PROJECTIVE LIMIT CONSTRUCTION OF
LOCALLY CONVEX SPACES BY SEMI–NORMS

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1. The construction by the weakest, locally convex topology for the continuity of the given semi–norms

Let $L$ be a linear space over $K (K = R$ or $K = C)$, then the notations $\varphi$ and $p$ are for a linear functional and semi–norm on $L$, respectively. The collection of all linear functionals (or semi–norms) on $L$ is denoted by $L^*$ (or SPECL), where SPECL is the abbreviation of algebraic spectrum of $L$. For any $\varphi$, its absolute value $p_\varphi$ is in SPECL, and is called the semi–norm associated with $\varphi$. For any $p$, the function $d(x, y) = p(x - y)$ for any $x, y \in L$ defines a pseudo–metric, and we can define the pseudo–metric topology of $L$ which consists of the sets of the form $V_p(x, \varepsilon) = \{y \in L : p(x - y) < \varepsilon\}$ for any $x \in L$ and $\varepsilon > 0$ as a base of open subsets. We can easily check that $p(x)$ is continuous on $L$ w.r.t. this topology, and any $x \in L$ has a base of neighborhoods consisting of convex subsets of $L$. $L$ with this topology is denoted by $L(p)$, and called a semi–normed linear space—a terminology which is an analogue of the so–called normed linear spaces discussed in the next paragraph.

If $L$ is a normed linear space and $L'$ is its topological dual, then $p_\varphi$ is a continuous semi–norm on $L$ for any $\varphi \in L'$. The weak topology of $L$ is defined by $\{V_{p_\varphi}(x, \varepsilon) : x \in L, \varphi \in L' \text{ and } \varepsilon > 0\}$ as a subbase of open subsets (Example 2.4. 20, [2]). Let $L_W$ be $L$ with the weak topology, then $p_\varphi$ is continuous on $L_W$ for any $\varphi \in L'$, and any $x \in L$ has a base of neighborhoods consisting of convex subsets of $L$ by construction 4 in the following. Furthermore, this topology is the weakest one which satisfies both conditions in the last statement. We say that $L_W$ is the locally convex, projective limit of (the collection of semi–normed linear spaces) $\{L_{(p_\varphi)} : \varphi \in L'\}$. The formal definition of this notion and its

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basic properties will be thoroughly discussed later. In this section, we will mainly prove that any locally convex spaces is arises by this way. For this purpose, we will consider its very basic background so that other interesting properties can also be obtained.

Let $X$ be a non-empty set with the topology $\tau$, then we write $X_{\tau}$ for $X$, or simply, $X$ if no confusion can possibly occur. $\omega \subseteq \tau$ is called a base of open subsets of $X$, or simply, a base of the topology $\tau$ if (1) $U\{W : W \in \omega\} = X$; and (2) for any $V \in \tau$ and $x \in V$, $x \in W \subseteq V$ for some $W \in \omega$ (Definition 2.2.1, [5]). In this case, $\tau$ is the collection of all unions of members of $\omega$ (Lemma 2.2.1, [5]).

Let $X$ be a set, then the non-empty collection $\omega$ of subsets of $X$ is called a base-to-be of a topology of $X$ if (1) $U\{W : W \in \omega\} = X$; and (2) for any $W_1, W_2 \in \omega$ and $x \in W_1 \cap W_2$, $x \in W \subseteq W_1 \cap W_2$ for some $W \in \omega$. In this case, the collection $\tau$ of all unions of members of $\omega$ is a topology of $X$, and $\omega$ is a base of $\tau$ (Proof on p. 79, [5]). Hence $\omega$ is called the canonical base of $\tau$.

If $\tau_1$ and $\tau_2$ are topologies of $X$ with bases $\omega_1$ and $\omega_2$, then $\tau_1 \subseteq \tau_2$ iff for any $x \in X$ and $W_1 \in \omega_1$ with $x \in W_1$, we have $x \in W_2 \subseteq W_1$ for some $W_2 \in \omega_2$ (Lemma 2.2.2, [5]).

If $X_{\tau}$ is a topological space and $\omega_s \subseteq \tau$ such that the collection of all finite intersections of members of $\omega_s$ is a base of $\tau$, then $\omega_s$ is called a subbase of the topology $\tau$. For any collection $\omega_s$ of subsets of $X$ covering $X$, the collection of all finite intersections of members of $\omega_s$ is a base-to-be of a topology of $X$ (Proof on p. 83, [5]).

If $X_{\tau}$ is a topological space, $V \subseteq X$, and $x \in V$ with $x \in W \subseteq V$ for some $W \in \tau$, then $V$ is called a neighborhood of $x$, and $x$ is called an interior point of $V$. Let $x \in X$, then a collection $\eta_x$ of subsets of $X$ is called a base of neighborhoods of $x$ if (1) any $W \in \eta_x$ is a neighborhood of $x$; and (2) for any neighborhood $V$ of $x$, $x \in W \subseteq V$ for some $W \in \eta_x$. $\eta_x$ is called an open base of neighborhoods of $x$ if any $W \in \eta_x$ is open in $X_{\tau}$. The following property will be applied frequently.

(i) If $\eta_x$ is an open base of neighborhoods of any $x \in X$, then $\omega = U\{\eta_x : x \in X\}$ is a base of open subsets of $X_{\tau}$.

A collection $(\eta_x)_s$ of subsets of $X$ is called a subbase of neighborhoods of $x$ if the collection of all finite intersections of members of $(\eta_x)_s$ is a base of neighborhoods of $x$. We can similarly define open subbase of neighborhoods of $x$. The following property will also be applied later.
(ii) If $\omega$ (or $\omega_s$) is a base (or subbase) of open subsets of $X_\tau$ and $x \in X$, then $\eta_\tau$ (or $(\eta_\tau)_s$), which is the collection $\{W \in \omega \text{ (or } W \in \omega_s) : x \in W\}$, is an open base (or subbase) of neighborhoods of $x$. We note that if $X = L$ is a locally convex space, then any $x \in L$ has a closed base of neighborhoods, i.e. any member of this base is closed in $L$ (Remark 5.3, [6]).

**Proposition 1.** Let $X \neq \emptyset$, $Y$ and $Y_\gamma$ be topological spaces for any $\gamma \in \Gamma$.

(i) If $f : X \to Y$ is a map and $\omega = \{f^{-1}(W) : W \text{ is a basic, open subset of } Y\}$, then $\omega$ is a base of a topology $\tau$ of $X$. Furthermore, $f : X_\tau \to Y$ is continuous, and $\tau$ is the weakest topology of $X$ satisfying this property.

(ii) If $f_\gamma : X \to Y$ is a map for any $\gamma \in \Gamma$ and $\omega_s = \{f_\gamma^{-1}(W) : W \text{ is a basic, open subset of } Y \text{ and } \gamma \in \Gamma\}$, then $\omega_s$ is a subbase of a topology $\tau$ of $X$. Furthermore, $f_\gamma : X_\tau \to Y$ is continuous for any $\gamma \in \Gamma$, and $\tau$ is the weakest topology of $X$ satisfying this property.

(iii) If $W_\gamma$ is a base of open subsets of $Y_\gamma$ and $f_\gamma : X \to Y_\gamma$ is a map for any $\gamma \in \Gamma$, then $\omega_s = \{f_\gamma^{-1}(W_\gamma) : W_\gamma \in \omega_\gamma \text{ and } \gamma \in \Gamma\}$ is a subbase of a topology $\tau$ of $X$. Furthermore, $f_\gamma : X_\tau \to Y_\gamma$ is continuous for any $\gamma \in \Gamma$, and $\tau$ is the weakest topology of $X$ satisfying this property.

We can easily check that the topologies $\tau$ of $X$ in Proposition 1 are independent of the choices of bases of open subsets of $Y$ and $Y_\gamma$ for $\gamma \in \Gamma$. In (ii), $\omega_s$ is a subbase of the topology $\tau$. For any $\gamma \in \Gamma$, $\{f_\gamma^{-1}(W) : W \text{ is a basic, open subset of } Y\}$ is a base of open subsets of $X_{\tau_\gamma}$ in (i).

**Definition 1.** The topological space $X_\tau$ constructed in Proposition 1 (ii) is called the projective limit of $\{X_{\tau_\gamma} : \gamma \in \Gamma\}$ induced by $\{f_\gamma : \gamma \in \Gamma\}$ and $Y$, where $X_{\tau_\gamma}$ is constructed in (i).

We now consider the case that $X = L$ is a linear space over $K$. We have the following computational rules.

**Lemma 1.** Let $L$ be a linear space, $p \in \text{SPECL}$ and $\varepsilon > 0$.

(i) If $V_p(x, \varepsilon) = \{y \in L : p(y - x) < \varepsilon\}$, then $V_p(x, \varepsilon) = V_{p^{-1}}(x, 1)$ for $x \in L$. 
(ii) If \( y \in V_p(x, \varepsilon) \), then \( |p(y) - p(x)| < \varepsilon \).

(iii) If \( y \in V_p(x, \varepsilon) \), then \( y \in V_p(y, \delta) \subseteq V_p(x, \varepsilon) \) for some \( \delta > 0 \).

(iv) If \( y \in P^{-1}((p(x) - \varepsilon, p(x) + \varepsilon)) \), then \( y \in p^{-1}((p(y) - \delta, p(y) + \delta)) \subseteq p^{-1}((p(x) - \varepsilon, p(x) + \varepsilon)) \) for some \( \delta > 0 \).

(v) \( V_p(x, \varepsilon) \subseteq p^{-1}((p(x) - \varepsilon, p(x) + \varepsilon)) \) for any \( x \in L \).

Let \( L_1, L_2 \) be linear spaces over \( K \), \( \varphi : L_1 \to L_2 \) be a linear map and \( q \in \text{SPECL}_2 \). Let \( \varepsilon > 0 \).

(vi) \( q \circ \varphi \in \text{SPECL}_1 \).

(vii) \( \varphi^{-1}(V_q(\varphi(x), \varepsilon)) = V_{q \circ \varphi}(x, \varepsilon) \) for any \( x \in L_1 \).

(viii) If \( x \in \varphi^{-1}(V_q(y, \varepsilon)) \), then \( x \in \varphi^{-1}(V_q(\varphi(x), \delta)) = V_{q \circ \varphi}(x, \delta) \subseteq \varphi^{-1}(V_q(y, \varepsilon)) \) for some \( \delta > 0 \).

**Proof.** We let \( \delta = \varepsilon - p(y - x) \) (or \( \delta = \varepsilon - |p(y) - p(x)| \)) in (iii) (or (iv)).

(v) follows the triangle inequality. For (vii), we note \( \varphi^{-1}(V_q(\varphi(x), \varepsilon)) = \{ x' \in L_1 : q(\varphi(x' - x)) = q(\varphi(x' - \varphi(x)) < \varepsilon \} = V_{q \circ \varphi}(x, \varepsilon) \). For (viii), we let \( \delta = \varepsilon - q(\varphi(x) - y) \).

If \( L \) is a linear space and \( p \in \text{SPECL} \), then there are many topologies of \( L \) for which \( p(x) \) to be continuous. Among them, we consider the following ones.

**Construction 1.** Let \( L_{\tau_p} \) be \( L \) with the weakest topology \( \tau_p \) for \( p(x) \) to be continuous. Since \( \{(a, b) : -\infty < a < b < +\infty \} \) is a base of open subsets of \( R \), the collection \( \{ p^{-1}((a, b)) : -\infty < a < b < +\infty \} \) forms a base of open subsets of \( L_{\tau_p} \) by Proposition 1 (i).

(i) The collection \( \{ p^{-1}((p(x) - \varepsilon, p(x) + \varepsilon)) : \varepsilon > 0 \} \) is an open base of neighborhoods of any \( x \in L \) w.r.t. \( \tau_p \). Each member of this base is convex in \( L \).

(ii) The collection \( \{ p^{-1}((p(x) - \varepsilon, p(x) + \varepsilon)) : x \in L \) and \( \varepsilon > 0 \} \) is a base of open subsets of \( L_{\tau_p} \) by property (i) on p.2.

(iii) \( p^{-1}((-\varepsilon, \varepsilon)) = p^{-1}((p(0) - \varepsilon, p(0) + \varepsilon)) = V_p(0, \varepsilon) \) for any \( \varepsilon > 0 \). But \( p^{-1}((p(x) - \varepsilon, p(x) + \varepsilon)) = x + p^{-1}((-\varepsilon, \varepsilon)) \) is not necessarily true.
By (iii), the vector addition of \( L \) is not necessarily continuous w.r.t. \( \tau_p \). Hence, though \( L_{\tau_p} \) has a base of neighborhoods of any \( x \in L \) which consists of convex subsets of \( L \), \( L_{\tau_p} \) is not a locally convex space in the sense of Definition 5.2, [6].

**Construction 2.** Let \( \omega = \{ V_p(x, \varepsilon) : x \in L \) and \( \varepsilon > 0 \} \), then \( \omega \) is a base of a topology of \( L \) by Lemma 1 (iii). Let \( L(\rho) \) be \( L \) with the topology generated by \( \omega \) as a base.

(i) For any \( x \in L \) and \( \varepsilon > 0 \), we have \( V_p(x, \varepsilon) \subseteq p^{-1}((p(x) - \varepsilon, p(x) + \varepsilon)) \) by Lemma 1(v). Thus \( L_{\tau_p} \subseteq L(p) \) in the sense that the topology of the former space is weaker than that of the latter. Since \( p(x) \) is continuous on \( L_{\tau_p} \), so is on \( L(p) \).

(ii) If \( V \) is a neighborhood of \( x \) in \( L(p) \), then \( x \in V_p(y, \varepsilon) \subseteq V \) for some \( y \in L \) and \( \varepsilon > 0 \). Thus \( x \in V_p(x, \delta) \subseteq V_p(y, \varepsilon) \subseteq V \) for some \( \delta > 0 \). Hence \( \{ V_p(x, \varepsilon) : \varepsilon > 0 \} \) is an open base of neighborhoods of \( x \) in \( L(p) \) for any \( x \in L \). It is clear that each member of this base is convex in \( L \).

(iii) If \( a, b \in L \), \( c = a + b \) and \( \varepsilon > 0 \), then, for any \( x \in V_p(a, \frac{1}{2} \varepsilon) \) and \( y \in V_p(b, \frac{1}{2} \varepsilon) \) we have \( p(x + y - c) \leq p(x - a) + p(y - b) < \varepsilon \). Hence \( x + y \in V_p(c, \varepsilon) \). This implies \( V_p(a, \frac{1}{2} \varepsilon) + V_p(b, \frac{1}{2} \varepsilon) \subseteq V_p(c, \varepsilon) \). Thus the vector addition of \( L \) is continuous on \( L(p) \). Also, if \( a \in L \), \( \alpha \in K \), \( b = \alpha a \) and \( \varepsilon > 0 \), then we let \( \delta = \frac{\varepsilon}{2}(p(a) + 1) \) and \( \eta = \frac{\varepsilon}{2}(\delta + |\alpha|) \). If \( |\gamma - \alpha| < \delta \) and \( x \in V_p(a, \eta) \), then \( \gamma x - b = \gamma(x - a) + (\gamma - \alpha)a \) implies \( p(\gamma x - b) \leq |\gamma|p(x - a) + |\gamma - \alpha|p(a) \leq (|\gamma - \alpha| + |\alpha|)p(x - a) + |\gamma - \alpha|p(a) \leq (\delta + |\alpha|)\eta + \delta p(a) < \varepsilon \) implies \( \gamma x \in V_p(b, \varepsilon) \). Hence the scalar multiplication of \( L \) is also continuous on \( L(p) \). Thus \( L(p) \) is a locally convex space.

