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# Development of Optimal Accelerated Life Test Plans for Weibull Distribution Under Intermittent Inspection+

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## ABSTRACT

For Weibull distributed lifetimes, this paper presents asymptotically optimal accelerated life test plans for practical applications under intermittent inspection and type-I censoring. Computational results show that the asymptotic variance of a low quantile at the design stress as optimal criterion is insensitive to the number of inspections at overstress levels.

Sensitivity analyses indicate that optimal plans are robust enough to moderate departures of estimated failure probabilities at the design and high stresses as input parameters to plan accelerated life tests from their true values.

Monte Carlo simulation for small sample study on optimal accelerated life test plans developed by the asymptotic maximum likelihood theory is conducted. Simulation results suggest that optimal plans are satisfactory for sample size in practice.

## 1. Introduction

This paper considers an optimal design of accelerated life test(ALT) plans for Weibull distributed lifetimes on the assumption of intermittent inspection and type-I censoring.

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+ 이 논문은 1988년도 문교부 지원 한국학술진흥재단의 자유공모과제 학술연구조성비에 의하여 연구되었음.

During the three preceding decades, Weibull distribution is probably the most popular model in the statistical analysis of failure time data(e.g., see Lawless(1983)). This model provides a good description of many types of lifetimes, since its hazard function could be monotone decreasing, increasing or constant on the proper choice of the parameter values.

Accelerated life tests quickly furnish information about the life distribution at the design (use or normal) stress of high reliable products and materials. But further reduction in testing efforts and administrative advantages may by achieved by periodic inspection (a regular interval) or intermittent inspection in which test items are examined at certain points in time rather than continuous inspection.

Also, some failure data may be obtained only by repeated inspections of the products. For example, when an unit can not be monitored continuously such a cracked part inside machine as turbin wheel, the exact failure times can not be observed.

The resulting information called "grouped" or "interval" data by above inspection scheme consists of the number of failures in each interval and the number of units that survive until the censoring time.

Much works on the optimal ALT plans have been carried out under the assumption of continuous inspection and have chosen levels of some stresses and allocations to satisfy specified optimality criterion.

Developments in this area were contributed by Chernoff(1962), Little and Jebe(1969), Mann (1972), Meeker and Nelson(1976), Nelson and Keilpinski(1975), Nelson and Meeker(1978), Meeker(1984) and Meeker and Hahn(1985).

On the other hand, studies on statistical analysis of grouped data or design of the inspection scheme have been largely concerned with univariate sample from a single lifetime distribution by Kulldorff(1961), Ehrenfeld(1962), Nelson(1977), Archer(1982) and Meeker(1986).

As exception, Choi(1987) presented optimal ALT plans for exponential life distribution under equally spaced inspection scheme.

This paper provides ALT plans designed for widely applicable Weibull distribution by combining these interesting and important aspects of life tests, namely acceleration and intermittent inspection.

For type-I censoring, the method of maximum likelihood estimation in its asymptotic setting is the most universal and perhaps the only work instrument in hands of any applied statistician and engineer.

The optimality criterion adopted to develop optimal ALT plans is the asymptotic minimum variance of maximum likelihood estimator of a low quantile of the lifetime distribution at the design stress.

The decision variables, i.e., the low stress level, the proportion of test units to be allocated to the low and high stress are derived and computed with respect to parameters involved.

Sensitivity analyses are performed to asses the effect of misspecifying the unknown parameters on the optimal plans.

When dealing field data, there is a doubt whether sample size is large enough to provide a reasonable accuracy of the asymptotic maximum likelihood theory and optimality of ALT plans.

A Monte Carlo study on the small and moderate sample properties of maximum likelihood estimators for optimal ALT Plans is conducted and discussed.

## 2. The Model and Maximum Likelihood Estimation

The results in this article are based on the following assumption :

1) The lifetimes of test units are independently and identically distributed as Weibull, that is, the probability density function of lifetime $T$ is given by

$$f(t)=\beta/\Theta(t/\Theta)^{\beta-1}\exp\{(-t/\Theta)^{\beta}\} \text{ for } t\geq 0, \beta,\ \theta>0 \quad (2.1)$$

2) The scale parameter is

$$\Theta=\exp(B_0+B_1\ x) \quad (2.2)$$

where $x$ is the (transformed) stress level.

3) The shape parameter $\beta$ dose not depend on the level of stress.

4) The design stress level $x_0$ and high stress level $x_2$ are prespecified, while low stress level $x_1$ is to be optimally determined.

5) $n_1$, $n_2$ units are tested independent of stress levels at the low and high stress simultaneously until some censoring time(Type-I censoring).

6) Inspections are conducted only at specified points $t_{i1}, t_{i2}, \cdots, t_{iK_i}$, $i=1,\ 2,\ K_i\geq\ 2$ and let $t_{i0}=0$ and $t_{i,K_i+1}=\infty$.

Eq. (2.2) is well known simple linear regression model such as the power rule model or Arrhenius model at life stress relationship in ALT(see e.g., Lawless(1982), Chapter 6 and Mann et al. (1974), Chapter 9).

If product life $T$ has a Weibull distribution, $Y=\ln\ T$ has a smallest extreme value distribution that the distribution function is written as

$$F(y)=1-\exp\left[-\exp\{(y-\mu)/\sigma\}\right],\quad -\infty<y<\infty \quad (2.3)$$

where $\sigma>0$, $-\infty<\mu<\infty$, $\sigma=1/\beta$, $\mu=\ln\ \Theta$

The number of test units allocated to $x_1$ and $x_2$ are respectively given by

$$n_1=\pi N,\quad n_2=(1-\pi)N=N-n \quad (2.4)$$

where $N$ is total number of test units preassigned, and $\pi$ is to be optimally determined.

