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Designing an Inventory Model of Parallel-Type Distribution System

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ABSTRACT

A one-upper level warehouse n-lower level retailer inventory distribution model is discussed. This paper presents the parallel-type inventory structure using an order-up-to-level invertory control system for analyzing the approximation of the expected units backordered and the measure of service. We find that the total expected backorder units in system can substitute the expected backorders in the last two periods for the expected backorders in total periods. The rate of total expected backorders which is the measure of disservice, is given by dividing the improved units of total expected backorder into the total demand during an order cycle. The average annual total cost in system is obtained by considering the results, but from the viewpoint of this study the cost analysis is not described.

1. Introduction

This paper presents a two-level distribution system with one-upper level ware-house (UW) in the first echelon and $n-$lower level retailer (LR's) in the second echelon. The various models have been developed for determining a mathematical representation of the system and the

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solution by Clark(1972), Hoadly and Heyman(1977). Pinkus, Gross and Soland(1981), Federgruen and Zipkin(1984), etc. In a one-warehouse n-retailer parallel-type inventory structure we show that the approximate service measure and the expected units backordered under an order-up-to-level (I_0, T) control system are analyzed. The system properties are different whether a direction of service level is management oriented or customer oriented, but there are essentially equivalent(1981) and this study is mainly related to design the backorder problem in system.

We use Figure 1 for explanation purposes in the 1−UW, n−LR distribution structure. The figure shows the behavior of inventory levels versus time. The solid line repersents total net inventory at the LR'S, the dotted line denotes the total system inventory position. When the UW receives the products, units are immediately allocated to each LR, its lead time is R_L and the lead time from the supplier to the UW is S_L. In(I_0, T) system demand is probabilistic and the average demand rate changes very little with time. The costs of the system do not depend upon the specific value of I_0 used.

We place order 1 at time t_0, in Figure 1. The next order will not be placed until time t_0+T. All previous orders, including order 1, must have arrived prior to time t_0+T+L. In other words, all I_0 of the inventory position at time to must have reached the stocking point before t_0+T+L. Therefore the service impact is determined by whether or not the total demand exceeds I_0, where the measure of service is a rate of system backorder. We show that only the expected backorders in the last periods can be significant. Using the results, we find the improved backorder units in system and the average annual total cost. The allocation quantities of LR's are obtained and from the objective of this study the cost analysis is not evaluated.

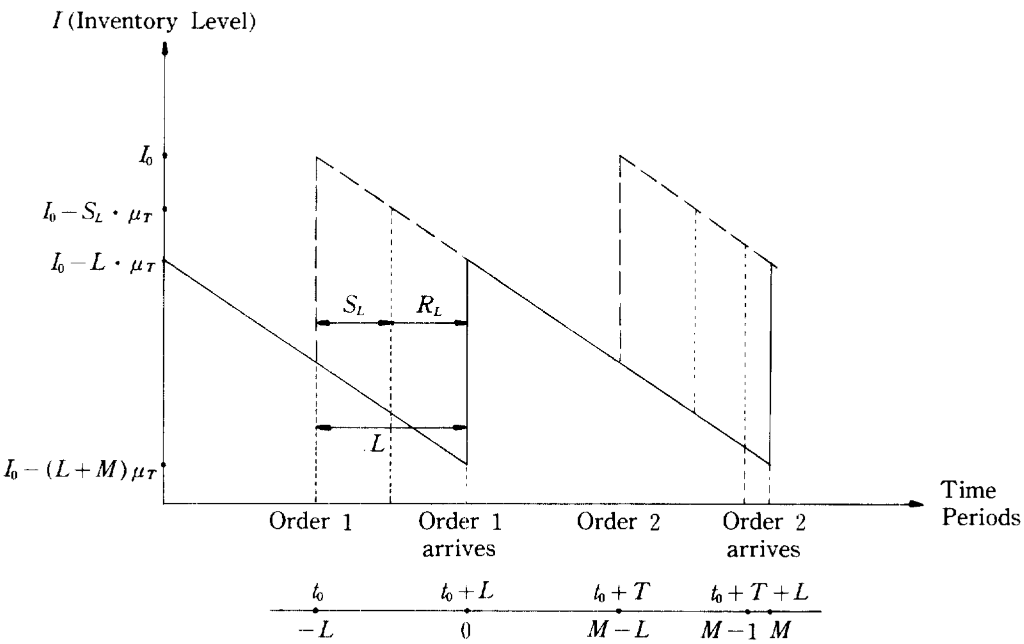

Figure 1. Behavior of Inventory Levels for the Time Period

(Notations)

