D$-set.

References