Define $m_{ij}$ = the number of units (at stress level $x_i$) failed

in $[t_{i,j-1}, t_{ij})$, $j=1,2,\cdots,K_i+1$

$P_{ij}$ = the probability of failure at stress level $x_i$

in $[t_{i,j-1}, t_{ij})$, $j=1,2,\cdots,K_i+1$

Then the resulting grouped data $\{m_{ij},\ i=1,\ 2,\ j=1,\ 2,\cdots,K_i+1\}$ are used to estimate $B_0$, $B_1$ and $\sigma$ and grouped data structure of intermittent inspection can be described as in Figure 1.

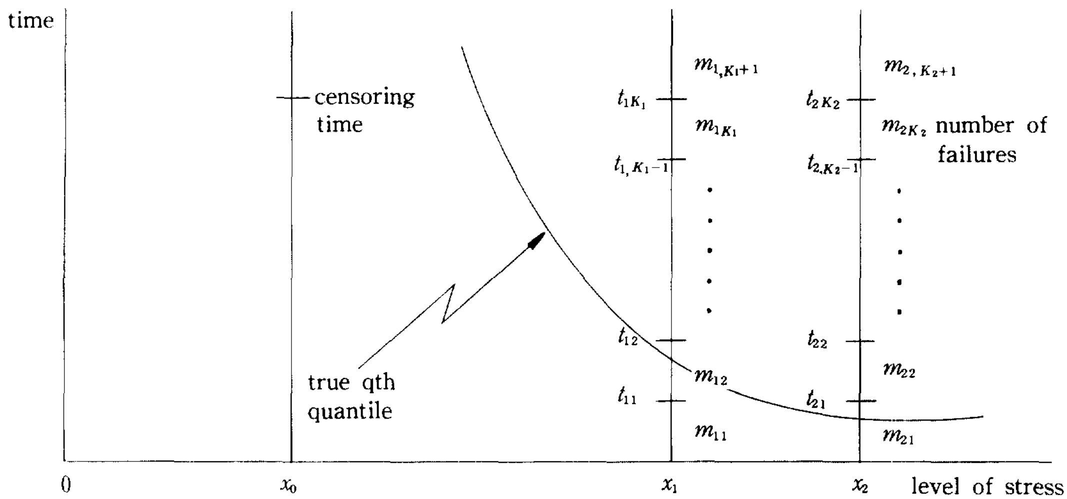


**Figure 1. Grouped Data structure**

A primary concern is to estimate the low $q$ th quantile, $y_q$ of extreme value distributon at the design stress as follow.

$$y_q = \ln\ t_q = B_0 + B_1 x + \sigma\ \ln\ \{-\ln(1-q)\} \tag{2.5}$$

Let $\hat{B}_0$, $\hat{B}_1$ and $\hat{\sigma}$ are maximum likelihood estimates of $B_0$, $B_1$ and $\sigma$ respectively, then estimate of $y_q(\hat{y}_q)$ is

$$\hat{y}_q = \hat{B}_0 + \hat{B}_1 x_0 + \sigma \ln\{(-\ln(1-q)\} \tag{2.6}$$

The problem of designing the optimal ALT plan under intermittent inspection can be stated as given $N$, $x_0$, $x_2$, inspection schemes at $x_1$ and $x_2$ to determine $\pi$ and $x_1$ such that large sample variance of $\hat{y}_q$ is minimized.

At each stress, the grouped data $\{m_{ij},\ j=1,\ 2,\cdots,K_i+1\}$ are multinomially distributed with parameters $n_i$ and $\{P_{ij},\ j=1,\ 2,\cdots,K_i+1\}$.

The likelihood function is given by

$$L=\prod_{i=1}^{2} L_i=\prod_{i=1}^{2}\prod_{j=1}^{K_i+1} n_i\ P_{ij}^{m_{ij}}(m_{ij})^{-1} \tag{2.7}$$

Then the log-likelihood function is

$$1(B_0,\ B_1,\ \sigma)=\sum_{i=1}^{2} \ln L_i$$

$$=C+\sum_{i=1}^{2}\sum_{j=1}^{K_i+1} m_{ij}\ \ln P_{ij} \tag{2.8}$$

where $C$ is constant and

$$P_{ij}=\exp\ \{-\exp(z_{i,\,j-1})\}-\exp\{-\exp(z_{ij})\}, \tag{2.9}$$

$$z_{ij}=(y_{ij}-B_0-B_1x_i)/\sigma \quad \text{for } i=1,2 \text{ and } j=0,1,\cdots,K_i+1 \tag{2.10}$$

The three partial derivatives of log-likelihood function are

$$\frac{\partial 1}{\partial B_0}=\sigma^{-1}\sum_{i=1}^{2}\sum_{j=1}^{K_i+1} m_{ij}(A_{i,\,j-1}-A_{ij})/P_{ij}$$

$$\frac{\partial 1}{\partial B_1}=\sigma^{-1}\sum_{i=1}^{2}\sum_{j=1}^{K_i+1} x_i\ m_{ij}\ (A_{i,\,j-1}-A_{ij})/P_{ij}$$

$$\frac{\partial 1}{\partial \sigma}=\sigma^{-1}\sum_{i=1}^{2}\sum_{j=1}^{K_i+1} m_{ij}\ (D_{i,\,j-1}-D_{ij})/P_{ij} \tag{2.11}$$

where $$A_{ij}\begin{cases}\exp\ \{z_{ij}-\exp(z_{ij})\}, & j=2,\cdots,K_i\\ 0, & j=0,K_i+1\end{cases} \tag{2.12}$$

$$D_{ij}=z_{ij}A_{ij} \tag{2.13}$$

when define $B_2=\sigma$ temporarily, then the Fisher information matrix for the grouped data at each stress level has the following form (See Rao(1973)).