From Constructions 1 and 2, we can conclude that \( L(p) \) is not with the weakest topology for \( p(x) \) to be continuous. But we have the following theorem.

**Theorem 1.** Let \( L \) be a linear space and \( p \in \text{SPECL} \), then \( L(p) \) is with the weakest, locally convex topology for \( p(x) \) to be continuous.

**Proof.** Let \( \tau \) be a linear topology of \( L \) for \( p \) to be continuous. If \( V \) is an open subset of \( L(p) \) and \( x \in V \), then \( x \in V_p(x, \varepsilon) = x + V_p(0, \varepsilon) \subseteq V \) for some \( \varepsilon > 0 \). Since \( p \) is continuous on \( L_\tau \) and \( L_\tau \) is a linear topological
space, \( V_p(0, \varepsilon) = p^{-1}([0, \varepsilon]) \) is an open subset of \( L_\tau \), and so is \( V_p(x, \varepsilon) \) by linear translates. This is true for any \( x \in V \). Hence \( V \) is open in \( L_\tau \), and we proved \( L(p) \subseteq L_\tau \). Thus \( L(p) \) is with the weakest, linear topology for \( p \) to be continuous. In particular, \( L(p) \) is with the weakest, locally convex topology for \( p \) to be continuous.

The space \( L(p) \) in Theorem 1 is called a semi–normed linear space.

**Construction 3.** Let \( L \) be a linear space and \( \emptyset \neq P \subseteq \text{SPECL} \). For any \( p \in P \), \( L_{tp} \) has a base of open subsets consisting of the sets of the form \( p^{-1}((p(x) - \varepsilon, p(x) + \varepsilon)) \) for any \( x \in L \) and \( \varepsilon > 0 \). Let \( L_{tp} \) be \( L \) with the weakest topology for any \( p \in P \) to be continuous.

(i) \( \{p^{-1}((p(x) - \varepsilon, p(x) + \varepsilon)) : x \in L, p \in P \text{ and } \varepsilon > 0\} \) is a subbase of open subsets of \( L_{tp} \) by Construction 1 and Proposition 1 (ii).

(ii) If \( V \) is a neighborhood of \( x \) in \( L_{tp} \), then there exist \( x_i \in L, p_i \in P \) and \( \varepsilon_i > 0 \) for \( i = 1, 2, \ldots, n \) with \( x \in \cap_{i=1}^n p_i^{-1}((p_i(x_i) - \varepsilon_i, p_i(x_i) + \varepsilon_i)) \subseteq V \). Thus, for any \( i = 1, 2, \ldots, n \), there is an \( \delta_i > 0 \) with \( x \in p_i^{-1}((p_i(x) - \delta_i, p_i(x) + \delta_i)) \subseteq p_i^{-1}((p_i(x_i) - \varepsilon_i, p_i(x_i) + \varepsilon_i)) \) by Lemma 1 (iv). Let \( \delta = \min\{\delta_i : i = 1, 2, \ldots, n\} \), then \( x \in \cap_{i=1}^n p_i^{-1}((p_i(x) - \delta, p_i(x) + \delta)) \subseteq \cap_{i=1}^n p_i^{-1}((p_i(x_i) - \delta_i, p_i(x_i) + \delta_i)) \subseteq \cap_{i=1}^n p_i^{-1}((p_i(x_i) - \varepsilon_i, p_i(x_i) + \varepsilon_i)) \subseteq V \). Hence \( L_{tp} \) has an open base of neighborhoods of any \( x \in L \) consisting of the sets of the form \( \cap_{i=1}^n p_i^{-1}((p_i(x) - \varepsilon, p_i(x) + \varepsilon)) \) for \( \varepsilon > 0 \) and \( p \in P \) for \( i = 1, 2, \ldots, n \).

(iii) By property (i) on p.2, the collection \( \{\cap_{i=1}^n p_i^{-1}((p_i(x_i) - \varepsilon, p_i(x_i) + \varepsilon)) : x \in L, \varepsilon > 0 \text{ and } p \in P \text{ for } i = 1, 2, \ldots, n\} \) is a base of open subsets of \( L_{tp} \).

Since \( L_{tp} \) is not a linear topological space, neither is \( L_{tp} \). The space \( L_{tp} \) is called the projective limit of \( \{L_{tp} : p \in P\} \) induced by \( P \) (and \( R \)).

**Construction 4.** Let \( L \) be a linear space and \( \emptyset \neq P \subseteq \text{SPECL} \). For any \( p \in P \), the collection \( \{V_p(x, \varepsilon) : x \in L \text{ and } \varepsilon > 0\} \) is a base of open subsets of \( L(p) \). Let \( \omega_s = \{V_p(x, \varepsilon) : x \in L, p \in P \text{ and } \varepsilon > 0\} \), then \( \omega_s \) can be a subbase of a topology of \( L \). Let \( L_P \) be \( L \) with the topology generated by \( \omega_s \) as a subbase.

(i) By Lemma 1 (ii) and the similar arguments in Construction 3, the collection \( \{\cap_{i=1}^n V_{p_i}(x, \varepsilon) : \varepsilon > 0 \text{ and } p_i \in P \text{ for } i = 1, 2, \ldots, n\} \) is an open base of neighborhoods of any \( x \in L \) in \( L_P \).
(ii) The collection \( \{ \cap_{i=1}^{n} V_{p_i}(x, \varepsilon) : x \in L, \varepsilon > 0 \text{ and } p_i \in P \text{ for } i = 1, 2, \ldots, n \} \) is a base of open subsets of \( L_P \).

**Definition 2.** The space \( L_P \) in Construction 4 is called the locally convex, projective limit (abbrev. by LCPL) of (the collection of semi–normed linear spaces) \( \{ L(p) : p \in P \} \) (induced by \( \mathcal{R} \)). The reason for this terminology is clear from the following theorem.

**Theorem 2.** Let \( L \) be a linear space and \( \emptyset \neq P \subset \text{SPECL} \), then \( L_P \) is with the weakest, locally convex topology for any \( p \in P \) to be continuous.

**Proof.** For any \( p \in P \), \( p \) is continuous on \( L_{\tau_p} \), and so is on \( L_P \) for \( L_{\tau_p} \subseteq L_P \). The proofs of continuity of vector addition and scalar multiplication of \( L \) on \( L_P \) are in Proposition 4.3, [6]. Since \( \cap_{i=1}^{n} V_{p_i}(x, \varepsilon) \) is convex in \( L \) for any \( x \in L, \varepsilon > 0 \) and \( p_i \in P \) for \( i = 1, 2, \ldots, n \), \( L_P \) is a locally convex space. The conclusion can be proved by the similar arguments in Theorem 1.

We have constructed a natural, locally convex topology of a linear space \( L \) over \( K \) through a given collection of semi–norms on \( L \). We now consider the case when the subcollection \( P \) of \( \text{SPECL} \) is replaced by a subset of \( L^* \).

**Construction 5.** Let \( L \) be a linear space, \( \varphi \in L^* \) and \( L_{\tau_{\varphi}} \) be \( L \) with the weakest topology for \( \varphi \) to be continuous. It is well–known that the collection \( \{ V(z, \varepsilon) = \{ w \in K : |w - z| < \varepsilon \} = z + V(0, \varepsilon) = z + \varepsilon D : z \in K \text{ and } \varepsilon > 0 \} \) is a base of the usual topology of \( K \), where \( D = \{ w \in K : |w| \leq 1 \} \) is the unit disk of \( K \). The collection \( \{ \varphi^{-1}(V(z, \varepsilon)) : z \in K \text{ and } \varepsilon > 0 \} \) is thus a base of open subsets of \( L \).

(i) We can easily check that the collection \( \{ \varphi^{-1}(\varphi(x) + \varepsilon D) : \varepsilon > 0 \} \) is an open base of neighborhoods of any \( x \in L \) in \( L_{\tau_{\varphi}} \).

(ii) Hence the collection \( \{ \varphi^{-1}(\varphi(x) + \varepsilon D) : x \in L \text{ and } \varepsilon > 0 \} \) is a base of open subsets of \( L_{\tau_{\varphi}} \).

**Theorem 3.** Let \( L \) be a linear space and \( \emptyset \neq A^* \subset L^* \).

(i) If \( \varphi \in L^* \), then \( L_{\tau_{\varphi}} \) is with the weakest, locally convex topology for \( \varphi(x) \) to be continuous, and \( L_{\tau_{\varphi}} = L(p_{\varphi}) \).
(ii) If \( L_{\tau_{A^*}} \) is \( L \) with the weakest topology \( \tau_{A^*} \) for any \( \varphi \in A^* \) to be continuous, then \( L_{\tau_{A^*}} \) is with the weakest, locally convex topology for any \( \varphi \in A^* \) to be continuous, and \( L_{\tau_{A^*}} = L_{P_{A^*}} \), where \( P_{A^*} = \{ p_{\varphi} : \varphi \in A^* \} \).

**Proof.** (i) Since \( |\varphi(y) - \varphi(x)| = p_{\varphi}(y - x) \) for \( x, y \in L \), \( \varphi \) is continuous on a locally convex space \( L \) iff \( p_{\varphi} \) is continuous. We note that the collection \( \{ \varphi^{-1}(\varphi(x) + \varepsilon D) \) (or \( V_{p_{\varphi}}(x, \varepsilon) \)) : \( x \in L \) and \( \varepsilon > 0 \} \) is a base of open subsets of \( L_{\tau_{\varphi}} \) (or \( L_{(p_{\varphi})} \)). By computation, \( \varphi^{-1}(\varphi(x) + \varepsilon D) = V_{p_{\varphi}}(x, \varepsilon) \) for any \( x \in L \) and \( \varepsilon > 0 \). Hence \( L_{\tau_{\varphi}} = L_{(p_{\varphi})} \), and \( L_{\tau_{\varphi}} \) is a locally convex space for \( \varphi \) to be continuous. But \( L_{\tau_{\varphi}} \) is with the weakest topology for \( \varphi \) to be continuous. This part is proved.

(ii) By Proposition 1 (ii), the collection \( \{ \varphi^{-1}(\varphi(x) + \varepsilon D) : x \in L, \varphi \in A^* \) and \( \varepsilon > 0 \} \) (or \( \{ V_{p_{\varphi}}(x, \varepsilon) : x \in L, \varphi \in A^* \) and \( \varepsilon > 0 \} \) ) is a subbase of open subsets of \( L_{\tau_{A^*}} \) (or \( L_{P_{A^*}} \)). The given identity is thus proved by \( \varphi^{-1}(\varphi(x) + \varepsilon D) = V_{p_{\varphi}}(x, \varepsilon) \).

We have proved that the LCPL of semi-normed linear spaces is locally convex. The converse is also true.

**Theorem 4.** \( L \) is a locally convex space iff \( L \) is a LCPL of semi-normed linear spaces.

**Proof.** This is Theorem 5.1, [6]. The converse is true by taking \( P = \text{Spec}L \) which is the topological spectrum of \( L \) consisting of all continuous semi-norms on \( L \).

By this theorem, the weak topology of a normed linear space \( L \) is locally convex. On the other hand, the idea of Theorem 4 is originated from the consideration of constructing weak topology of a normed linear space. Thus the method given in Construction 4 is also called the weak topology construction of locally convex spaces.

In the proof of Theorem 5.1, [6], the relation \( \{ x \in L : p(x) < 1 \} \subseteq V \subseteq \{ x \in L : p(x) \leq 1 \} \) for any convex, balanced neighborhood \( V \) of \( 0 \in L \) is applied. If \( q = 2p \), then \( q \in \text{Spec}L \) and \( \{ x \in L : q(x) \leq 1 \} = q^{-1}([0, 1]) \) is a closed, convex, balanced neighborhood of \( 0 \in L \), and \( \{ x \in L : q(x) \leq 1 \} = \{ x \in L : p(x) \leq \frac{1}{2} \} \subseteq V \). Hence \( L \) has a closed base of neighborhoods of \( 0 \in L \), consisting of convex, balanced subsets of \( L \) (Remark 5.3, [6]).

In the following paragraph, we list some examples of LCPL construction of locally convex spaces.
Example 1. (i) Let $\Gamma \neq \emptyset$ and $L = K^\Gamma = \{(x_\gamma)_{\gamma \in \Gamma} : x_\gamma \in K$ for $\gamma \in \Gamma\}$, then $L$ is a linear space under the coordinatewise vector addition and scalar multiplication. For any $\gamma \in \Gamma$, we let $p_\gamma : L \to [0, +\infty)$ be the semi-norm which maps any $x \in L$ to the absolute value of the $\gamma$-th entry of $x$. We can easily check that the product topology of $K^\Gamma$ is the LCPL of $\{L(p_\gamma) : \gamma \in \Gamma\}$. A special case that $K = R$ and $\Gamma = \mathbb{Z}_+$ is discussed in Corollary 4, [3].

(ii) If $L$ is a (Hausdorff) locally convex space, then we let $L^+$ (or $L^b$) be the linear space of all $s$–continuous (or ess–bounded) linear functionals on $L$ (Definition in §1, [4]), and $L_w$ (or $Lwb$) be the LCPL of $\{L(p_\gamma) : \varphi \in L^+ \text{ (or } \varphi \in L^b\}$). Since $L' \subseteq L^+ \subseteq L^b$, we have $L_w \subseteq L_w \subseteq Lwb$, where $L_w$ is the LCPL of $\{L(\varphi) : \varphi \in L'\}$. In Lemma 1, [4], we also have the identities $(L_w)' = L^+$ and $(Lwb)' = L^b$.

(iii) If $L$ is a locally convex space, then we let $P_1$ (or $\rho_e$) be the collection of lower semi–continuous semi–norms on $L$ (or barrels of $L$, i.e. closed, convex, balanced, absorbing subsets of $L$), then $P_1 = P_{\rho_e} = \{p_A : A \in \rho_e\}$, where $p_A(x)$ is the Minkowski functional associated with $A$ on $L$. We will prove later that $L_{P_1}$ has a closed base of neighborhoods of $0 \in L$ consisting of all barrels of $L$. This implies $L \subseteq L_{P_1}$ and $L_{P_1}$ is called the barrelled extension of $L$. Similarly, if $P_{b_1}$ (or $\rho_b$) is the collection of all ess–bounded, lower–semi–continuous semi–norms on $L$ (or all quasibarrels of $L$, i.e. closed, convex, balanced bornivores of $L$ (Definition 3.5.3, [2])), then $P_{b_1} = P_{\rho_b}$. We can check that $\rho_b$ is a closed base of neighborhoods of $0 \in L_{P_{b_1}}$. Hence $L \subseteq L_{P_{b_1}}$, and $L_{P_{b_1}}$, is called the quasibarrelled extension of $L$.