$D_j(T)$: demand at location j during T periods,

$D(T)$: system demand during T periods,

$EB_j(t)$: expected backorder units by LR j at the end of period t,

EB_{LR} : expected backorder units in the system during an order cycle,

$I(t)$: inventory level at the end of period t,

$I_j(t)$: inventory level at location j at the end of period t,

L : total lead time,

μ_j : average demand per period at j,

μ_s : $\sum_{j=1}^{n}\mu_j$,

σ_j : standard deviation of demand per period at j,

σ_s : $\sum_{j=1}^{n}\sigma_j$,

σ_T : $\sqrt{\sum_{j=1}^{n}\sigma_j^2}$

2. Approximation of System Backorder and Measure of Service Level

Consider a normally distributed variable x with mean $\hat{x}$ and standard deviation σ_x, $p.d.f.$ $f(x)$. The expected shortage per replenishment cycle (ESPR) becomes

$$\text{ESPR} = \int_{\hat{x}+k\sigma_x}^{\infty}(x-\hat{x}-k\sigma_x)f(x)\,dx$$

$$= \int_{\hat{x}+k\sigma_x}^{\infty}(x-\hat{x}-k\sigma_x)\frac{1}{\sigma_x\sqrt{2\pi}}\exp[-(x-\hat{x})^2/2\sigma_x^2]\,dx$$

Here, let $u=(x-\hat{x})/\sigma_x$, the normal loss function $G(k)=\int_k^{\infty}(u-k)f(u)\,du$.

Now ESPR becomes

$$\text{ESPR} = \sigma_x\int_k^{\infty}(u-k)\frac{1}{2\pi}\exp(-u^2/2)\,du$$

$$= \sigma_x \cdot Gu(k) \tag{1}$$

Suppose that a total system inventory I_0 is to be allocated among the LR's so as to satisfy the total demand in T periods and let S_j be the allocation quantities. Then the expected total backorder units at the end of the T periods (EB) are obtained by

$$EB=\sum_{j=1}^{n-1}\sqrt{T}\,\sigma_j G\left[\frac{S_j-T\mu_j}{\sqrt{T}\,\sigma_j}\right]+\sqrt{T}\,\sigma_n\,G\left[\frac{I_0-\sum_{j=1}^{n-1}S_j-T\mu_n}{\sqrt{T}\,\sigma_n}\right] \tag{2}$$

and optimal allocation quantity, S_j^* is found by sloving the equation (2). It will be average demand plus standard deviation over T time periods, namely

$$S_j^* = T \cdot \mu_j + \frac{\sigma_j}{\sigma_s}[I_0 - T \cdot \mu_s] \tag{3}$$

Especially we know the expected backorder units by LR j at the end of period t, period by period through the T periods, are given by the equation (2). Then

$$EB_j(t) = \sqrt{t}\sigma_j G[(S_j^* - t \cdot \mu_j)/\sqrt{t} \cdot \sigma_j]$$

and substituting (3) into the above equation yields

$$EB_j(t) = \sqrt{t} \cdot \sigma_j G\left[\frac{(I_0 - T \cdot \mu_s)}{\sqrt{t}\sigma_s} + \frac{(T-t)\mu_j}{\sqrt{t}\sigma_j}\right]$$

$$= \sqrt{t} \cdot \sigma_j G\left[\frac{(I_0 - T \cdot \mu_s)}{\sigma_s\sqrt{T}}\sqrt{\frac{T}{t}} + \frac{(T-t)}{\sqrt{t}\sigma_j/\mu_j}\right] \tag{4}$$