$$F_i=n_i\left(f_{gh}^{(i)}\right)_{g,\ h=0,1,2} \qquad i=1,2$$

$$f_{gh}^{(i)}=\sum_{j=1}^{K_i+1}\frac{\left(\frac{\partial P_{ij}}{\partial B_g}\right)\left(\frac{\partial P_{ij}}{\partial B_h}\right)}{P_{ij}} \tag{2.14}$$

After some algebraric manipulation, I obtain

$$f_{00}^{(i)}=\sigma^{-2}\sum_{j=1}^{K_i+1}(A_{i,j-1}-A_{ij})^2/P_{ij}$$

$$f_{01}^{(i)}=x_i f_{00}^{(i)}$$

$$f_{11}^{(i)}=x_i^2 f_{00}^{(i)}$$

$$f_{02}^{(i)}=\sigma^{-2}\sum_{j=1}^{K_i+1}(A_{i,j-1}-A_{ij})(D_{i,j-1}-D_{ij})/P_{ij}$$

$$f_{12}^{(i)}=x_i f_{02}^{(i)}$$

$$f_{22}^{(i)}=\sigma^{-2}\sum_{j=1}^{K_i+1}(D_{i,j-1}-D_{ij})/P_{ij} \tag{2.15}$$

The total Fisher information matrix for any plan with a sample of $N$ independent observations is given by

$$F=\sum_{i=1}^{2}F_i=N\left(\pi f_{gh}^{(1)}+(1-\pi)f_{gh}^{(2)}\right)g,\ h=0,1,2 \tag{2.16}$$

By taking inverse of $F$, the asymptotic covariance matrix is obtained by

$$V=F^{-1} \tag{2.17}$$

Newton–Raphson method or scoring method may be appropriate and convenient for deriving maximum likelihood estimates of $B_0$, $B_1$, and $\sigma$. Burridge (1980) discussed conditions for regression model with grouped data that log–likelihood is concave. Also, the existence and uniqueness of maximum likelihood estimates for Weibull grouped data at the design stress was given by Cheng and Chen (1988).

Further, the asymptotic variance of the estimator of the given $q$ th quantile at $x_0$ is

$$\text{Avar}\{\hat{y}_q(x_0)\}=[1\ x_0\ Uq]\ F^{-1}\ [1\ x_0\ Uq]'\ \text{where}\ Uq=\ln\{-\ln(1-q)\} \tag{2.18}$$

where Avar denotes the asymptotic variance, and is a function of $\sigma$, $B_0$, $B_1$, $q$, $\pi$, $N$ and $x_i$, $i=0, 1, 2$.

## 3. Optimal Plans and Computational Results

An optimal ALT plan provides the asymptotic best estimate of the quantile at the design stress.

As a consequence in section 2, the optimal plan is to be determine optimal $x_1$ and $\pi(x_1^*, \pi^*)$ for the given values of $B_0$, $B_1$, $q$ and prespecified inspection scheme to minimize standardized asymptotic variance as the follow.

$$\underset{x_1, \pi}{\text{Min}} \left[ \frac{N}{\sigma^2} \text{Avar}\{\hat{y}_q(x_0)\} \right] = V_{q,k} \tag{3.1}$$

subject to $0<\pi<1$, $x_0 \le x_1 < x_2$

The constrained optimization problem can be avoided by simple transformation of variables to unconstrained problem, i.e., $v_1 = \ln\{(x_1 - x_0)/(x_2 - x_1)\}$and $v_2 = \ln\{\pi/(1-\pi)\}$ and solved by the Powell's conjugate direction method(1964).

To develop optimal ALT plans, we set $t_{k1} = t_{k2} = t_c$, $K_1 = K_2 = k$. Further, the parameters are standardized such that let $t_c = 1$ and let range of new parameters for stress level [0, 1].

Under the above reparameterization, adjusted parameters($s$, $b_0$, $b_1$) can be easily calculated by orginal parameters($x$, $B_0$, $B_1$)

$$s = (x - x_0)/(x_2 - x_0)$$

$$b_1 = (x_2 - x_0) B_1$$

$$b_0 = B_0 + B_1 x - \ln t_c \tag{3.2}$$

However, no generality is lost under the above standardization.

It is important that experimenters choose the inspection scheme for ALT plan. Though equally spaced inspection times are common at the high and low stress and are widely accepted by the ease of implementation, it is statistically inefficient in case of decreasing and constant failure rate and additional estimation of $\sigma$ to design ALT plans must be needed.

Therefore, I take a choice the equal probability inspection scheme that is reasonably statistically efficient (Meeker(1986) gave guidelines for choosing statistically efficient inspection times and discussed the inspection scheme and Hassanein(1972) gave optimum inspection times with type-I censoring for a particular criterion from Weibull data).

Inputs required to develop an optimal plan are "guess" values for $b_0$, $b_1$ and $\sigma$.

In actual experiments, the following defined quantities can be used instead of $b_0$, $b_1$ and $\sigma$ by information of product design, or engineering judgement of preliminary test because of the convenience of estimation.