(iv) Let $L$ be a (Hasudorff) locally convex space and $P_s$ (or $\eta_{cs}$) be the collection of all $s$–continuous semi–norms on $L$ (or all convex, balanced $s$–neighborhoods of $0 \in L$ (Definition in §2, [4])), then $P_s = P_{\eta_{cs}}$, and $L_{P_s} = L_{P_{\eta_{cs}}}$). In [4], we proved that $\eta_{cs}$ is a base of neighborhoods of $0 \in L_{P_s}$. Thus $I \subseteq L_{P_s}$, and $L_{P_s}$ is called the $c$–sequential extension of $L$. Similarly, if $P_b$ (or $\eta_{cb}$) is the collection of all ess–bounded semi–norms on $L$ (or all convex, balanced bornivores of $L$), then $P_b = P_{\eta_{cb}}$, and $\eta_{cb}$ is a base of neighborhoods of $0 \in L_{P_b}$. Thus $L \subseteq L_{P_b}$, and $L_{P_b}$ is called the bornological extension of $L$.

(v) If $L$ is a (Hausdorff) locally convex space, then $L$ is called strictly $s$–barrelled, strictly $s$–quasibarrelled and is said to satisfy the strict condition of ess–uniform boundedness if $(L_{P_s})_P = L_{P_s}$, $(L_{P_s})_{P_{b_1}} = L_{P_s}$ and
(L_{P_b})_{P_1} = L_{P_b} are true, respectively (Theorems 5 and 6, [4]).

(vi) If L is a linear space and $\rho$ is the collection of all convex, balanced, absorbing subsets of L, then $\text{SPECL} = P_\rho$, and $L_{P_\rho}$ is called the super-barrelled extension of L since $\rho$ is a base of neighborhoods of $0 \in L_{P_\rho}$. $L_{P_\rho}$ is clearly with the finest, locally convex topology.

(vii) If L is a (Hausdorff) locally convex space and $\mathcal{A}$ is a bounded subset of L, then $T_\mathcal{A} : L^b \to (0, +\infty)$ defined by $\varphi \mapsto \sup\{ |\varphi(x)| : x \in \mathcal{A} \}$ is a semi-norm. If $\omega$ is a collection of bounded subsets of L, then we define $L'_\omega (L^+_{\omega} \text{ or } L^b_{\omega})$ to be the LCPL of $(L'_{(T_A)} (L^+_A) \text{ or } (L^b_{(T_A)})) : A \in \omega$.

We now present some applications of Theorem 4.

Construction 6. Let $L_1$ be a linear space and $L_2$ be a locally convex space over $K$ with $L_2 = (L_2)_{Q_0}$ for some $Q \subset \text{SPECL}_2$. Hence the collection $\{ \cap_{i=1}^n V_{q_i}(y, \varepsilon) : y \in L_2, \varepsilon > 0 \text{ and } q_i \in Q \text{ for } i = 1, 2, \ldots, n \}$ is a base of open subsets of $L_2$. Let $\varphi : L_1 \to L_2$ be a linear map and $(L_1)_{T_{\varphi}}$ be $L_1$ with the weakest topology $T_{\varphi}$ for $\varphi$ to be continuous.

(i) The collection $\{ \varphi^{-1}(\cap_{i=1}^n V_{q_i}(y, \varepsilon)) : y \in L_2, \varepsilon > 0 \text{ and } q_i \in Q \text{ for } i = 1, 2, \ldots, n \}$ is a base of open subsets of $(L_1)_{T_{\varphi}}$ by proposition 1(i).

(ii) By the similar arguments in Construction 3 and Lemma 1 (vii), $(L_1)_{T_{\varphi}}$ has an open base of neighborhoods of any $x \in L_1$ consisting of the sets of the form $\cap_{i=1}^n V_{q_i,0,\varphi}(x, \varepsilon)$, where $\varepsilon > 0$ and $q_i \in Q$ for $i = 1, 2, \ldots, n$.

(iii) By property (i) on p.2, the collection $\{ \cap_{i=1}^n V_{q_i,0,\varphi}(x, \varepsilon) : x \in L_1, \varepsilon > 0 \text{ and } q_i \in Q \text{ for } i = 1, 2, \ldots, n \}$ is a base of open subsets of $(L_1)_{T_{\varphi}}$.

We note $q \circ \varphi \in \text{SPECL}_1$ for any $q \in \text{SPECL}_2$.

In fact, the construction of $(L_1)_{T_{\varphi}}$ is a generalization of that in Theorem 3(i), and $L_{T_{\varphi}}$ and $(L_1)_{T_{\varphi}}$ have the same nice properties. The following theorem generalizes the idea of Construction 6 much further.

Theorem 5. Let $L, L_1$ be linear space, and $L_2$ and $L_\gamma$ for any $\gamma \in \Gamma$ be locally convex spaces over $K$.

(i) If $\varphi : L_1 \to L_2$ is a linear map and $(L_1)_{T_{\varphi}}$ is $L_1$ with the weakest topology for $\varphi$ to be continuous, then $(L_1)_{T_{\varphi}}$ is with the weakest, locally convex topology for $\varphi$ to be continuous. In fact, if $Q \subseteq \text{SPECL}_2$
with $L_2 = (L_2)_Q$ and $P = \{q \circ \varphi : q \in Q\}$, then $P \subseteq \text{SPECL}_1$ and $(L_1)_{\tau_{\varphi}} = (L_1)_P$.

(ii) If $\Lambda \neq \emptyset$ is a collection of linear maps from $L_1$ into $L_2$, and $(L_1)_{\tau_{\Lambda}}$ is $L_1$ with the weakest topology for any $\varphi \in \Lambda$ to be continuous, then $(L_1)_{\tau_{\Lambda}}$ is with the weakest, locally convex topology for any $\varphi \in \Lambda$ to be continuous. If $Q$ is defined in (i) and $P = \{q \circ \varphi : \varphi \in \Lambda$ and $q \in Q\}$, then $P \subseteq \text{SPECL}_1$ and $(L_1)_{\tau_{\Lambda}} = (L_1)_P$.

(iii) If $\varphi_\gamma : L \to L_\gamma$ is a linear map for any $\gamma \in \Gamma$ and $L_\gamma$ is $L$ with the weakest topology for any $\varphi_\gamma$ to be continuous, then $L_\gamma$ is with the weakest, locally convex topology for any $\varphi_\gamma$ to be continuous. In fact, if $P_\gamma \subseteq \text{SPECL}_\gamma$ with $L_\gamma = (L_\gamma)_{P_\gamma}$ for any $\gamma \in \Gamma$ and $P = \{p_\gamma \circ \varphi_\gamma : p_\gamma \in P_\gamma$ and $\gamma \in \Gamma\}$, then $P \subseteq \text{SPECL}$ and $L_\gamma = L_P$.

Proof. The identity in (i) follows directly from Constructions 4 (ii) and 6 (iii). For (ii), we let $Q = \text{Spec}L_2$. Thus the collection $\{V_q(0,1) : q \in Q\}$ is a base of neighborhoods of $0 \in L_2$ and the collection $\{V_q(y,1) = y + V_q(0,1) : q \in Q\}$ is a base of open subsets of $L_2$. The conclusions can be obtained by Proposition 1 (ii) and the routine computations. This is also the case for (iii).

At the end of this section, we will discuss some special properties of semi-norms on a linear space $L$. If $p \in \text{SPECL}$, then we will give a sufficient condition for $L_{\tau_p}$ to be locally convex. On the other hand, if $\varphi \in L^*$, then $L_{\tau_\varphi}$ is locally convex and $L_{\tau_\varphi} = L_{(p_\varphi)}$. Thus there must exist distinctions between usual semi-norms on $L$, and the semi-norms associated with linear functionals on $L$. We will also consider these distinctions.

Let $L$ be a linear space and $M$ be a proper, linear subspace of $L$ such that $N = M$ of $N = L$ for any linear subspace $N$ of $L$ with $M \subseteq N \subseteq L$, then $M$ is called a hyperplane of $L$ (Definition on p. 41, [2]). Hence the following conditions are equivalent: (i) $M$ is a hyperplane of $L$, (ii) the quotient space $L/M$ has dimension 1, and (iii) $M = \ker \varphi$ for some non-trivial $\varphi \in L^*$ (p. 41, [2]). In this case, for any fixed $x_0 \in L\setminus M$, $L = M \oplus Kx_0$, i.e. for any $y \in L$, there exist unique $x \in M$ and $\alpha \in K$ with $y = x + \alpha x_0$.

Definition 3. Let $L$ be a linear space and $p \in \text{SPECL}$, then $p$ is called
extremal if, for any \( q \in \text{SPECL} \) with \( q \leq p \) on \( L \), \( q = \gamma p \) for some \( \gamma \geq 0 \).

We have the following characterizations of extremal semi-norms on \( L \).

**Proposition 2.** Let \( L \) be a linear space and \( p \in \text{SPECL} \) be non-trivial, then the following statements are equivalent.

(i) \( p = p_\varphi \) for some non-trivial \( \varphi \in L^* \).

(ii) \( \ker p \) is a hyperplane of \( L \).

(iii) \( p \) is extremal.

**Proof.** This is Proposition 15.1, [7].

We have the following corollaries.

**Corollary 1.** If \( L \) is a linear space and \( p \in \text{SPECL} \) is extremal, then \( L_{r_p} = L_{(p)} \).

**Proof.** Since \( p \) is extremal, we have \( p = p_\varphi \) for some non-trivial \( \varphi \in L^* \). Hence \( L_{r_\varphi} = L_{(p_\varphi)} = L_{(p)} \) by Theorem 3 (i). We note that \( L_{r_\varphi} \) is with the weakest topology for \( \varphi \) to be continuous, and \( \varphi \) is continuous iff \( p_\varphi \) is continuous iff \( p \) is continuous. Thus \( L_{r_\varphi} \) is with the weakest topology for \( p \) to be continuous. This implies \( L_{r_\varphi} = L_{r_p} = L_{(p)} \).

We may wonder whether the given condition in Corollary 1 is necessary for \( p \) to be extremal. We give a partial and related answer for this question.

**Corollary 2.** If \( L \) is an infinite dimensional, linear space, then any non-trivial norm \( p \) on \( L \) is not extremal.

**Proof.** For the brevity of notation, we write \( L \) for \( L_{(p)} \). Thus \( L \neq L_w \) (1, p. 58, [1]). If \( p \) is extremal, then \( p = p_\varphi \) on \( L \) for some non-trivial \( \varphi \in L^* \). Since \( p \) is continuous on \( L \), so is \( p_\varphi \). Hence \( \varphi \in L' = (L_{(p)})' \), and \( p_\varphi \) is a continuous semi-norm on \( L_w \). This implies that \( p \) is continuous on \( L_w \). Hence \( L = L_{(p)} \) is not with the weakest, locally convex topology for \( p \) to be continuous. This is a contradiction. Thus \( p \) is not extremal.

Thus if \( L \) is an infinite dimensional, linear space, then even a norm satisfying the identity in Corollary 1 may not be extremal.
Corollary 3. If $L$ is a linear space and $\varphi \in L^*$ is non-trivial, then $L(p) = L_{\tau_{\varphi}} = L(p_{\varphi})$ for any non-trivial $p \in \text{SPECL}$ with $p \leq p_{\varphi}$ on $L$.

Proof. We note that $p = \gamma p_{\varphi}$ on $L$ for some $\gamma > 0$, and $L_{\tau_{\varphi}} = L(p_{\varphi})$. The conclusion follows directly from Construction 2 (ii).

Although not every semi-norm $p$ on a linear space $L$ is extremal, $p$ is indeed the upper envelope of a collection of extremal semi-norms on $L$ in the sense of Proposition 3.

Lemma 2. Let $L$ be a linear (or locally convex) space over $K$, $p \in \text{SPECL}$ (or $p \in \text{SpecL}$) and $x_0 \in L$, then there is an $\varphi \in L^*$ (or $\varphi \in L'$) with $\varphi(x_0) = p(x_0)$ and $p_{\varphi} \leq p$ on $L$.

Proof. For both case, we let $M = Kx_0$ and $\varphi_0 : M \to K$ by $\alpha x_0 \mapsto \alpha p(x_0)$. We can easily check that $\varphi_0 \in M^*$ and $\varphi_0(x_0) = p(x_0)$. Also, if $x = \alpha x_0 \in M$, then $|\varphi_0(x)| = |\alpha \varphi_0(x_0)| = |\alpha| p(x_0) = p(\alpha x_0) = p(x)$. By Hahn–Banach theorem, there is an $\varphi \in L^*$ with $\varphi|_M = \varphi_0$ and $p_{\varphi} \leq p$ on $L$ (Theorem 3.1.1, [2]). If $p \in \text{SpecL}$, then $\varphi_0 \in M'$, where $M$ is with the relative topology induced by $L$. Since $p_{\varphi} \leq p$ on $L$, we have $\varphi \in L'$.

Proposition 3. Let $L$ be a linear (or locally convex) space and $p \in \text{SPECL}$ (or $p \in \text{SpecL}$), then $p(x) = \sup\{p_{\varphi}(x) : \varphi \in L^*$ (or $\varphi \in L'$) with $p_{\varphi} \leq p\}$ for any $x \in L$.

Proof. By Lemma 2, there is an $\varphi_0 \in L^*$ (or $\varphi_0 \in L'$) with $\varphi_0(x) = p(x)$ and $p_{\varphi_0} \leq p$ on $L$. Thus $\sup\{p_{\varphi}(x) : \varphi \in L^*$ (or $\varphi \in L'$) with $p_{\varphi} \leq p\} \geq p_{\varphi_0}(x) = p(x)$. The inverse inequality is clear.

Corollary 1. Let $L$ be a linear (or locally convex) space and $p \in \text{SPECL}$ (or $p \in \text{SpecL}$) be non-trivial and non-extremal, then there exist at least two non-trivial $\varphi \in L^*$ (or $\varphi \in L'$) with $p_{\varphi} \leq p$ on $L$.

Definition 4. If $L$ is a locally convex space and $\emptyset \neq A \subseteq L$, then we define $A^0$ (or $A^0'$) be the set $\{\varphi \in L' : |\varphi(x)| \leq 1 \text{ for } x \in A\}$ (or $\{p \in \text{SpecL} : p(x) \leq 1 \text{ for } x \in A\}$) which is called the polar of $A$ in $L'$ (or SpecL). Similarly, if $A' \subseteq L'$ (or $p \subseteq \text{SpecL}$) is non-empty, then we define $^0(A') = \{x \in L : |\varphi(x)| \leq 1 \text{ for } \varphi \in A'\}$ (or $^0P = \{x \in L : p(x) \leq 1 \text{ for } p \in P\}$) which is called the pre-polar of $A'$ (or $P$) in $L$.

We have the following variation of bipolar theorem.