Equation (4), dividing by $\sigma_j \cdot \sqrt{T}$, leads to the expected backorder units normalized (NEB). we have

$$\text{NEB}_j(t) = \sqrt{\frac{t}{T}} G\left[\frac{(I_0 - T \cdot \mu_s)}{\sigma_s\sqrt{T}}\sqrt{\frac{T}{t}} + \frac{(T-t)}{\sqrt{t}\sigma_j/\mu_j}\right] \tag{5}$$

Considering $Gu(K) = f_u(K) - KP_{u\geq}(K)$ and $k = (I_0 - T \cdot \mu_s)/\sigma_s\sqrt{T}$, we can solve(5). The normalized values are given in Table 1 for three values of T, four values of k(safety factor), two values of coefficients of variation. We find the expected backorder units period by period drop off rapidly. With $T=6$ periods, error by neglecting the values below $T-1$ periods is only 1.06%. With $T=12$ periods, error by neglecting the values below $T-2$ period is no more than 0.11%. This result means that the expected backorder units require only the values until $T-1$ period. It is assumed that the total expected backorder units of system can substitute $\sum_{j=1}^{n}\sum_{t=M}^{M} EB_j(t)$ for $\sum_{j=1}^{n}\sum_{t=1}^{M} EB_j(t)$. The longer order cycle, of course, the higher is error by neglecting. But it is unlikely that the impractical order cycle would be utilized. Also, we can find the error would be larger for smaller K values, but it is not probable as low service levels in system are considered.

The rate of total expected backorder (BR) is obtained by dividing the improved units of total expected backorder into the total demand during T periods and it shows that the result of the approximate equation can be used by the measure of service (SM) for a level of system

Table 1. The *NEB* for the t Periods of a Cycle of the T Periods

k(Pr. of Stockout)		2.33(1%)		2.05(2%)		1.64(5%)		1.28(10%)	
T	t \ c.v	0.1	0.3	0.1	0.3	0.1	0.3	0.1	0.3
3	3	.003343279	.003343279	.007398914	.007398914	.021092782	.021092782	.047438432	.047438432
	2		.000000014		.000000088		.000001085		.000008117
6	6	.003343279	.003343279	.007398914	.007398914	.021092782	.021092782	.047438432	.047438432
	5		.000005379		.000020305		.000121730	.000000003	.000505605
	4						.000000007		.000000074
12	12	003343279	.003343279	.007398914	.007398914	.021092782	.021092782	.047438432	.047438432
	11	.000000004	.000071239	.000000029	.000215846	.000000212	.000954600	.000001357	.003089703
	10		.000000285		.000001282		.000009899	.000000001	.000051155
	9				.000000001		.000000013		.000000125

inventory operation. Namely,

$$\mathrm{BR} = \Sigma_{j=1}^{n}\Sigma_{t=M-1}^{M} EB_j(t)/T \cdot \Sigma_{j=1}^{n}\mu_j \tag{6}$$

$$\mathrm{SM} = 1 - \mathrm{BR} \tag{7}$$

Each of that is in inverse proportion and it can be applied to the inventory distribution system as this case, considering the overall inventory policies and the level of customer services.

3. Backordered Units in System and Cost Function

Figure 1 shows that order 1 at time t_0 is placed at lead time $-L$ and it arrives at time $-R_L$ and inventory distribution is made for the $n-$lower level retailer. The system inventory position at time $-R_L$ is given by

$$I(-R_L) = I_0 - D(S_L) \tag{8}$$

The next replenishment will arrive at the LR'S at time $M(t_0+T+L)$, accordingly the range of review periods becomes $M+R_L$. In this periods equation (3), which is the minimum of the total expected backorders, leads to finding the allocation of LR's, S_j. That is

$$S_j = (M+R_L)\mu_j + \sigma_j/\Sigma_{j=1}^{n}\sigma_j[I(-R_L) - (M+R_L)\Sigma_{j=1}^{n}\mu_j] \tag{9}$$