$P_d$ = probability that an unit fails in $[0, t_c)$ at the design stress

$P_h$ = probability that an unit fails in $[0, t_c)$ at the high stress

Then the corresponding standardized intercept and standardized slope can be determined as follows.

$$b_0/\sigma = -\ln\{-\ln(1-P_d)\} + \ln t_c/\sigma \tag{3.3}$$

$$b_1/\sigma = -\ln\{-\ln(1-P_h)\} - b_0/\sigma + \ln t_c/\sigma \tag{3.4}$$

Accordingly to $P_d$ and $P_h$, the probability that an unit will fail at $s_1$ is given by

$$P_l = 1 - \exp[-\exp\{z_{0k}(1-s_1) + s_1 \; z_{2k}\}] \tag{3.5}$$

where $z_{ik} = \{\ln t_c - (b_0 + b_1 s_i)\}/\sigma$ for $i=0,2$ can be determined by Eqs. (3.3) and (3.4).

Hence inspection times $\{t_{i1}, \cdots, t_{ik} \; ; \; i=1,2\}$ can be calculated by Eqs. (3.5) and (3.6).

$$y_{ij}/\sigma = \ln t_{ij}/\sigma = (b_0 + b_1 s_i)/\sigma + \ln\{-\ln(1 - jP_i/k)\} \tag{3.6}$$

where $P_1 = P_l$ and $P_2 = P_h$

While inputs to design optimal ALT plans are only $P_d$ and $P_h$ under equal probability inspection scheme, additional estimate of $\sigma$ or $\beta$ is needed to determine inspection times at each stress level.

Partial listings of optimal ALT plans that are summarized by Tables 1−2 and these tables provide for the combinations of

$k = 2, \; 3, \; 5, \; 10, \; \infty$

$P_d = 0.0001, \; 0.001, \; 0.01, \; 0.05, \; 0.1$

$P_h = 0.1, \; 0.25, \; 0.05, \; 0.7, \; 0.9, \; 0.99$

$q = 0.01, \; 0.1$

$$\text{subject to } P_d \le q < P_h \tag{3.7}$$

To limit the derived extrapolation incurred in implementation of ALT plans, Eq. (3.7) is assigned.

The tabulations show standardized intercept and slope of location parameter ($b_0/\sigma$ and $b_1/\sigma$), optimal level of low stress($s_1^*$), optimal proportion allocated to low stress($\pi^*$), standardized asymptotic minimum variance($V_{q,k}$) ratio of $V_{q,k}$ to continuous inspection ($V_{q,k}/V_{q,\infty}$), and probability of failure until $t_c$ at $s_1$ ($P_l$).