Proposition 4. If $L$ is a locally convex space and $\emptyset \neq A \subseteq L$, then $^0(A^0) = ^0(A^0')$ which is the closed, absolutely convex hull of $A$ in $L$. 
Proof. We note that if \( \varphi \in A^o \), then \( p_{\varphi} \in A^{o'} \). If \( x \in o(A^{o'}) \), then \( p(x) \leq 1 \) for any \( p \in A^{o'} \). If \( \varphi \subseteq A^o \), then \( p_{\varphi} \in A^{o'} \) implies \( |\varphi(x)| = p_{\varphi}(x) \leq 1 \). This is true for any \( \varphi \in A^o \). Thus \( x \in o(A^o) \). Hence \( o(A^{o'}) \subseteq o(A^o) \).

Conversely, we note \( o(A^{o'}) = \cap \{ p^{-1}([0,1]) : p \in A^{o'} \} \) is closed, convex, balanced in \( L \). If \( x_0 \not\in o(A^{o'}) \), then there is an \( \varphi_0 \in L' \) with \( |\varphi_0(x_0)| > 1 \) and \( |\varphi_0(x)| \leq 1 \) for any \( x \in o(A^{o'}) \) (Corollary 15.3, [7]). If \( x \in A \), then \( x \in o(A^{o'}) \) implies \( |\varphi_0(x)| \leq 1 \). This is true for any \( x \in A \). Thus \( \varphi_0 \in A^o \).

Since \( |\varphi_0(x_0)| > 1 \), we have \( x_0 \not\in o(A^o) \). This proves \( o(A^o) \subseteq o(A^{o'}) \). Hence the given identity is proved.

2. Bases of continuous semi-norms on locally convex spaces: Inner structure of projective limit construction of locally convex spaces

In this section, the projective limit construction of locally convex spaces by semi-norms means the LCPL construction of these spaces defined in Construction 4, and we will consider the inner structure of them. The prototypes of questions are in the following:

(i) If \( L \) is a linear space and \( P, Q \subseteq \text{SPECL} \) are non-empty, then what are the necessary and sufficient conditions for \( P \) and \( Q \) with \( L_P = L_Q \) ?

(ii) For any \( \emptyset \neq P \subseteq \text{SPECL} \), is there a maximal (or minimal) \( Q \subseteq \text{SPECL} \) with \( L_P = L_Q \) ? If such an \( Q \) exists, what are the relations between \( P \) and \( Q \) ?

(iii) For any \( \emptyset \neq P \subseteq \text{SPECL} \), what are the general forms for continuous semi-norms on \( L_P \), and convex, balanced neighborhoods of \( 0 \in L_P \) ?

The above questions compose an essential part of projective limit construction of locally convex spaces, and closely related to the various notions of bases of continuous semi-norms on locally convex spaces whose meanings will be precisely defined later. On the other hand, since the Minkowski functionals associated with convex, balanced, absorbing subsets of \( L \) are the only semi-norms on \( L \), we will consider the similar questions as above with \( P \) replaced by a collection \( \rho \) of convex, balanced,
absorbing subsets of $L$. However, in order to investigate more sophisticated questions, we will take a roundabout approach (Theorem 8 in the following) for the second part of questions.

We first prove the following lemma.

**Lemma 3.** Let $L$ be a linear space, $p, q \in \text{SPEC}L$ and $p_i \in \text{SPEC}L$ for $i = 1, 2, \ldots, n$.

(i) $p + q \in \text{SPEC}L$ and $\sup\{p, q\} \in \text{SPEC}L$, where $\sup\{p, q\}(x) = \max\{p(x), q(x)\}$ for any $x \in L$.

(ii) $p \leq \sum_{i=1}^{n} \gamma_i p_i$ on $L$ for some $\gamma_1, \gamma_2, \ldots, \gamma_n \geq 0$ iff for any $\varepsilon > 0$, there is an $\delta > 0$ with $\cap_{i=1}^{n} V_{p_i}(0, \delta) \subseteq V_p(0, \varepsilon)$ iff for any $\varepsilon > 0$, there is an $\delta > 0$ with $\cap_{i=1}^{n} V_{p_i}(0, \delta) \subseteq V_p(0, \varepsilon)$.

Let $A$ and $B$ be convex, balanced, absorbing subsets of $L$, and $p_A(x)$ and $p_B(x)$ be the Minkowski functionals associated with $A$ and $B$ on $L$, respectively.

(iii) $\gamma p_A = p_{\gamma A}$ on $L$ for any $\gamma > 0$.

(iv) $p_{A \cap B} = \sup\{p_A, p_B\}$ on $L$. We note that $A \cap B$ is also convex, balanced, absorbing in $L$.

(v) $p_{A+B} \leq \sup\{p_A, p_B\} \leq p_A + p_B$ on $L$, we note that $A + B$ is also convex, balanced, absorbing in $L$.

**Proof.** (ii) Let these three statements be denoted by (a), (b) and (c).

(a) $\Rightarrow$ (b) Without loss of generality, we can assume $p \neq 0$ on $L$ and $\gamma_i > 0$ for $i = 1, 2, \ldots, n$. If $\varepsilon > 0$ is given, then we let $\delta_i = \varepsilon / n \gamma_i$ for $i = 1, 2, \ldots, n$ and $\delta = \min\{\delta_i : i = 1, 2, \ldots, n\}$. Since $\cap_{i=1}^{n} V_{p_i}(0, \delta_i) \subseteq V_p(0, \varepsilon)$, we have $\cap_{i=1}^{n} V_{p_i}(0, \delta) \subseteq V_p(0, \varepsilon)$.

(b) $\Rightarrow$ (c) If $\varepsilon > 0$ is given, then there is an $\delta > 0$ with $\cap_{i=1}^{n} V_{p_i}(0, \delta) \subseteq \cap_{i=1}^{n} V_{p_i}(0, 2\delta) \subseteq V_p(0, \varepsilon) \subseteq V_p(0, \varepsilon)$.

(c) $\Rightarrow$ (a) We prove the existence of $\gamma_i$'s by induction on $n$. If $n = 1$, then $V_{p_1}(0, \delta) \subseteq V_p(0, 1)$ for some $\delta > 0$. We can check $p(x) \leq \delta^{-1} p_1(x)$ for $x \in L$. Let $\gamma_1 = \delta^{-1}$. Suppose that the conclusion is true for $n$. Let $p, p_i \in \text{SPEC}L$ for $i = 1, 2, \ldots, n + 1$ satisfying (c), then $\cap_{i=1}^{n+1} V_{p_i}(0, \delta) \subseteq V_p(0, 1)$ for some $\delta > 0$. Thus $(\cap_{i=1}^{n} V_{p_i}(0, \delta)) \cap V_{p_{n+1}}(0, \delta) \subseteq V_p(0, 1)$. Let $q_i = p_i + p_{n+1}$, then $q_i \in \text{SPEC}L$ and $V_{q_i}(0, \delta) \subseteq V_{p_i}(0, \delta) \cap V_{p_{n+1}}(0, \delta)$ for $i = 1, 2, \ldots, n$. Hence there is an $\beta_i \geq 0$ for $i = 1, 2, \ldots, n$ with
\[ p \leq \sum_{i=1}^{n} \beta_i q_i \] by the inductive hypothesis. Thus \( p \leq \sum_{i=1}^{n+1} \gamma_i p_i \), where \( \gamma_i = \beta_i \) for \( i = 1, 2, \ldots, n \) and \( \gamma_{n+1} = \sum_{i=1}^{n} \beta_i \).

(iii) If \( x \in L \) and \( \epsilon > 0 \) is given, then \( x \in \alpha A \) and \( p_A(x) + \epsilon > \alpha \) for some \( \alpha > 0 \). Hence \( \gamma^{-1} x \in \alpha (\gamma^{-1} A) \) implies \( p_{\gamma^{-1} A}(\gamma^{-1} x) \leq \alpha < p_A(x) + \epsilon \) and \( p_{\gamma^{-1} A}(x) < \gamma p_A(x) + \gamma \epsilon \) for any \( \epsilon > 0 \). Thus \( p_{\gamma^{-1} A}(x) \leq \gamma p_A(x) \).

But \( p_A(x) = p_{\gamma^{-1} A}(\gamma x) \leq \gamma p_A(x) \) implies \( \gamma^{-1} p_A(x) \leq p_A(x) \). The given identity is proved.

(iv) \( A \cap B \) is clearly convex, balanced, absorbing in \( L \). Since \( A \cap B \subseteq A \), we have \( p_A \leq p_{A \cap B} \). This implies \( \sup\{p_A, p_B\} \leq p_{A \cap B} \) on \( L \). Conversely, if \( \epsilon > 0 \), there exist \( \alpha, \beta > 0 \) with \( x \in \alpha A, p_A(x) + \epsilon > \alpha \) and \( x \in \beta B, p_B(x) + \epsilon > \beta \). Thus \( x \in \max\{\alpha, \beta\}(A \cap B) \) implies \( p_{A \cap B}(x) \leq \max\{\alpha, \beta\} < \max\{p_A(x) + \epsilon, p_B(x) + \epsilon\} = \sup\{p_A, p_B\}(x) + \epsilon \) for any \( \epsilon > 0 \). Hence \( p_{A \cap B}(x) \leq \sup\{p_A, p_B\}(x) \). The given identity is proved.

If \( L \) is a linear space, \( P \subseteq \text{SPECL} \) and \( Q \subseteq P \) are non-empty, then \( Q \) is said to direct \( P \) whenever, for any \( p_1, p_2 \in P \), there exist an \( q \in Q \) and \( \gamma \geq 0 \) with \( p_i \leq \gamma q \) for \( i = 1, 2 \) (Definition 4.2, [6]).

**Proposition 5.** Let \( L \) be a linear space, \( P \subseteq \text{SPECL} \) and \( Q \subseteq P \) be non-empty, then the following statements (i)~(iii) are equivalent.

(i) \( Q \) directs \( P \).

(ii) For any \( p_1, p_2 \in P \), there exist \( q \in Q \) and \( \gamma \geq 0 \) with \( \sup\{p_1, p_2\} \leq \gamma q \).

(iii) For any \( p_1, p_2 \in P \), there exist \( q \in Q \) and \( \gamma \geq 0 \) with \( p_1 + p_2 \leq \gamma q \).

Also, the following statements (iv) and (v) are equivalent.

(iv) \( L_P \) has a subbase of neighborhoods of \( 0 \in L \) consisting of the sets of the form \( V_q(0, \epsilon) \) for \( q \in Q \) and \( \epsilon > 0 \).

(v) \( L_Q = L_P \).

(vi) Each of the statements (i)~(iii) implies each of (iv)~(v).

**Proof.** The equivalences among (i), (ii) and (iii) are clear, and so is \( L_Q \subseteq L_P \). (iv) implies \( L_P \subseteq L_Q \). Thus (iv) implies (v). We can easily check that (v) implies (iv). For (vi), we assume that (i) is true. If \( p \in P \) and \( \epsilon > 0 \), then \( p \leq \gamma q \) for some \( q \in Q \) and \( \gamma \geq 0 \). If \( p \neq 0 \) on \( L \), then \( \gamma \neq 0 \), and \( V_q(0, \epsilon/\gamma) \subseteq V_p(0, \epsilon) \). Thus we proved (iv).
Remarks. (i) The condition $Q \subseteq P$ is essential for the equivalences between (iv) and (v) since, without it, (iv) only implies $L_p \subseteq L_Q$. (v) clearly implies (iv).

(ii) In general, (iv) does not imply (i) under the condition $Q \subseteq P$: If $p \in P$, then (iv) implies $p \leq \sum_{i=1}^{n} \gamma_i q_i$ by Lemma 3 (ii), where $q_i \in Q$ and $\gamma_i \geq 0$ for $i = 1, 2, \ldots, n$. But $\sum_{i=1}^{n} \gamma_i q_i \in Q$ is not necessarily true, and neither is (i).

The following theorem gives necessary and sufficient conditions for subcollection $Q$ of $P$ with $L_p = L_Q$.

**Theorem 6.** Let $L$ be a linear space, $P \subseteq \text{SPECL}$ and $Q \subseteq P$ be non-empty, then the following statements are equivalent

(i) $L_p = L_Q$.

(ii) For any $p \in P$ and $\varepsilon > 0$, there exist an $q_i \in Q$ for $i = 1, 2, \ldots, n$ and $\delta > 0$ with $\bigcap_{i=1}^{n} V_{q_i}(0, \delta) \subseteq V_p(0, \varepsilon)$.

(iii) $L_p$ has a subbase of neighborhoods of $0 \in L$ consisting of the sets of the form $V_{q}(0, \varepsilon)$ for any $q \in Q$ and $\varepsilon > 0$.

(iv) For any $p \in P$, there exist an $q_i \in Q$ and $\gamma_i \geq 0$ for $i = 1, 2, \ldots, n$ with $p \leq \sum_{i=1}^{n} \gamma_i q_i$.

**Proof.** The equivalences among (i), (ii) and (iii) follow directly from Proposition 5. (iii) implies (iv) by Lemma 3 (ii). $L_Q \subseteq L_P$ is clear. By routine computations, we can check that (iv) implies $L_P \subseteq L_Q$. Thus $L_P = L_Q$ and (iv) implies (i).

**Remark.** If $\emptyset \neq P \subseteq \text{SPECL}$ and $\emptyset \neq Q \subseteq P$ directs $P$, then $L_P = L_Q$. But $L_P = L_Q$ and $Q \subseteq P$ do not necessarily imply that $Q$ directs $P$. Hence the characterizations in Theorem 6 are more general than those in Proposition 5.

We now consider the questions posed in the beginning of this section.

Let $L$ be a linear space and $\emptyset \neq P \subseteq \text{SPECL}$, then $P$ is called irreducible if (i) $p \in P$ and $\gamma \geq 0 \Rightarrow \gamma p \in P'$; (ii) $q \in \text{SPECL}$ and $q \leq p$ with $p \in P \Rightarrow q \in P'$; and (iii) $p_1, p_2 \in P \Rightarrow p_1 + p_2 \in P'$ (Definition 2.1, [7]). In this case, the trivial semi-norm which is identically zero on $L$ is in $P$. The intersection of irreducible subcollections of SPECL is clearly irreducible.
Definition 5. If $\emptyset \neq P \subseteq \text{SPECL}$, then we let $\text{irr}(P)$ be the smallest, irreducible subcollection of $\text{SPECL}$ containing $P$ which is the intersection of all irreducible subcollections of $\text{SPECL}$ containing $P$ and is called the irreducible hull of $P$.

Lemma 4. Let $L$ be a linear space and $\emptyset \neq P \subseteq \text{SPECL}$.

(i) $P$ is irreducible iff (1) $p \in P$ and $\gamma \geq 0 \Rightarrow \gamma p \in P$; (2) $q \in \text{SPECL}$ and $q \leq p$ with $p \in P \Rightarrow q \in P$; and (3) $p_1, p_2 \in P \Rightarrow \sup\{p_1, p_2\} \in P$.