As previously analyzed, we are only interested in the expected backorder units at the last two periods. The inventory levels at the ends of periods $M-1$ and M, $I_j(M-1)$ and $I_j(M)$, are obtained by subtracting the demands $D_j(M+R_L-1)$ and $D_j(M+R_L)$ from S_j and substituting (8) into (9). Thus the expected inventory levels become respectively

$$E[I_j(M-1)]=\mu_j+\sigma_j/\sum_{j=1}^{n}\sigma_j[I_0-(M+L)\sum_{j=1}^{n}\mu_j] \tag{10}$$

Then

$$E[I_j(M)]=\sigma_j/\sum_{j=1}^{n}\sigma_j[I_0-(M+L)\sum_{j=1}^{n}\mu_j] \tag{11}$$

and the variances of the coresponding inventory levels are given by

$$V[I_j(M-1)]=\sigma_j^2[(\sigma_T/\sum_{j=1}^{n}\sigma_j)^2S_L+M+R_L-1] \tag{12}$$

Then

$$V[I_j(M)]=\sigma_j^2[(\sigma_T/\sum_{j=1}^{n}\sigma_j)^2S_L+M+R_L] \tag{13}$$

Now, the expected backorder units in the system per order cycle are obtained by considering the equation (6) and the expected values and variances of the inventory levels. We have

$$EB_{LR}=\sum_{j=1}^{n}\sum_{t=M-1}^{M}\sqrt{V[I_j(t)}\cdot G[E[I_j(t)]/\sqrt{V[I_j(t)]}] \tag{14}$$

On the other hand, under an order-up-to-level (I_0, T) control system and $1-$UW, $n-$LR distribution system, the average annual cost in system will be different from the existing inventory control system of order-up-to-level by considering a two echelon system and the approximate cost factor. The average annual cost included here is only the review cost, the ordering cost, the carrying cost and the backorder cost. Where, let A be the cost of placing an order and I be the inventory carrying charge. The cost J of making a review is independent of I_0, T and the unit cost C of the item is constant independent of the quantity ordered. The average annual demand and the backorder cost of an unit are respectively D, B.

The average annual cost of review and ordering (AROC) is

$$\text{AROC}=\frac{1}{T}\sum_{j=1}^{n}V_j,\quad V_j=A_j+J_j \tag{15}$$

The average annual cost of holding inventory (AHC) is obtained by using the mean of inventory position just after the arrival of a replenishment and inventory position just prior to the arrival of a replenishment. Hence the expected carrying cost per period is $\text{ICT}[I_0-\sum_{j=1}^{n}\mu_j+DT/2]$, the average annual cost of holding inventory is given by

$$AHC = IC[I_0 - \Sigma_{j=1}^{n} \mu_j + DT/2] \quad (16)$$

where the inventory carrying charge being the same for all LR's.

The average annual cost of backorders (ABC) gets dividing (14) by $1/T$. That is

$$ABC = \frac{B}{T} EB_{LR} \quad (17)$$

Therefore the average annual total cost (ATC) in system becomes

$$ATC = AROC + AHC + ABC$$

$$= \frac{1}{T} \Sigma_{j=1}^{n} V_j + IC[I_0 - \Sigma_{j=1}^{n} \mu_j + DT/2]$$

$$+ \frac{B}{T} \Big[\Sigma_{j=1}^{n} \Sigma_{t=M-1}^{M} \sqrt{V\{I_j(t)\}} \cdot G[E\{I_j(t)\}/\sqrt{V\{I_j(t)\}}] \Big] \quad (18)$$

From the viewpoint of this paper, the cost analysis in this system is not described.

4. Conclusion

This study presents the inventory distribution system of one - warehouse n - retailer parallel - type. Under an order - up - to - level inventory control system and on the assumption the demand in the order cycle is normally distributed, we find the total expected backorder units in system and the measure of service level. The expected backorder units normalized show that the expected backorders in the last two periods may be only significant. By this result the total expected backorder units in system is obtained. The rate of total expected backorder, the measure of disservice, is given by dividing the improved units of total expected backorder into the total demand during an order cycle.

The average annual cost will be different from the existing inventory control system of order - up - to - level considering a two level distribution system and the approximation of the expected backorder units in system. The average annual total cost in system is given by using the above results, but from the objective of this study the cost analysis is not described. Finally, it is expected a studies on the cost analysis and a control system in which redistribution is permitted during the order cycle.

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