**Table 1. Optimal ALT Plans for Weibull Grouped Data Under Equal Probability Inspection Scheme** ($q = 0.01$)

| $P_d$ | $P_h$ | $b_0/\sigma$ | $b_1/\sigma$ | $k$ | $s_1^*$ | $\pi^*$ | $V_{q,k}$ | ratio[§] | $P_l$ |
|---|---|---|---|---|---|---|---|---|---|
| .0001 | .9900 | 9.2103 | -10.7375 | 2 | .7248 | .6948 | 130.7717 | 1.1219 | .2133 |
| .0001 | .9900 | 9.2103 | -10.7375 | 3 | .7250 | .7150 | 123.2560 | 1.0574 | .2137 |
| .0001 | .9900 | 9.2103 | -10.7375 | 5 | .7260 | .7254 | 119.4980 | 1.0252 | .2157 |
| .0001 | .9900 | 9.2103 | -10.7375 | 10 | .7267 | .7309 | 117.6098 | 1.0090 | .2172 |
| .0001 | .9900 | 9.2103 | -10.7375 | ∞ | .7268 | .7344 | 116.5619 | 1.0000 | .2172 |
| .0001 | .9000 | 9.2103 | -10.0443 | 2 | .7505 | .6878 | 210.8054 | 1.1492 | .1712 |
| .0001 | .9000 | 9.2103 | -10.0443 | 3 | .7449 | .7114 | 196.1589 | 1.0695 | .1627 |
| .0001 | .9000 | 9.2103 | -10.0443 | 5 | .7417 | .7238 | 189.2252 | 1.0316 | .1580 |
| .0001 | .9000 | 9.2103 | -10.0443 | 10 | .7397 | .7306 | 185.7367 | 1.0126 | .1552 |
| .0001 | .9000 | 9.2103 | -10.0443 | ∞ | .7380 | .7354 | 183.4289 | 1.0000 | .1528 |
| .0001 | .7000 | 9.2103 | -9.3959 | 2 | .7528 | .6775 | 348.0181 | 1.1876 | .1113 |
| .0001 | .7000 | 9.2103 | -9.3959 | 3 | .7467 | .7036 | 319.5947 | 1.0906 | .1054 |
| .0001 | .7000 | 9.2103 | -9.3959 | 5 | .7427 | .7181 | 305.6709 | 1.0431 | .1018 |
| .0001 | .7000 | 9.2103 | -9.3959 | 10 | .7403 | .7264 | 298.3375 | 1.0180 | .0996 |
| .0001 | .7000 | 9.2103 | -9.3959 | ∞ | .7384 | .7326 | 293.0518 | 1.0000 | .0980 |
| .0001 | .5000 | 9.2103 | -8.8438 | 2 | .7486 | .6679 | 538.6126 | 1.2101 | .0723 |
| .0001 | .5000 | 9.2103 | -8.8438 | 3 | .7415 | .6956 | 491.5312 | 1.1043 | .0680 |
| .0001 | .5000 | 9.2103 | -8.8438 | 5 | .7375 | .7113 | 467.7861 | 1.0510 | .0657 |
| .0001 | .5000 | 9.2103 | -8.8438 | 10 | .7349 | .7206 | 454.8605 | 1.0220 | .0643 |
| .0001 | .5000 | 9.2103 | -8.8438 | ∞ | .7329 | .7279 | 445.0875 | 1.0000 | .0632 |
| .0001 | .2500 | 9.2103 | -7.9644 | 2 | .7320 | .6508 | 1090.1309 | 1.2390 | .0335 |
| .0001 | .2500 | 9.2103 | -7.9644 | 3 | .7250 | .6803 | 987.2476 | 1.1221 | .0317 |
| .0001 | .2500 | 9.2103 | -7.9644 | 5 | .7208 | .6975 | 933.7790 | 1.0613 | .0306 |
| .0001 | .2500 | 9.2103 | -7.9644 | 10 | .7182 | .7081 | 903.6893 | 1.0271 | .0300 |
| .0001 | .2500 | 9.2103 | -7.9644 | ∞ | .7159 | .7169 | 879.8471 | 1.0000 | .0295 |
| .0001 | .1000 | 9.2103 | -6.3599 | 2 | .7033 | .6259 | 2451.6696 | 1.2754 | .0133 |
| .0001 | .1000 | 9.2103 | -6.3599 | 3 | .6970 | .6583 | 2196.4671 | 1.1426 | .0127 |
| .0001 | .1000 | 9.2103 | -6.3599 | 5 | .6921 | .6772 | 2061.8967 | 1.0726 | .0123 |
| .0001 | .1000 | 9.2103 | -6.3599 | 10 | .6892 | .6890 | 1984.8728 | 1.0325 | .0120 |
| .0001 | .1000 | 9.2103 | -6.3599 | ∞ | .6866 | .6991 | 1922.3297 | 1.0000 | .0118 |
| .0010 | .9900 | 6.9073 | -8.4344 | 2 | .6192 | .7571 | 71.2079 | 1.0996 | .1694 |
| .0010 | .9900 | 6.9073 | -8.4344 | 3 | .6251 | .7692 | 68.0276 | 1.0505 | .1771 |
| .0010 | .9900 | 6.9073 | -8.4344 | 5 | .6301 | .7752 | 66.2282 | 1.0227 | .1840 |
| .0010 | .9900 | 6.9073 | -8.4344 | 10 | .6333 | .7784 | 65.2604 | 1.0078 | .1886 |
| .0010 | .9900 | 6.9073 | -8.4344 | ∞ | .6350 | .7802 | 64.7585 | 1.0000 | .1910 |
| .0010 | .9000 | 6.9073 | -7.7413 | 2 | .6548 | .7557 | 103.0296 | 1.0824 | .1471 |
| .0010 | .9000 | 6.9073 | -7.7413 | 3 | .6499 | .7718 | 98.7240 | 1.0371 | .1420 |
| .0010 | .9000 | 6.9073 | -7.7413 | 5 | .6473 | .7799 | 96.7250 | 1.0161 | .1394 |
| .0010 | .9000 | 6.9073 | -7.7413 | 10 | .6458 | .7840 | 95.7637 | 1.0060 | .1379 |
| .0010 | .9000 | 6.9073 | -7.7413 | ∞ | .6444 | .7868 | 95.1885 | 1.0000 | .1366 |
| .0010 | .7000 | 6.9073 | -7.0929 | 2 | .6548 | .7497 | 159.4616 | 1.1121 | .0989 |
| .0010 | .7000 | 6.9073 | -7.0929 | 3 | .6477 | .7688 | 151.0123 | 1.0532 | .0942 |
| .0010 | .7000 | 6.9073 | -7.0929 | 5 | .6435 | .7788 | 146.9501 | 1.0249 | .0916 |