(ii) $\text{irr}(P) = \{p \in \text{SPECL} : \text{there exist an } p_i \in P \text{ and } \gamma_i \geq 0 \text{ for } i = 1, 2, \ldots, n \text{ with } p \leq \sum_{i=1}^{n} \gamma_i p_i\} = \{p \in \text{SPECL} : \text{there exist an } p_i \in P \text{ for } i = 1, 2, \ldots, n \text{ and } \gamma \geq 0 \text{ with } p \leq \sum_{i=1}^{n} \gamma p_i\} = \{p \in \text{SPECL} : \text{there exist an } p_i \in P \text{ for } i = 1, 2, \ldots, n \text{ and } \gamma \geq 0 \text{ with } p \leq \sup\{\gamma p_i : i = 1, 2, \ldots, n\}\} = \{p \in \text{SPECL} : \text{there exist an } p_i \in P \text{ for } i = 1, 2, \ldots, n \text{ and } \gamma \geq 0 \text{ with } p \leq \sup\{\gamma p_i : i = 1, 2, \ldots, n\}\}.

Proof. The equivalences in (i) are clear. In (ii), the four collections are identical by (i). We can easily check the first collection after $\text{irr}(P)$ is irreducible and contains $P$. Hence it is equal to $\text{irr}(P)$.

Let $L$ be a linear space and $\emptyset \neq P \subseteq \text{SPECL}$, then Construction 4 gives the exterior construction of $L_P$. More precisely, for any $p \in P$, we equip $L$ with the weakest, locally convex topology for $p$ to be continuous. Then the topology of $L_P$ is defined by these topologies through Proposition 1 (ii). We now gives the interior construction of $L_P$ which defines a base of neighborhoods of $0 \in L$ directly. The only requirement for this approach is that $P$ be irreducible.

Theorem 7. Let $L$ be a linear space over $K$ and $P \subseteq \text{SPECL}$ be irreducible. Let $\eta_0 = \{V \subseteq L : V_p(0, 1) \subseteq V \text{ for some } p \in P\}$ and $\eta_x = x + \eta_0 = \{x + V : V \in \eta_0\}$ for any $x \in L$.

(i) For any $x \in L$, $\eta_x$ is a filter (of subsets) of $L$ (Definition on p. 75, [2]), and any member of $\eta_x$ contains $x$. Hence if $\tau = \{V \subseteq L : x \in V \Rightarrow V \in \eta_x\}$, then $\tau$ is a topology of $L$. For any $x \in L$, the collection $\{V_p(x, 1) : p \in P\}$ is an open base of neighborhoods of $x$ in $L_\tau$. Hence the collection $\{V_p(x, 1) : x \in L \text{ and } p \in P\}$ is a base of open subsets of $L_\tau$, and $\eta_x$ is the filter of all neighborhoods of
any \( x \in L \) in \( L_\tau \). Moreover, for any \( x \in L \) and \( W \in \eta_x \), there is an \( V \in \eta_x \) with \( W \in \eta_y \) for \( y \in V \).

(ii) \( L_\tau \) is a locally convex space and \( L_\tau = L_P \).

(iii) \( p \in \text{Spec} L_\tau \) iff \( p \in P \). Hence \( V \) is a convex, balanced neighborhood of \( 0 \in L_\tau \) iff \( V_p(0, 1) \subseteq V \subseteq \overset{\text{y}}{V}(0, 1) \) for some \( p \in P \).

Proof. (i) We consider the following steps.

(1) It is clear that \( 0 \in V \) for any \( V \in \eta_0 \), and \( \eta_0 \) is closed under super-sets. If \( V, W \in \eta_0 \), then \( V_p(0, 1) \subseteq V \) and \( V_p(0, 1) \subseteq W \) for \( p_1, p_2 \in P \). If \( p = p_1 + p_2 \), then \( p \in P \) and \( V_p(0, 1) \subseteq V_p(0, 1) \cap V_p(0, 1) \subseteq V \cap W \) implies \( V \cap W \in \eta_0 \). Hence \( \eta_0 \) is a filter on \( L \). If \( x \in L \) is fixed and \( x + V \in \eta_x \) with \( x + V \subseteq W \), then \( V \subseteq -x + W \) implies \( -x + W \in \eta_0 \) since \( \eta_0 \) is a filter on \( L \). Hence \( W \in \eta_x \). Also, if \( x + V' \in \eta_x \), then \( (x + V) \cap (x + V') = x + (V \cap V') \). Since \( V \cap V' \in \eta_0 \), we have \( (x + V) \cap (x + V') \in \eta_x \). Hence \( \eta_x \) is a filter on \( L \) for any \( x \in L \).

(2) It is clear that \( 0, L \in \tau \). If \( V, W \in \tau \) and \( V \cap W \in \tau \). Otherwise, for any \( x \in V \cap W \), \( V, W \in \eta_x \) implies \( V \cap W \in \eta_x \). This is true for any \( x \in V \cap W \). Hence \( V \cap W \in \tau \). Similarly, we can prove that \( \tau \) is closed under arbitrary union. Thus \( \tau \) is a topology of \( L \).

(3) If \( W \) is a neighborhood of \( x \) in \( L_\tau \), then \( x \in V \subseteq W \) for some \( V \in \tau \). Since \( V \in \eta_x \), we have \( W \in \eta_x \), and \( -x + W \in \eta_0 \). This implies \( V_p(0, 1) \subseteq -x + W \) for some \( p \in P \) and \( V_p(x, 1) = x + V_p(0, 1) \subseteq W \). We claim \( V_p(x, 1) \in \tau \). If \( y \in V_p(x, 1) \), then \( p(y - x) < 1 \) implies \( \varepsilon = 1 - p(y - x) > 0 \). We can check \( V_p(y, \varepsilon) \subseteq V_p(x, 1) \) and \( V_p(y, \varepsilon) = y + V_p(0, \varepsilon) = y + V_{\varepsilon - 1} \in \eta_y \). Hence \( V_p(x, 1) \in \eta_y \) for any \( y \in V_p(x, 1) \) implies \( V_p(x, 1) \in \tau \). This proves that \( \{V_p(x, 1) : p \in P \} \) is an open base of neighborhoods of any \( x \in L \) in \( L_\tau \). Hence \( \{V_p(x, 1) : x \in L \) and \( p \in P \} \) is a base of open subsets of \( L \).

(4) If \( W \) is a neighborhood of \( x \) in \( L_\tau \), then \( V_p(x, 1) \subseteq W \) for some \( p \in P \) by (3), and \( V_p(x, 1) = x + V_p(0, 1) \in \eta_x \). Thus \( X \in \eta_x \). Conversely, if \( W \in \eta_x \), then \( x + V_p(0, 1) = V_p(x, 1) \subseteq W \) for some \( p \in P \). Since \( V_p(x, 1) \) is a neighborhood of \( x \) in \( L_\tau \), so is \( W \). Thus \( \eta_x \) is the filter of all neighborhoods of any \( x \in L \) in \( L \tau \) we now prove the last statement of this part. If \( W \in \eta_x \), then \( W \) is a neighborhood of \( x \) in \( L_\tau \), and \( x \in V \subseteq W \) for some \( V \in \tau \). By the definition of \( \tau \), \( V \in \eta_y \), and hence \( W \in \eta_y \) for any \( y \in V \). The last statement is proved.

(ii) We consider the following steps.
(1) If \( a, b \in L, c = a + b \) and \( c + V_p(0, 1) \) is a basic, open neighborhood of \( c \) in \( L_\tau \), then \( a + V_p(0, \frac{1}{2}) = a + V_{2p}(0, 1) \in \eta_a \) and \( b + V_p(0, \frac{1}{2}) \in \eta_b \). We can check \((a + V_p(0, \frac{1}{2})) + (b + V_p(0, \frac{1}{2})) \subseteq c + V_p(0, 1)\). Hence the vector addition of \( L \) is continuous w.r.t. \( \tau \). Also, if \( a \in L, \alpha \in K, b = \alpha a \) and \( b + V_p(0, 1) \) is a basic, open neighborhood of \( b \) in \( L_\tau \), then we let \( \varepsilon = \frac{1}{2}(p(a) + 1) \) and \( \delta = \frac{1}{2}(1) \). We can check that \( \gamma x \in b + V_p(0, 1) \) for any \( x \in a + V_p(0, \delta) = a + V_{\delta-1_p}(0, 1) \in \eta_a \) and \( |\gamma - \alpha| < \varepsilon \). Thus the scalar multiplication of \( L \) is continuous w.r.t. \( \tau \).

(2) By the proof in (1), \( L_\tau \) is a linear topological space. Since \( V_p(x, 1) \) is convex in \( L \) for any \( x \in L \) and \( p \in P \), \( L_\tau \) is a locally convex space.

(3) The collection \( \{ \bigcap_{i=1}^n V_{p_i}(x, \varepsilon) : x \in L, \varepsilon > 0 \text{ and } p_i \in P \text{ for } i = 1, 2, \ldots, n \} \) is a base of open subsets of \( L_P \) by Construction 4. In particular, \( V_p(x, 1) \) is open in \( L_P \) for any \( x \in L \) and \( p \in P \). This implies \( L_\tau \subseteq L_P \). Conversely, we note \( \bigcap_{i=1}^n V_{p_i}(x, \varepsilon) = \bigcap_{i=1}^n V_{\varepsilon^{-1}p_i}(x, 1) \) is certainly open in \( L_\tau \). Hence we have \( L_\tau \subseteq L_P \). This proves \( L_\tau = L_P \).

(iii) If \( p \in \text{Spec}L_\tau \), then \( V_p(0, 1) = p^{-1}([0, 1]) \) is a neighborhood of \( 0 \in L_\tau \), and hence \( V_{\varepsilon^{-1}p_i}(0, 1) \subseteq V_p(0, 1) \) for some \( p_i \in P \). We can check that \( p \leq p_i \) on \( L_\tau \). Since \( L_\tau \) is irreducible, we have \( p \in P \). Conversely, if \( p \in P \), then \( p \in \text{Spec}L_\tau = \text{Spec}_L \) by (ii). We now prove the second statement. If \( V \) is a convex, balanced neighborhood of \( 0 \in L_\tau \), then the Minkowski functional \( p(x) = P_V(x) \) associated with \( V \) on \( L \) is in \( \text{Spec}L_\tau = P \) and \( V_p(0, 1) \subseteq V \subseteq V_p(0, 1) \) (Proof on p.95, [2], and Lemma 5.1(5), [6]). Conversely, if \( V \) is a convex, balanced, absorbing subset of \( L \) with \( V_p(0, 1) \subseteq V \subseteq V_p(0, 1) \) for some \( p \in P \), then \( V \) is a neighborhood of \( 0 \in L_\tau \) (for \( p \in P = \text{Spec}L_\tau \)), and so is \( V \).

Remark. In the proof of Theorem 7, the notation \( \eta_x \) is intended to indicate the filter of all neighborhoods of \( x \)—“\( \eta \)” means neighborhoods. By the proof in (i), \( \eta_x \) indeed is. We have also found a subcollection of \( \eta_x \), namely, the collection of the sets of the form \( x + V_p(0, 1) = V_p(x, 1) \) for any \( p \in P \) to be an open base of neighborhoods of any \( x \in L \) in \( L_\tau \).

We now give two examples of the construction in Theorem 7.

Example 2. (i) The simplest one is that \( L \) is a linear space and \( P = \text{SPECL} \). The space \( L_P \) is called the super-barrelled extension of \( L \).

(ii) If \( L \) is a (Hausdorff) locally convex space and \( P_s, L_P \), are defined in Example 1 (iv), then \( P_s \) is irreducible, and \( \text{Spec}L_s = P_s \). Also, \( \{ V_p(x, 1) : p \in P_s \} \) is an open base of neighborhoods of \( 0 \in L \) in \( L_P \). Similarly, \( P_b \)
is irreducible and \( \text{SpecL}_{P_b} = P_b \).

Theorem 7 (iii) states \( \text{SpecL}_P = P \) if \( P \subseteq \text{SPECL} \) is irreducible. The following corollary considers the general case of \( P \) (which may satisfy some additional conditions).

**Corollary 1.** Let \( L \) be a linear space and \( \emptyset \neq P \subseteq \text{SPECL} \).

(i) \( L_P \) has an open (or closed) base of neighborhoods of \( 0 \in L \) consisting of the sets of the form \( \bigcap_{i=1}^{n} V_{p_i}(0, \varepsilon) \) (or \( \bigcap_{i=1}^{n} \bar{V}_{p_i}(0, \varepsilon) = \varepsilon^0 \{p_i : i = 1, 2, \ldots, n\} \)) for \( \varepsilon > 0 \) and \( p_i \in P \) for \( i = 1, 2, \ldots, n \). Hence \( P \in \text{SpecL}_P \) iff \( p \in \text{irr}(P) \) iff \( p \leq \sum_{i=1}^{n} \gamma_i p_i \) for some \( p_i \in P \) and \( \gamma_i \geq 0 \) for \( i = 1, 2, \ldots, n \).

(ii) If \( \gamma p \in P \) for any \( p \in P \) and \( \gamma \geq 0 \), then \( L_P \) has an open (or closed) base of neighborhoods of \( 0 \in L \) consisting of the sets of the form \( \bigcap_{i=1}^{n} V_{p_i}(0, 1) \) (or \( 0 \{p_i : i = 1, 2, \ldots, n\} \)) for \( p_i \in P \) for \( i = 1, 2, \ldots, n \). Thus \( p \in \text{SpecL}_P \) iff \( p \leq \sum_{i=1}^{n} p_i \) for \( p_i \in P \) for \( i = 1, 2, \ldots, n \).

(iii) If (1) \( p \in P \) and \( \gamma \geq 0 \Rightarrow \gamma p \in P \) and (2) \( p_1, p_2 \in P \Rightarrow p_1 + p_2 \in P \), then \( L_P \) has an open (or closed) base of neighborhoods of \( 0 \in L \) consisting of the sets of the form \( V_{p_i}(0, 1) \) (or \( \bar{V}_{p_i}(0, 1) = 0 \{p\} \)) for any \( p \in P \). Thus \( p \in \text{SpecL}_P \) iff \( p \leq p_1 \) for some \( p_1 \in P \).

**Proof.** (i) The conclusion on closed base of neighborhoods of \( 0 \in L \) follows from the set-containment \( V_{p_1}(0, \varepsilon) \subseteq \bar{V}_{p_2}(0, \varepsilon) \subseteq V_{p_r}(0, 2\varepsilon) \) for any \( p \in P \) and \( \varepsilon > 0 \). If \( p \in \text{SpecL}_P \), then \( \bigcap_{i=1}^{n} V_{p_i}(0, \varepsilon) \subseteq V_{p_r}(0, 1) \) for some \( p_i \in P \) for \( i = 1, 2, \ldots, n \) and \( \varepsilon > 0 \). By Lemma 3 (ii), \( p \in \text{irr}(P) \). Conversely, if \( p \in \text{irr}(P) \), then \( p \in \text{SpecL}_P \) is clear. This proves \( \text{SpecL}_P = \text{irr}(P) \).

(ii) The conclusions follow directly from (i) and the conditions on \( P \).