| .0010 | .7000 | 6.9073 | -7.0929 | 10 | .6410 | .7843 | 144.8545 | 1.0103 | .0900 |
| .0010 | .7000 | 6.9073 | -7.0929 | ∞ | .6390 | .7885 | 143.3829 | 1.0000 | .0888 |
| .0010 | .5000 | 6.9073 | -6.5407 | 2 | .6430 | .7425 | 234.7117 | 1.1351 | .0649 |
| .0010 | .5000 | 6.9073 | -6.5407 | 3 | .6352 | .7636 | 220.5429 | 1.0666 | .0618 |
| .0010 | .5000 | 6.9073 | -6.5407 | 5 | .6305 | .7752 | 213.4583 | 1.0323 | .0600 |
| .0010 | .5000 | 6.9073 | -6.5407 | 10 | .6276 | .7818 | 209.6367 | 1.0138 | .0589 |
| .0010 | .5000 | 6.9073 | -6.5407 | ∞ | .6251 | .7870 | 206.7750 | 1.0000 | .0580 |
| .0010 | .2500 | 6.9073 | -5.6614 | 2 | .6096 | .7267 | 435.8113 | 1.1698 | .0311 |
| .0010 | .2500 | 6.9073 | -5.6614 | 3 | .6004 | .7512 | 404.8491 | 1.0867 | .0295 |
| .0010 | .2500 | 6.9073 | -5.6614 | 5 | .5949 | .7651 | 388.7536 | 1.0435 | .0286 |
| .0010 | .2500 | 6.9073 | -5.6614 | 10 | .5914 | .7734 | 379.7031 | 1.0192 | .0281 |
| .0010 | .2500 | 6.9073 | -5.6614 | ∞ | .5884 | .7802 | 372.5376 | 1.0000 | .0276 |
| .0010 | .1000 | 6.9073 | -4.6569 | 2 | .5501 | .7000 | 872.2240 | 1.2188 | .0129 |
| .0010 | .1000 | 6.9073 | -4.6569 | 3 | .5400 | .7288 | 797.0359 | 1.1137 | .0123 |
| .0010 | .1000 | 6.9073 | -4.6569 | 5 | .5332 | .7459 | 757.1734 | 1.0580 | .0119 |
| .0010 | .1000 | 6.9073 | -4.6569 | 10 | .5289 | .7561 | 734.2798 | 1.0260 | .0117 |
| .0010 | .1000 | 6.9073 | -4.6569 | ∞ | .5251 | .7651 | 715.6390 | 1.0000 | .0115 |
| .0100 | .9900 | 4.6001 | -6.1273 | 2 | .4237 | .7976 | 39.5213 | 1.2029 | .1261 |
| .0100 | .9900 | 4.6001 | -6.1273 | 3 | .4454 | .8065 | 36.5945 | 1.1138 | .1427 |
| .0100 | .9900 | 4.6001 | -6.1273 | 5 | .4595 | .8136 | 34.6938 | 1.0559 | .1546 |
| .0100 | .9900 | 4.6001 | -6.1273 | 10 | .4679 | .8184 | 33.5639 | 1.0215 | .1620 |
| .0100 | .9900 | 4.6001 | -6.1273 | ∞ | .4731 | .8216 | 32.8552 | 1.0000 | .1667 |
| .0100 | .9000 | 4.6001 | -5.4342 | 2 | .4519 | .8488 | 42.2730 | 1.0288 | .1105 |
| .0100 | .9000 | 4.6001 | -5.4342 | 3 | .4560 | .8512 | 41.7564 | 1.0161 | .1129 |
| .0100 | .9000 | 4.6001 | -5.4342 | 5 | .4588 | .8528 | 41.4027 | 1.0075 | .1145 |
| .0100 | .9000 | 4.6001 | -5.4342 | 10 | .4602 | .8536 | 41.2063 | 1.0027 | .1153 |
| .0100 | .9000 | 4.6001 | -5.4342 | ∞ | .4609 | .8541 | 41.0949 | 1.0000 | .1157 |
| .0100 | .7000 | 4.6001 | -4.7858 | 2 | .4212 | .8770 | 53.4733 | 1.0050 | .0727 |
| .0100 | .7000 | 4.6001 | -4.7858 | 3 | .4221 | .8775 | 53.3540 | 1.0028 | .0730 |
| .0100 | .7000 | 4.6001 | -4.7858 | 5 | .4227 | .8778 | 53.2772 | 1.0013 | .0732 |
| .0100 | .7000 | 4.6001 | -4.7858 | 10 | .4230 | .8779 | 53.2344 | 1.0005 | .0733 |
| .0100 | .7000 | 4.6001 | -4.7858 | ∞ | .4232 | .8780 | 53.2061 | 1.0000 | .0734 |
| .0100 | .5000 | 4.6001 | -4.2336 | 2 | .3672 | .8986 | 66.6149 | 1.0012 | .0464 |
| .0100 | .5000 | 4.6001 | -4.2336 | 3 | .3675 | .8987 | 66.5786 | 1.0007 | .0465 |
| .0100 | .5000 | 4.6001 | -4.2336 | 5 | .3677 | .8988 | 66.5557 | 1.0003 | .0465 |
| .0100 | .5000 | 4.6001 | -4.2336 | 10 | .3678 | .8988 | 66.5429 | 1.0001 | .0466 |
| .0100 | .5000 | 4.6001 | -4.2336 | ∞ | .3678 | .8988 | 66.5340 | 1.0000 | .0466 |
| .0100 | .2500 | 4.6001 | -3.3542 | 2 | .2225 | .9394 | 89.3754 | 1.0001 | .0210 |
| .0100 | .2500 | 4.6001 | -3.3542 | 3 | .2226 | .9394 | 89.3702 | 1.0001 | .0210 |
| .0100 | .2500 | 4.6001 | -3.3542 | 5 | .2226 | .9394 | 89.3672 | 1.0000 | .0210 |
| .0100 | .2500 | 4.6001 | -3.3542 | 10 | .2227 | .9394 | 89.3658 | 1.0000 | .0210 |
| .0100 | .2500 | 4.6001 | -3.3542 | ∞ | .2227 | .9394 | 89.3649 | 1.0000 | .0210 |
| .0100 | .1000 | 4.6001 | -2.3498 | 2 | .0000 | 1.0000 | 100.0537 | 1.0000 | .0100 |
| .0100 | .1000 | 4.6001 | -2.3498 | 3 | .0000 | 1.0000 | 100.0537 | 1.0000 | .0100 |
| .0100 | .1000 | 4.6001 | -2.3498 | 5 | .0000 | 1.0000 | 100.0537 | 1.0000 | .0100 |
| .0100 | .1000 | 4.6001 | -2.3498 | 10 | .0000 | 1.0000 | 100.0536 | 1.0000 | .0100 |
| .0100 | .1000 | 4.6001 | -2.3498 | ∞ | .0000 | 1.0000 | 100.0535 | 1.0000 | .0100 |