(iii) The conclusions on open (or closed) base of neighborhoods of \( 0 \in L \) follow from (ii) by taking \( p = \sum_{i=1}^{n} p_i \in P \). Since \( p \in \text{SpecL}_P \), \( V_{p_1}(0, 1) \) and \( \bar{V}_{p_1}(0, 1) \) are open and closed neighborhoods of \( 0 \in L_P \), respectively. The second statement is also clear from (ii).

**Remark.** In Corollary 1(iii), we proved that the collection \( \{V_{p_1}(0, 1) : p \in P\} \) is an open base of neighborhoods of \( 0 \in L_P \), and this was proved in Theorem 7 (i). But if \( P \) does not satisfy the second condition of irreducibility, we can not have the identity \( \text{SpecL}_P = P \).
We now give some applications of Corollary 1.

**Example 3.** It $L$ is a locally convex space and $P_1$ is the collection of all lower semi-continuous semi-norms on $L$, then we can easily check that $P_1$ satisfies the first and third conditions of irreducibility. Hence $P_1$ satisfies the conditions in Corollary 1(iii). The barrelled extension $L_{P_1}$ of $L$ has all the barrels as a closed base of neighborhoods of $0 \in L$. The same conclusions are true for the collection $P_{b_1}$ of all ess-bounded, lower semi-continuous semi-norms on $L$, and the quasibarrelled extension $L_{P_{b_1}}$ of $L$.

**Corollary 2.** If $L$ is a locally convex space, then $P_{L'} = \{ p_\varphi : \varphi \in L' \}$ satisfies the first and second conditions of irreducibility.

*Proof.* If $p_\varphi \in P_L$, and $\gamma \geq 0$, then $\gamma p = p_{\gamma \varphi} \in P_{L'}$. Also, if $p \in \text{SPEC}L$ and $p \leq p_\varphi$ for some $\varphi \in L'$, then we can assume $p \neq 0$ on $L$. Thus $p_\varphi$ is an extremal semi-norm on $L$, and $p = \gamma p_\varphi = p_{\gamma \varphi}$ for some $\gamma \geq 0$ by Proposition 2 (iii).

**Corollary 3.** Let $L$ be a linear space and $\emptyset \neq P \subseteq \text{SPEC}L$, then $\text{Spec}L_P = \text{irr}(P)$ and $L_P = L_{\text{irr}(P)}$.

*Proof.* The first identity is proved in Corollary 1(i). Since $P \subseteq \text{irr}(P)$, we have $L_P \subseteq L_{\text{irr}(P)}$. Conversely, $\text{irr}(P)$ is irreducible, and the collection $\{ V_P(0,1) : p \in \text{irr}(P) \}$ is an open base of neighborhoods of $0 \in L_{\text{irr}(P)}$. If $p \in \text{irr}(P)$, then $\cap_{i=1}^a V_{p_i}(0,\delta) \subseteq V_p(0,1)$ for some $\delta > 0$ and $p_i \in P$ for $i = 1, 2, \ldots, n$ by Lemma 3 (ii). Thus $V_p(0,1)$ is a neighborhood of $0 \in L_P$. This proves $L_{\text{irr}(P)} \subseteq L_P$.

*Remark.* Corollary 3 is a generalization of Theorem 4—By Theorem 2, $L_P$ is a locally convex space. In the proof of Theorem 4, $L_P$ is the LCPL of $\{ L(p) : p \in \text{Spec}L_P = \text{irr}(P) \} = L_{\text{irr}(P)}$. However, the proof of Theorem 4 uses some basic properties of Minkowski functional (e.g. Lemma 5.1, [6]), but the proof of Corollary 3 does not.

In Theorem 6, we presented the characterizations of all $Q \subseteq P$ with $L_P = L_Q$, where $P \subseteq \text{SPEC}L$ is given. It is natural to ask whether there are other subcollections of $\text{SPEC}L$ satisfying this identity. We thus have the following result.

**Corollary 4.** Let $L$ be a linear space and $\emptyset \neq P \subseteq \text{SPEC}L$.

(i) If $\emptyset \neq Q \subseteq \text{SPEC}L$, then the following statements are equivalent:
(1) \( L_P = L_Q \); (2) \( \text{irr}(P) = \text{irr}(Q) \); (3) \( L_P = L_{\text{irr}(Q)} \); and (4) \( L_Q = L_{\text{irr}(P)} \). Furthermore, each of these statements implies (5) \( Q \subseteq \text{irr}(P) \).

(ii) If \( P \subseteq Q \subseteq \text{SPECL} \), then the statements (1)~(4) in (1) are equivalent to \( P \subseteq Q \subseteq \text{irr}(P) \).

Proof. (i) The implications (1)\(\Rightarrow\)(2) and (1)\(\Rightarrow\)(5) follow from Corollary 1(i), while the implications (2)\(\Rightarrow\)(3) \(\Rightarrow\)(4) \(\Rightarrow\)(1) follow from Corollary 3. All are routine computations.

(ii) If (i) is true, then \( P \subseteq Q \subseteq \text{irr}(P) \) by (i)(5). Conversely, if \( p \subseteq Q \subseteq \text{irr}(P) \), then \( L_P \subseteq L_Q \subseteq L_{\text{irr}(P)} = L_P \) by Corollary 3.

Remarks: (i) Corollary 4 is mainly concerned about the following question: If \( L \) is a linear space and \( \emptyset \neq P \subseteq \text{SPECL} \), do there exist minimal and maximal subcollections \( Q \) of \( \text{SPECL} \) with \( L_P = L_Q \). Corollary 3 and Corollary 4 (i)(5) prove that \( \text{irr}(P) \) is the maximal one.

(ii) Corollary 4(i) also confirms the possibility of the existence of subcollection \( Q \) of \( \text{SPECL} \) which is not properly contained in \( P \), nor properly contains \( P \), but satisfies the identity \( L_P = L_Q \) (and the set–containment \( Q \subseteq \text{irr}(P) \)), e.g. two different norms on a finite dimensional, linear space \( L \) satisfy the given identity and Corollary 4(i)(5) by Definition 1.1.3 and Theorem 1.2.1, [1].

(iii) If \( Q \subseteq \text{irr}(P) \) and \( L_P \neq L_Q \), then \( P \subseteq Q \) must not be true.

Definition 6. If \( \emptyset = P \subseteq \text{SPECL} \), then \( P \) is called “the” base of continuous semi–norms on \( L_P \) for the following reasons: (i) any \( p \in \text{SpecL}P \) is dominated by a linear combination of members of \( P \); (ii) the subcollections \( Q \) of \( P \) can not serve as “the” base since they may not satisfy the required identity; and (iii) any \( Q \) with \( P \subseteq Q \subseteq \text{irr}(P) \) can not be the base, either (because of the lack of the uniqueness) although the given identity is satisfied.

Colloquially speaking, Theorem 6 (or Corollary 4 (ii)) proves the characterizations of sub–(or super–) collection \( Q \) of \( P \) with \( L_P = L_Q \). However, Corollary 4 (i) gives the characterizations of all \( Q \subseteq \text{SPECL} \) with \( L_P = L_Q \). Since \( P \) and \( Q \) are arbitrary, this part gives the necessary and sufficient conditions for non–empty subcollections \( P, Q \) of \( \text{SPECL} \) with \( L_P = L_Q \).
Any locally convex space $L$ has a base of neighborhoods of $0 \in L$ consisting of convex, balanced subsets of $L$. Hence it is natural to ask whether locally convex spaces can be constructed by a given collection of convex, balanced, absorbing subsets, and the construction is initialized by the process similar to that of Theorem 7. For this question, we have to introduce the following notion.

**Definition 7.** Let $L$ be a linear space $\rho$ be a collection of convex, balanced, absorbing subsets of $L$ such that (i) $A \in \rho$ and $\gamma > 0 \Rightarrow \gamma A \in \rho$; (ii) $B$ is convex, balanced, absorbing in $L$ and $A \subseteq B$ for some $A \in \rho \Rightarrow B \in \rho$; and (iii) $A, B \in \rho \Rightarrow A \cap B \in \rho$, then $\rho$ is called irreducible. The condition (i) can be replaced by $\gamma A \in \rho$ for any $A \in \rho$ and $\gamma \in K$ with $\gamma \neq 0$. For any $\rho \neq \emptyset$ of convex, balanced, absorbing subsets of $L$, we let $\text{irr}(\rho)$ be the intersection of all irreducible collections of convex, balanced, absorbing subsets of $L$ containing $\rho$. $\text{irr}(\rho)$ is irreducible, and is called the irreducible hull of $\rho$.

**Lemma 5.** Let $L$ be a linear space and $\rho \neq \emptyset$ be a collection of convex, balanced, absorbing subsets of $L$.

(i) $\text{irr}(\rho) = \{A \subseteq L : A \text{ is convex, balanced, absorbing in } L \text{ with } \cap_{i=1}^n \gamma_i A_i \subseteq A \text{ for some } A_i \in \rho \text{ and } \gamma_i > 0 \text{ for } i = 1, 2, \ldots, n\} = \{A \subseteq L : A \text{ is convex, balanced, absorbing in } L \text{ with } \cap_{i=1}^n \gamma_i A_i \subseteq A \text{ for some } \gamma > 0 \text{ and } A_i \in \rho \text{ for } i = 1, 2, \ldots, n\}$.

(ii) If $\rho$ is irreducible, then $P_\rho = \{p_A : A \in \rho\} \subseteq \text{SPECL}$ is irreducible.

(iii) $P_{\text{irr}(\rho)} = \text{irr}(P_\rho)$.

**Proof.** (ii) If $A \in \rho$ and $\gamma > 0$, then $\gamma p_A = p_{\gamma^{-1}A}$ by Lemma 3(iii). If $\gamma = 0$, then $\gamma p_A = 0 = p_L$, where $L \in \rho$ since $\rho$ is irreducible. Hence $\gamma p_A \in P_\rho$ for any $A \in \rho$ and $\gamma \geq 0$. Next, if $A \in \rho$ and $p \in \text{SPECL}$ with $p \leq p_A$, then $A \subseteq B \bar{V}_{p_A}(0,1) \subseteq \bar{V}_p(0,1) = B$ implies $B \in \rho$. We can easily check that $p = p_B$ on $L$. Thus $p \in P_\rho$. Finally, if $A, B \in \rho$, then $A \cap B \in \rho$ and $\sup\{p_A, p_B\} = p_{A \cap B}$ on $L$ by Lemma 3 (iv). Thus $\sup\{p_A, p_B\} \in P_\rho$, and we proved that $P_\rho$ is irreducible.

(iii) Since $\text{irr}(\rho)$ is irreducible, $P_{\text{irr}(\rho)}$ is irreducible by (ii) and $P_\rho \subseteq P_{\text{irr}(\rho)}$. Thus $\text{irr}(P_\rho) \subseteq P_{\text{irr}(\rho)}$. Conversely, if $A \in \text{irr}(\rho)$, then $\cap_{i=1}^n \gamma_i A_i \subseteq A$ for some $A_i \in \rho$ and $\gamma_i > 0$ for $i = 1, 2, \ldots, n$. This implies $p_A \leq p_{\cap_{i=1}^n \gamma_i A_i} = \sup\{p_{\gamma_i A_i} = \gamma_i^{-1}p_{A_i} : i = 1, 2, \ldots, n\}$ by Lemma 3(iii). Since
\( p_{A_i} \in P_{\rho}, \) we have \( \gamma_i^{-1}p_{A_i} \in \text{irr}(P_{\rho}) \) for \( i = 1, 2, \ldots, n \) and hence \( p_A \in \text{irr}(P_{\rho}) \) by Lemma 4(i). Thus \( P_{\text{irr}(\rho)} \subseteq \text{irr}(P_{\rho}) \). The given identity is proved.

Remarks. We consider whether \( P \) and \( \rho \) in Lemma 5(ii) and (iii) can be interchanged.

(i) If \( P \subset \text{SPECL} \) is irreducible and \( A_p = p^{-1}([0,1]) \) for any \( p \in P \), then \( \rho_P = \{ A_p : p \in P \} \) satisfies the first and third conditions of irreducibility. Thus \( \rho_P \) is not necessarily irreducible. \( \bar{\rho}_P = \{ \bar{A}_p : p \in P \} \) is not irreducible, either, where \( \bar{A}_p = p^{-1}([0,1]) \) for \( p \in P \).

(ii) If \( \emptyset \neq P \subset \text{SPECL} \), then the identity \( \rho_{\text{irr}(P)} = \text{irr}(\rho_P) \) is not necessarily true, and neither is \( \bar{\rho}_{\text{irr}(P)} = \text{irr}(\bar{\rho}_P) \).

The following is another main theorem of this section.

Theorem 8. \( L \) be a linear space and \( \rho \) be an irreducible collection of convex, balanced, absorbing subsets of \( L \). Let \( \eta_0 = \{ V \subseteq L : A \subset V \text{ for some } A \in \rho \} \) and \( \eta_x = x + \eta_0 = \{ x + V : V \in \eta_0 \} \) for any \( x \in L \).

(i) For any \( x \in L \), \( \eta_x \) is a filter (of subsets) of \( L \), and any member of \( \eta_x \) contains \( x \). Hence if \( \tau = \{ V \subseteq L : x \in V \Rightarrow V \in \eta_x \} \), then \( \tau \) is a topology of \( L \). For any \( x \in L \), the collection \( \{ V_{p_A}(x,1) = x + V_{p_A}(0,1) : A \in \rho \} \) is an open base of neighborhoods of \( x \) in \( L_\tau \), where \( p_A(x) \) is the Minkowski functional associated with \( A \) on \( L \). Hence the collection \( \{ V_{p_A}(x,1) : x \in L \text{ and } A \in \rho \} \) is a base of open subsets of \( L_\tau \), and \( \eta_x \) is the filter of neighborhoods of any \( x \in L \) in \( L_\tau \). Moreover, for any \( x \in L \) and \( W \in \eta_x \), there is an \( V \in \eta_x \) with \( W \in \eta_y \) for any \( y \in V \).

(ii) \( L_\tau \) is a locally convex space.

(iii) \( p \in \text{Spec}_L \iff p = p_A \) for some \( A \in \rho \) iff \( A_p, \bar{A}_p \in \rho \), where \( A_p, \bar{A}_p \) are defined in the above remarks. Hence \( V \) is a convex, balanced neighborhood of \( 0 \in L_\tau \iff V \in \rho \).

Proof. (i) We consider the following steps.

(1) It is clear that \( 0 \in V \) for any \( V \in \eta_0 \), and \( \eta_0 \) is a filter on \( L \), and so is \( \rho \subseteq \eta_0 \). If \( A \in \rho \), then \( \frac{1}{2}A \subseteq \{ x \in L : p_{\bar{A}}(x) = 2p_A(x) \leq 1 \} \subseteq \{ x \in L : P_A(x) < 1 \} = V_{p_A}(0,1) \) implies \( V_{p_A}(0,1) \in \rho \subseteq \eta_0 \). For any \( x \in L \), the proof of \( \eta_x = x + \eta_0 \) to be a filter on \( L \) is similar to that of
Theorem 7 (i) (1). In particular, \( x + V_{PA}(0, 1) = V_{PA}(x, 1) \), \( x + A \) and \( x + \bar{V}_{PA}(0, 1) = \bar{V}_{PA}(x, 1) \) are in \( \eta_x \) for any \( A \in \rho \) and \( x \in L \).

(2) \( \tau \) is clearly a topology of \( L \). We claim that \( V_{PA}(x, 1) \in \tau \) for any \( x \in L \) and \( A \in \rho \). Indeed, if \( y \in V_{PA}(x, 1) \), then \( p_A(y - x) < 1 \) implies \( \varepsilon = 1 - p_A(y - x) > 0 \). Hence \( V_{PA}(y, \varepsilon) \subseteq V_{PA}(x, 1) \), and \( V_{PA}(y, \varepsilon) = y + V_{PA}(0, \varepsilon) = y + V_{PA,A}(0, 1) = y + V_{PA}(0, 1) \in \eta_y \). Hence \( V_{PA}(x, 1) \in \eta_y \) for any \( y \in V_{PA}(x, 1) \) implies \( V_{PA}(x, 1) \in \tau \). If \( W \) is a neighborhood of \( x \) in \( L_\tau \), then \( x \in V \subseteq W \) for some \( V \in \tau \). Thus \( V \in \eta_x \) and \( x + A \subseteq V \) for some \( A \in \rho \). This implies \( x + V_{PA}(0, 1) \subseteq x + A \subseteq V \subseteq W \). Hence \{ \( V_{PA}(x, 1) = x + V_{PA}(0, 1) ; A \in \rho \} \) is an open base of neighborhoods of any \( x \in L \) in \( L_\tau \).

(3) If \( W \) is a neighborhood of \( x \) in \( L_\tau \), then \( W \in \eta_x \) by the proof at the end of (2). Conversely, if \( W \in \eta_x \), then \( x + A \in W \) for some \( A \in \rho \) implies \( x + V_{PA}(0, 1) \subseteq x + A \subseteq W \). Thus \( W \) is a neighborhood of \( x \) in \( L_\tau \). Hence \( \eta_x \) is the filter of all neighborhoods of any \( x \in L \) in \( L_\tau \). The proof of the last statement of this part is similar to that of Theorem 7(i)(4).

(ii) The proofs of the continuity of vector addition and scalar multiplication of \( L \) w.r.t. \( \tau \) are similar to those of Theorem 7(ii) (since \{ \( V_{PA}(x, 1) ; A \in \rho \} \) is an open base of neighborhoods of any \( x \in L \) in \( L_\tau \)). Since each set of this base is convex in \( L \), \( L_\tau \) is locally convex.

(iii) Let the three statements about SpecL \( \tau \) be denoted by (1), (2) and (3).

(1) \( \Rightarrow \) (2) \( V_p(0, 1) \) is a neighborhood of \( 0 \in L_\tau \), and \( V_{PA}(0, 1) \subseteq V_p(0, 1) \) for some \( A \in \rho \). Hence \( p \leq p_A \) on \( L \). If \( B = \bar{V}_p(0, 1) \), then \( A \subseteq V_{PA}(0, 1) \subseteq \bar{V}_p(0, 1) = B \) implies \( B \in \rho \). We can check \( p = P_B \) on \( L \).

(2) \( \Rightarrow \) (3) If \( p = P_A \) for some \( A \in \rho \), then \( \frac{1}{2} A \subseteq A_p \) implies \( A_p \in \rho \) implies \( \bar{A}_p \in \rho \).

(3) \( \Rightarrow \) (1) Let \( B = \bar{A}_p \in \rho \), then \( p = P_B \), \( A_p = V_{PB}(0, 1) \) is a basic, open neighborhood of \( 0 \in L_\tau \), and \( p(x) < \varepsilon \) for any \( x \in \varepsilon V_{PB}(0, 1) \) and \( \varepsilon > 0 \). Thus \( p \in \text{Spec} L_\tau \). If \( V \) is a convex, balanced neighborhood of \( 0 \in L_\tau \), then \( V_{PA}(0, 1) \subseteq V \) for some \( A \in \rho \) implies \( V \in \rho \) (for \( \frac{1}{2} A \subseteq V_{PA}(0, 1) \)). Conversely, if \( V \in \rho \), then \( \{ x \in L ; P_V(x) < 1 \} \), and so is \( V \), is a convex, balanced neighborhood of \( 0 \in L_\tau \).

Remark. By Theorem 8(iii), \( \rho \) can be a base of neighborhoods of \( 0 \in L_\tau \). However, we choose the subcollection \{ \( V_{PA}(0, 1) ; A \in \rho \} \) of \( \rho \) as a base of neighborhoods of \( 0 \in L_\tau \).
By following corollary, the constructions in Theorems 7 and 8 are compatible.

**Corollary 1.** Let $L$ be a linear space and $\rho$ be an irreducible collection of convex, balanced, absorbing subsets of $L$, then $P_\rho = \{p_A : A \in \rho\} \subseteq $SPECL is irreducible. Furthermore, the space $L_\tau$ in Theorem 8 can be constructed by the method given in Theorem 7 through $P_\rho$.

**Proof.** $P_\rho$ is irreducible by Lemma 5(ii). Let $L_\tau$, be the space constructed in Theorem 7 by $P_\rho$, and $\eta_0 = \{V \subseteq L : A \subseteq V \text{ for some } A \in \rho\}$ and $\eta'_0 = \{V \subseteq L : V_{PA}(0,1) \subseteq V \text{ for some } A \in \rho\}$ be the filters of all neighborhoods of $0 \in L_\tau$, and $0 \in L_{\tau'}$, respectively. We have $L_\tau \subseteq L_{\tau'}$, for $\eta_0 \subseteq \eta'_0$. Conversely, if $V \in \eta'_0$, then $V_{PA}(0,1) \subseteq V$ for some $A \in \rho$ implies $\frac{1}{2}A \subseteq V$ and $\frac{1}{2}A \in \rho$. Thus $\eta'_0 \subseteq \eta_0$, and $L_{\tau'} \subseteq L_\tau$. This proves the given identity.

**Remark.** The space $L_\tau$ in Theorem 8 can be constructed by the method in Theorem 7 by $P_\rho$. But the space in Theorem 7 can not be constructed by the method in Theorem 8 by $\rho_P$ since $\rho_P$ is not necessarily irreducible.

**Notation.** We write $L_{\tau_\rho}$ for the space $L_\tau$ in Theorem 8 for the emphasis of $\rho$. Hence $L_{\tau_\rho} = L_{P_\rho}$ by Corollary 1, where $L_{P_\rho}$ is the space either constructed by the method in Theorem 7 through $P_\rho$ or the LCPL of $\{L(p_A) : A \in \rho\}$—these two meanings are identical by Theorem 7(ii).

**Remarks**

(i) The space $L_{\tau_\rho}$ has the collection $\{V_{PA}(0,1) : A \in \rho\}$ as a base of neighborhoods of $0 \in L$. But $\frac{1}{2}A \in \rho$ and $V_{PA}(0,1) \subseteq \frac{1}{2}A \subseteq V_{PA}(0,1)$. Thus $\rho$ can also be a base of neighborhoods of $0 \in L_{\tau_\rho}$.

(ii) By Corollary 1, some results in Theorem 8 can be easily derived. For example, $p \in \text{Spec}L_{\tau_\rho}$ iff $p \in \text{Spec}L_{P_\rho} = p_\rho$ iff $p = p_A$ for some $A \in \rho$ by Theorem 7.

(iii) Also, $V$ is a convex, balanced neighborhood of $0 \in L_{\tau_\rho}$ iff $V_{PA}(0,1) \subseteq V \subseteq V_{PA}(0,1)$ for some $A \in \rho$. Since $\frac{1}{2}A \subseteq V_{PA}(0,1)$ and $\frac{1}{2}A \in \rho$, we have $V \in \rho$ as we proved in Theorem 8(iii).

We now give several examples of the constructions of $L_{\tau_\rho}$.

**Example 4.** (i) The simplest one is the space $L_{\tau_\rho}$, where $\rho$ is the collection of all convex, balanced, absorbing subsets of the linear space $L$. Thus $\rho$ is the collection of all convex, balanced neighborhoods of $0 \in L_{\tau_\rho}$.
by Theorem 8(iii).

(ii) If \( L \) is a (Hausdorff) locally convex space and \( \eta_{0,cs} \) is defined in Example 1(iv). The space \( L_{\tau_{\eta_{0,cs}}} \) is denoted by \( L_{\tau_{cs}} \) in \( \S2 \), [4], and is called the \( c \)-sequential extension of \( L \). By Theorem 8(iii), we have \( \text{Spec}_L_{\tau_{cs}} = P_{\tau_{\eta_{0,cs}}} = \{ p_V : V \in \eta_{0,cs} \} \), where \( p_V(x) \) is the Minkowski functional associated with \( V \) on \( L \). We can check that \( P_{\eta_{0,cs}} = P_s \) which is the collection of all \( s \)-continuous semi-norms on \( L \). This result is consistent with that in Example 2(ii). Similarly, we denote the bornological extension \( L_{\tau_{ncb}} \) by \( L_{\tau_b} \) in \( \S2 \), [4].

Notations. For any (Hausdorff) locally convex space \( L, L_P, L_{\eta_{0,cs}}, L_{\tau_{\eta_{0,cs}}}, \) and \( L_{\tau_{cs}} \) denote the \( c \)-sequential extension of \( L \), while \( L_{P_b}, L_{\eta_{ncb}}, L_{\tau_{cb}} \) and \( L_{\tau_b} \) denote the bornological extension of \( L \).

The following corollary considers the same properties in Theorem 8 for any collection \( \rho \) of convex, balanced, absorbing subsets of \( L \) (which may satisfy some additional conditions).

Corollary 2. Let \( L \) be a linear space and \( \rho \neq \emptyset \) be a collection of convex, balanced, absorbing subsets of \( L \). Let \( L_{P_\rho} \) be LCPL of \( \{ L(P_A) : A \in \rho \} \).

(i) The collection \( \{ \varepsilon \cap_{i=1}^{n} A_i : \varepsilon > 0 \text{ and } A_i \in \rho \text{ for } i = 1, 2, \cdots, n \} \) is a base of neighborhoods of \( 0 \in L_{P_\rho} \). Hence \( p \in \text{Spec}_L_{P_\rho} \) iff \( p \leq \sum_{i=1}^{n} \gamma_i p_{A_i} \) for some \( \gamma_i \geq 0 \) and \( A_i \in \rho \) for \( i = 1, 2, \cdots, n \); and \( V \) is a convex, balanced neighborhood of \( 0 \in L_{P_\rho} \) iff \( V \in \text{irr}(\rho) \).

(ii) If \( \gamma A \in \rho \) for any \( A \in \rho \) and \( \gamma > 0 \), then the collection \( \{ \cap_{i=1}^{n} A_i : A_i \in \rho \text{ for } i = 1, 2, \cdots, n \} \) is a base of neighborhoods of \( 0 \in L_{P_\rho} \). Hence \( p \in \text{Spec}_L_{P_\rho} \) iff \( p \leq \sum_{i=1}^{n} p_{A_i} \) for some \( A_i \in \rho \) for \( i = 1, 2, \cdots, n \).

(iii) If \( A \cap B \in \rho \) for any \( A, B \in \rho \), then the collection \( \{ \varepsilon A : A \in \rho \text{ and } \varepsilon > 0 \} \) is a base of neighborhoods of \( 0 \in L_{P_\rho} \). Hence \( p \in \text{Spec}_L_{P_\rho} \) iff \( p \leq \gamma p_A \) for some \( A \in \rho \) and \( \gamma > 0 \).

(iv) If (1) \( A \in \rho \) and \( \gamma > 0 \Rightarrow \gamma A \in \rho \); and (2) \( A, B \in \rho \Rightarrow A \cap B \in \rho \), then \( \rho \) is a base of neighborhoods of \( 0 \in L_{P_\rho} \). Hence \( p \in \text{Spec}_L_{P_\rho} \) iff \( p \leq p_A \) for some \( A \in \rho \).

Proof. (i) The statement of the base of neighborhoods of \( 0 \in L_{P_\rho} \) is clear since \( V_{PA}(0, \varepsilon) \subseteq \varepsilon A \subseteq V_{PA}(0, \varepsilon) \) for any \( A \in \rho \) and \( \varepsilon > 0 \). If \( V \) is a
convex, balanced neighborhood of $0 \in L_{P_{\rho}}$, then $\varepsilon \cap_{i=1}^{n} A_{i} \subseteq V$ for some $\varepsilon > 0$ and $A_{i} \in \rho$ for $i = 1, 2, \ldots, n$ implies $V \in \text{irr}(\rho)$ by Lemma 5 (i). The converse follows that lemma, too.

(iii) The statement of $\text{Spec} L_{P_{\rho}}$ follows from the given condition on $\rho$ and the repeated uses of Lemma 3(iii) and (iv).

**Remark.** If $L$ is a linear space and $\rho \neq \emptyset$ is a collection of convex, balanced, absorbing subsets of $L$, then $\text{irr}(\rho)$ can certainly be a base of neighborhoods of $0 \in L_{P_{\rho}}$ by Corollary 2(i). But in practical applications, we use the subcollection $\{\varepsilon \cap_{i=1}^{n} A_{i} : A_{i} \in \rho \text{ for } i = 1, 2, \ldots, n \text{ and } \varepsilon > 0\} \text{irr}(\rho)$ as a base of neighborhoods of $0 \in L_{P_{\rho}}$, and this has been indicated in Corollary 2(i).

**Corollary 3.** Let $L$ be a linear space and $\rho \neq \emptyset$ be a collection of convex, balanced, absorbing subsets of $L$. If $L_{P_{\rho}}$ is the LCPL of $\{L_{(P_{A})} : A \in \rho\}$ and $L_{\text{irr}(\rho)}$ is the space constructed in Theorem 8 by $\text{irr}(\rho)$, then $L_{P_{\rho}} = L_{\text{irr}(\rho)}$.

**Proof.** The first proof is easy since $L_{P_{\rho}} = L_{\text{irr}(P_{\rho})} = L_{P_{\text{irr}(\rho)}} = L_{\text{irr}(\rho)}$ by Corollary 3 of Theorem 7, Lemma 5(ii) and Corollary 1. The second proof will be used in the next corollary. By Theorem 8(i) (or Corollary 2(i)), $\text{irr}(\rho)$ is the collection of all convex, balanced neighborhoods of $0 \in L_{\text{irr}(\rho)}$ (or $0 \in L_{P_{\rho}}$). Thus these two spaces have the same base of neighborhoods of $0 \in L$, and are identical.

**Notation.** If $L$ is a linear space and $\rho \neq \emptyset$ is a collection of convex, balanced, absorbing subsets of $L$, then the space $L_{P_{\rho}}$ is the space $L_{\text{irr}(\rho)}$ in Theorem 8. In order to emphasize the role of $\rho$ in the construction of $L_{P_{\rho}}$, we usually write $L_{\tau_{\rho}}$ for $L_{P_{\rho}}$. Hence the notation $L_{\tau_{\rho}}$ has two identical meanings: one is simply the projective limit space $L_{P_{\rho}}$, and the other indirectly indicates the space $L_{\text{irr}(\rho)}$. Thus the real meaning of the identity $L_{\tau_{\rho}} = L_{P_{\rho}}$ has been extended.