§) ratio $= V_{q,k} / V_{q,\infty}$

**Table 2. Optimal ALT Plans for Weibull Grouped Data Under Equal Probability Inspection Scheme ($q=0.1$)**

| $P_d$ | $P_h$ | $b_0/\sigma$ | $b_1/\sigma$ | $k$ | $s_1^*$ | $\pi^*$ | $V_{q,k}$ | ratio | $P_l$ |
|---|---|---|---|---|---|---|---|---|---|
| .0001 | .9900 | 9.2103 | -10.7375 | 2 | .7623 | .6246 | 163.6684 | 1.1937 | .3016 |
| .0001 | .9900 | 9.2103 | -10.7375 | 3 | .7572 | .6514 | 149.6630 | 1.0915 | .2879 |
| .0001 | .9900 | 9.2103 | -10.7375 | 5 | .7541 | .6660 | 142.8650 | 1.0419 | .2799 |
| .0001 | .9900 | 9.2103 | -10.7375 | 10 | .7522 | .6741 | 139.4049 | 1.0167 | .2752 |
| .0001 | .9900 | 9.2103 | -10.7375 | ∞ | .7502 | .6799 | 137.1146 | 1.0000 | .2703 |
| .0001 | .9000 | 9.2103 | -10.0443 | 2 | .7737 | .6104 | 278.7394 | 1.2565 | .2111 |
| .0001 | .9000 | 9.2103 | -10.0443 | 3 | .7670 | .6407 | 249.1907 | 1.1233 | .1989 |
| .0001 | .9000 | 9.2103 | -10.0443 | 5 | .7627 | .6578 | 234.7208 | 1.0581 | .1913 |
| .0001 | .9000 | 9.2103 | -10.0443 | 10 | .7599 | .6676 | 227.1697 | 1.0240 | .1866 |
| .0001 | .9000 | 9.2103 | -10.0443 | ∞ | .7576 | .6751 | 221.8382 | 1.0000 | .1827 |
| .0001 | .7000 | 9.2103 | -9.3959 | 2 | .7719 | .5938 | 469.3490 | 1.3016 | .1316 |
| .0001 | .7000 | 9.2103 | -9.3959 | 3 | .7654 | .6274 | 413.9690 | 1.1481 | .1243 |
| .0001 | .7000 | 9.2103 | -9.3959 | 5 | .7611 | .6469 | 386.3668 | 1.0715 | .1198 |
| .0001 | .7000 | 9.2103 | -9.3959 | 10 | .7584 | .6583 | 271.5387 | 1.0304 | .1170 |
| .0001 | .7000 | 9.2103 | -9.3959 | ∞ | .7561 | .6675 | 360.5839 | 1.0000 | .1146 |
| .0001 | .5000 | 9.2103 | -8.8438 | 2 | .7650 | .5807 | 735.6878 | 1.3252 | .0831 |
| .0001 | .5000 | 9.2103 | -8.8438 | 3 | .7589 | .6155 | 645.5491 | 1.1628 | .0789 |
| .0001 | .5000 | 9.2103 | -8.8438 | 5 | .7548 | .6361 | 599.6964 | 1.0802 | .0762 |
| .0001 | .5000 | 9.2103 | -8.8438 | 10 | .7522 | .6486 | 574.4751 | 1.0348 | .0745 |
| .0001 | .5000 | 9.2103 | -8.8438 | ∞ | .7494 | .6591 | 555.1706 | 1.0000 | .0728 |
| .0001 | .2500 | 9.2103 | -7.9644 | 2 | .7481 | .5588 | 1522.7826 | 1.3547 | .0380 |
| .0001 | .2500 | 9.2103 | -7.9644 | 3 | .7422 | .5946 | 1328.0102 | 1.1814 | .0362 |
| .0001 | .2500 | 9.2103 | -7.9644 | 5 | .7382 | .6164 | 1226.7136 | 1.0913 | .0351 |
| .0001 | .2500 | 9.2103 | -7.9644 | 10 | .7355 | .6299 | 1169.5452 | 1.0405 | .0344 |
| .0001 | .2500 | 9.2103 | -7.9644 | ∞ | .7333 | .6415 | 1124.0687 | 1.0000 | .0338 |
| .0010 | .9900 | 6.9073 | -8.4344 | 2 | .6882 | .6596 | 89.5749 | 1.1666 | .2825 |
| .0010 | .9900 | 6.9073 | -8.4344 | 3 | .6829 | .6851 | 82.8183 | 1.0786 | .2721 |
| .0010 | .9900 | 6.9073 | -8.4344 | 5 | .6800 | .6989 | 79.5338 | 1.0358 | .2664 |
| .0010 | .9900 | 6.9073 | -8.4344 | 10 | .6782 | .7065 | 77.8674 | 1.0141 | .2629 |
| .0010 | .9900 | 6.9073 | -8.4344 | ∞ | .6761 | .7118 | 76.7830 | 1.0000 | .2591 |
| .0010 | .9000 | 6.9073 | -7.7413 | 2 | .7019 | .6443 | 147.1947 | 1.2271 | .2047 |
| .0010 | .9000 | 6.9073 | -7.7413 | 3 | .6934 | .6736 | 133.0561 | 1.1093 | .1930 |
| .0010 | .9000 | 6.9073 | -7.7413 | 5 | .6881 | .6898 | 126.1297 | 1.0515 | .1860 |
| .0010 | .9000 | 6.9073 | -7.7413 | 10 | .6848 | .6991 | 122.5065 | 1.0213 | .1818 |