**Corollary 4.** If $L$ is a linear space and $\rho \neq \emptyset$ is a collection of convex, balanced, absorbing subsets of $L$, then $P_{\text{irr}(\rho)} = \text{irr}(P_{\rho})$.

**Proof.** Since $P_{\text{irr}(\rho)}$ is irreducible, we have $\text{irr}(P_{\rho}) \subseteq P_{\text{irr}(\rho)}$. If $A \in \text{irr}(\rho)$, then $p_{A} \in \text{Spec} L_{P_{\text{irr}(\rho)}} = \text{Spec} L_{\text{irr}(\rho)} = \text{Spec} L_{P_{\rho}} = \text{irr}(P_{\rho})$ by Corollary 3. Thus $P_{\text{irr}(\rho)} \subseteq \text{irr}(P_{\rho})$.

We now give some examples of the space $L_{P_{\rho}}$ in Corollary 2.
Example 5. (i) If \( L \) is a locally convex space, then the collection \( \rho_c \) of all barrels of \( L \) satisfies the first and third conditions of irreducibility, and is thus a closed base of neighborhoods of \( 0 \in L_{\tau_{pc}} \). If \( P_{\rho_c} = \{ p_A : A \in \rho_c \} \), then \( P_{\rho_c} = P_1 \), which is the collection of all lower semi-continuous semi-norms on \( L \), and \( L_{\tau_{pc}} = L_{P_{\rho_c}} \). \( L_{\tau_{pc}} \) is called the barrelled extension of \( L \). The same conclusions hold for \( \rho_b \) (the collection of all quasibarrels of \( L \)), \( L_{\tau_{pb}} \) (the quasi-barrelled extension of \( L \)) and \( P_{b_1} \) (the collection of all ess-bounded, lower semi-continuous semi-norms on \( L \)).

(ii) By the notations so far, the space \( (L_{P_1})_{P_1} \) in Example 1(v) can be written as \( (L_{\tau_{cs}})_{\tau_{pc}} \). \( L \) is called strictly \( s \)-barrelled if \( (L_{\tau_{cs}})_{\tau_{pc}} = L_{\tau_{cs}} \). Similarly, we have \( (L_{P_1})_{P_{b_1}} = (L_{\tau_{cs}})_{\tau_{pb}} \). \( L \) is called strictly \( s \)-quasibarrelled if \( (L_{\tau_{cs}})_{\tau_{pb}} = L_{\tau_{cs}} \) (Theorem 5, [4]).

(iii) If \( L \) is a locally convex space, then we can write \( (L_{\tau_b})_{\tau_{pc}} \) for \( (L_{P_1})_{P_1} \). \( L \) is said to satisfy the strict condition on ess-uniform boundedness if \( (L_{\tau_b})_{\tau_{pc}} = L_{\tau_b} \) (Theorem 6, [4]).

Corollary 5. Let \( L \) be a linear space, \( \rho \neq \emptyset \) be a collection of convex, balanced, absorbing subsets of \( L \), then the following statements are equivalent.

(i) \( L_{\tau_\rho} = L_{\tau_\rho'} \).

(ii) For any \( A \in \rho \), there exist an \( A_i \in \rho' \) and \( \gamma_i \geq 0 \) for \( i = 1, 2, \ldots, n \) with \( p_A \leq \sum_{i=1}^{n} \gamma_i p_{A_i} \) on \( L \).

(iii) \( L_{\tau_\rho} \) has a subbase of neighborhoods of \( 0 \in L \) consisting of the sets of the form \( \epsilon A \) for any \( A \in \rho' \) and \( \epsilon > 0 \).

(iv) For any \( A \in \rho \), there exist an \( A_i \in \rho' \) and \( \gamma_i > 0 \) for \( i = 1, 2, \ldots, n \) with \( \cap_{i=1}^{n} \gamma_i A_i \subseteq A \).

Proof. (i)\( \Rightarrow \) (ii) If \( A \in \rho \), then \( p_A \in \text{Spec}_{L_{\tau_\rho}} = \text{Spec}_{L_{\tau_\rho}} \). By Corollary 1 of Theorem 7, we proved (ii).

(ii)\( \Rightarrow \) (iii) If \( A \in \rho \) and (ii) is true, then we can assume \( \gamma_i \neq 0 \) for \( i = 1, 2, \ldots, n \). Let \( \delta_i = \frac{1}{n\gamma_i} \) for \( i = 1, 2, \ldots, n \) and \( \delta = \min\{ \delta_i : i = 1, 2, \ldots, n \} \), then \( \cap_{i=1}^{n} V_{p_{A_i}}(0, \delta) \subseteq \cap_{i=1}^{n} V_{p_{A_i}}(0, \delta_i) \subseteq \{ x \in L : \sum_{i=1}^{n} \gamma_i p_{A_i}(x) < 1 \} \subseteq A \). But \( \frac{1}{2} \delta A_i \subseteq \frac{1}{2} \delta V_{p_{A_i}}(0, 1) = V_{p_{A_i}}(0, \frac{1}{2} \delta) \subseteq V_{p_{A_i}}(0, \delta) \) for \( i = 1, 2, \ldots, n \) implies \( \cap_{i=1}^{n} \frac{1}{2} \delta A_i \subseteq A \).

(iii) is proved.
(iii)⇒(iv) If $A \in \rho$, then $A$ is a neighborhood of $0 \in L_{\tau_\rho}$ by Corollary 2. Thus we proved (iv).

(iv)⇒(i) It is clear that $L_{\tau_{\rho'}} \subseteq L_{\tau_\rho}$. If $A \in \rho$, then (iv) implies that $A$ is a neighborhood of $0 \in L_{\tau_{\rho'}}$ (by Corollary 2(i) on $L_{\tau_\rho}$), and so is any basic neighborhood $\varepsilon \cap \bigcap_{i=1}^n A_i$ of $0 \in L_{\tau_{\rho'}}$, where $\varepsilon > 0$ and $A_i \in \rho$ for $i = 1, 2, \ldots, n$ are arbitrary. Thus $L_{\tau_\rho} \subseteq L_{\tau_{\rho'}}$.

Remark. Corollary 5 gives the characterizations of subcollections $\rho'$ of $\rho$ with $L_{\tau_\rho} = L_{\tau_{\rho'}}$. However, such an $\rho'$ may not exist. But there indeed exist other collections $\rho'$ of convex, balanced, absorbing subsets of $L$ with $L_{\tau_\rho} = L_{\tau_{\rho'}}$ (e.g. $\text{irr}(\rho)$).

We characterize all such collections $\rho'$ in the above remark.

**Corollary 6.** Let $L$ be a linear space and $\rho \neq \emptyset$ be a collection of convex, balanced, absorbing subsets of $L$.

(i) If $\rho'$ is another such a collection, then the following statements are equivalent: (1) $L_{\tau_\rho} = L_{\tau_{\rho'}}$; (2) $\text{irr}(\rho) = \text{irr}(\rho')$; (3) $L_{\tau_\rho} = L_{\tau_{\rho'}}$; (4) $L_{\tau_{\rho'}} = L_{\tau_{\text{irr}(\rho)}}$; (5) $\text{irr}(P_\rho) = \text{irr}(P_{\rho'})$. Furthermore, each of the above statements implies (6) $\rho' \subseteq \text{irr}(\rho)$.

(ii) If $\rho \subseteq \rho'$, then the statements (1)~(5) in (i) are equivalent to $\rho \subseteq \rho' \subseteq \text{irr}(\rho)$.

**Proof.** (i) By Corollary 2(i), $\text{irr}(\rho)$ (or $\text{irr}(\rho')$) is the collection of all convex, balanced neighborhoods of $0 \in L_{\tau_\rho}$ (or $0 \in L_{\tau_{\rho'}}$). This proves (1)⇒(2). The implications (2)⇒(3)⇒(4) follow from Corollary 3 and the Notation after it. Since $\text{irr}(P_{\rho'}) = \text{Spec}_{L_{\rho'}} = \text{Spec}_{L_{\tau_{\rho'}}} = \text{Spec}_{L_{\tau_{\text{irr}(\rho)}}} = \text{Spec}_{L_{\rho'}} = \text{irr}(P_{\rho})$, we proved (4)⇒(5). Similarly, $L_{\tau_\rho} = L_{\tau_{\rho'}} = L_{\tau_{\text{irr}(\rho)}} = L_{\tau_{\rho'}} = L_{\tau_{\rho'}}$ proves (5)⇒(1). If (1) is true, then $\text{irr}(\rho) = \text{irr}(\rho')$ implies $\rho' \subseteq \text{irr}(\rho') = \text{irr}(\rho)$. This proves (6).

(ii) If (i)(1) is true, then $\rho \subseteq \rho' \subseteq \text{irr}(\rho)$. Conversely, if $\rho \subseteq \rho' \subseteq \text{irr}(\rho)$, then $L_{\tau_\rho} = L_{\rho'} \subseteq L_{\rho'} \subseteq L_{\rho'}$, $L_{\tau_{\rho'}} = L_{\rho'} = L_{\rho'}$ implies $L_{\tau_\rho} = L_{\tau_{\rho'}}$.

Remarks. (i) Corollary 6(i) states that if $L_{\tau_\rho} = L_{\tau_{\rho'}}$, then $\rho' \subseteq \text{irr}(\rho)$. By Corollary 3, $L_{\tau_\rho} = L_{\tau_{\text{irr}(\rho)}}$ is always true. Hence $\text{irr}(\rho)$ is the maximal collection of convex, balanced, absorbing subsets of $L$ satisfying this identity.
(ii) But the minimal one of such \( p' \) may not exist—Corollary 5(i) confirms the possibility of proper subcollection \( p' \) of \( p \) with \( L_{\tau_p} = L_{\tau_{p'}} \).

(iii) Corollary 6(i) asserts the possibility of existence of \( p' \) of convex, balanced, absorbing subsets of \( L \) which is not properly contained in \( p \), nor properly contains \( p \), but satisfies \( L_{\tau_p} = L_{\tau_{p'}} \) and \( p' \subseteq \text{irr}(p) \).

(iv) If \( p' \subseteq \text{irr}(p) \) and \( L_{\tau_p} \neq L_{\tau_{p'}} \), then \( p \subseteq p' \) must not be true.

**Definition 8.** If \( L \) is a linear space and \( p \neq \emptyset \) is a collection of convex, balanced, absorbing subsets of \( L \), then \( p \) is called "the" base of convex, balanced neighborhoods of \( 0 \in L_{\tau_p} \).

Corollary 5 (or 6(ii)) characterizes sub–(or super–) collection \( p' \) of convex, balanced, absorbing subsets of \( L \) with \( L_{\tau_p} = L_{\tau_{p'}} \). But Corollary 6(i) characterizes all collections \( p' \) satisfying this identity. Since \( p \) and \( p' \) are arbitrary in this part, it presents the necessary and sufficient conditions for two collections of convex, balanced, absorbing subsets of \( L \) which satisfies the required identity.

### 3. Appendix

A few years after the completion of this paper, the author found other intrinsic properties of extremity of semi–norms which are presented in this Appendix. We assume that \( L \) is a linear space over \( K \) (\( K = \mathbb{R} \) or \( K = \mathbb{C} \)).

**Theorem 9.** Let \( p \in \text{SPECL} \) be non-extremal and \( \varphi \in L^* \) be non–trivial, then the following statements are equivalent.

(i) \( p \leq p_\varphi \) on \( L \).

(ii) \( p_\varphi(x) \leq 1 \) implies \( p(x) \leq 1 \).

(iii) \( p_\varphi(x) < 1 \) implies \( p(x) < 1 \).

(iv) \( p(x) = \gamma p_\varphi(x) \) for any \( x \in L \) and some \( 0 \leq \gamma \leq 1 \).

**Proof.** The equivalences between (i) and (ii) are clear, and so are those between (i) and (iii). If (i) is true and \( x_0 \in L \setminus \ker \varphi \) with \( \varphi(x_0) = 1 \), then \( L = \ker \varphi \oplus Kx_0 \), and hence, for any \( y \in L \), there exist a unique \( x \in \ker \varphi \) and \( \alpha \in K \) with \( y = x + \alpha x_0 \). Thus \( \varphi(y) = \alpha \) and \( p(y) \leq p(x) + |\alpha|p(x_0) = |\alpha|p(x_0) \) (since \( p(x) \leq p_\varphi(x) = 0 \)). But
\[ p(y) = p(\alpha x_0 - (-x)) \geq ||\alpha|p(x_0) - p(-x)| = ||\alpha|p(x_0) - p(x)| = ||\alpha|p(x_0). \]

Thus \( p(y) = |\alpha|p(x_0) = p(x_0)|\varphi(y)| = p(x_0)p_{\varphi}(y) = \gamma p_{\varphi}(y) \), where \( \gamma = p(x_0) \leq p_{\varphi}(x_0) = 1. \) Conversely, if (iv) is true, then so is (i).

We have the following remarks.

(i) If \( p, q \in \text{SPECL} \), then each of (i), (ii) and (iii) in the above theorem is equivalent to \( p \leq q \) on \( L \), where \( p_{\varphi} \) is replaced by \( q \). Therefore, we illustrate a difference between ordinary and extremal semi–norms on \( L \).

(ii) Let \( \varphi, \psi \in L^* \) be non–trivial, then \( \psi = \gamma \varphi \) on \( L \) for some \( \gamma \neq 0 \) iff \( \ker \varphi = \ker \psi \). Thus the open (or closed) unit disk of \( p \in \text{SPECL} \), i.e. the set \( \{ x \in I : p(x) < 1 \) (or \( p(x) \leq 1 \}) \) plays the similar role as \( \ker \psi \) when we compare two semi–norms on \( L \).

(iii) The condition of non–extremity of \( p \) in Theorem 9 is essential. Otherwise we will derive an interesting mistake which can be considered as a corollary of Theorem 9.

**Corollary 1.** If \( \varphi, \psi \in L^* \) are non–trivial then the following statements are equivalent.

(i) \( p_{\varphi} \leq p_{\psi} \) or \( p_{\psi} \leq p_{\varphi} \) on \( L \).

(ii) \( \ker \varphi = \ker \psi \).

**Proof.** If \( \ker \varphi = \ker \psi \), then \( \psi = \gamma \varphi \) on \( L \) for some \( \gamma \neq 0 \). If \( |\gamma| \leq 1 \), then \( p_{\psi} \leq p_{\varphi} \). If \( |\gamma| \geq 1 \), then \( p_{\varphi} \leq p_{\psi} \). Conversely, if \( p_{\psi} \leq p_{\varphi} \) on \( L \), then \( p_{\psi} = \gamma p_{\varphi} \) on \( L \) for some \( 0 < \gamma \leq 1 \) by Theorem 9. This implies \( \ker \varphi = \ker \psi \).

We can easily check that, for non–trivial \( \varphi, \psi \in L^* \), (ii) of the above corollary implies (i). Therefore, the implication of (i) to (ii) is generally not correct.

**References**


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