| .0010 | .9000 | 6.9073 | -7.7413 | ∞ | .6821 | .7061 | 119.9507 | 1.0000 | .1784 |
| .0010 | .7000 | 6.9073 | -7.0929 | 2 | .6959 | .6257 | 240.0574 | 1.2758 | .1300 |
| .0010 | .7000 | 6.9073 | -7.0929 | 3 | .6871 | .6582 | 213.7027 | 1.1357 | .1226 |
| .0010 | .7000 | 6.9073 | -7.0929 | 5 | .6810 | .6773 | 200.5237 | 1.0657 | .1178 |
| .0010 | .7000 | 6.9073 | -7.0929 | 10 | .6774 | .6883 | 193.4246 | 1.0280 | .1150 |
| .0010 | .7000 | 6.9073 | -7.0929 | ∞ | .6751 | .6968 | 188.1619 | 1.0000 | .1133 |
| .0010 | .5000 | 6.9073 | -6.5407 | 2 | .6824 | .6093 | 365.6027 | 1.3050 | .0832 |
| .0010 | .5000 | 6.9073 | -6.5407 | 3 | .6741 | .6436 | 323.0686 | 1.1532 | .0789 |
| .0010 | .5000 | 6.9073 | -6.5407 | 5 | .6679 | .6640 | 301.3474 | 1.0757 | .0759 |
| .0010 | .5000 | 6.9073 | -6.5407 | 10 | .6650 | .6759 | 289.2592 | 1.0329 | .0746 |
| .0010 | .5000 | 6.9073 | -6.5407 | ∞ | .6619 | .6859 | 280.1508 | 1.0000 | .0731 |
| .0010 | .2500 | 6.9073 | -5.6614 | 2 | .6495 | .5796 | 719.2338 | 1.3469 | .0388 |
| .0010 | .2500 | 6.9073 | -5.6614 | 3 | .6408 | .6156 | 629.0110 | 1.1779 | .0369 |
| .0010 | .2500 | 6.9073 | -5.6614 | 5 | .6358 | .6378 | 581.9170 | 1.0897 | .0359 |
| .0010 | .2500 | 6.9073 | -5.6614 | 10 | .6318 | .6513 | 555.2590 | 1.0398 | .0352 |
| .0010 | .2500 | 6.9073 | -5.6614 | ∞ | .6282 | .6639 | 533.9912 | 1.0000 | .0345 |
| .0100 | .9900 | 4.6001 | -6.1273 | 2 | .5435 | .7324 | 38.0512 | 1.1177 | .2448 |
| .0100 | .9900 | 4.6001 | -6.1273 | 3 | .5405 | .7530 | 35.9359 | 1.0556 | .2410 |
| .0100 | .9900 | 4.6001 | -6.1273 | 5 | .5391 | .7644 | 34.8959 | 1.0250 | .2392 |
| .0100 | .9900 | 4.6001 | -6.1273 | 10 | .5383 | .7704 | 34.3695 | 1.0096 | .2382 |
| .0100 | .9900 | 4.6001 | -6.1273 | ∞ | .5368 | .7746 | 34.0438 | 1.0000 | .2363 |
| .0100 | .9000 | 4.6001 | -5.4342 | 2 | .5598 | .7182 | 57.4116 | 1.1667 | .1899 |
| .0100 | .9000 | 4.6001 | -5.4342 | 3 | .5483 | .7436 | 53.1568 | 1.0802 | .1795 |
| .0100 | .9000 | 4.6001 | -5.4342 | 5 | .5423 | .7571 | 51.0697 | 1.0378 | .1742 |
| .0100 | .9000 | 4.6001 | -5.4342 | 10 | .5381 | .7649 | 49.9785 | 1.0156 | .1706 |
| .0100 | .9000 | 4.6001 | -5.4342 | ∞ | .5344 | .7709 | 49.2094 | 1.0000 | .1675 |
| .0100 | .7000 | 4.6001 | -4.7858 | 2 | .5412 | .6987 | 86.7961 | 1.2187 | .1254 |
| .0100 | .7000 | 4.6001 | -4.7858 | 3 | .5288 | .7282 | 78.9348 | 1.1084 | .1186 |
| .0100 | .7000 | 4.6001 | -4.7858 | 5 | .5210 | .7448 | 74.9724 | 1.0527 | .1145 |
| .0100 | .7000 | 4.6001 | -4.7858 | 10 | .5160 | .7546 | 72.8239 | 1.0225 | .1120 |
| .0100 | .7000 | 4.6001 | -4.7858 | ∞ | .5118 | .7624 | 71.2179 | 1.0000 | .1099 |
| .0100 | .5000 | 4.6001 | -4.2336 | 2 | .5079 | .6786 | 123.2949 | 1.2576 | .0827 |
| .0100 | .5000 | 4.6001 | -4.2336 | 3 | .4962 | .7110 | 110.8267 | 1.1305 | .0789 |
| .0100 | .5000 | 4.6001 | -4.2336 | 5 | .4880 | .7299 | 104.3942 | 1.0648 | .0763 |
| .0100 | .5000 | 4.6001 | -4.2336 | 10 | .4827 | .7411 | 100.8141 | 1.0283 | .0746 |
| .0100 | .5000 | 4.6001 | -4.2336 | ∞ | .4777 | .7507 | 98.0369 | 1.0000 | .0731 |
| .0100 | .2500 | 4.6001 | -3.3542 | 2 | .4248 | .6356 | 213.3770 | 1.3248 | .0409 |
| .0100 | .2500 | 4.6001 | -3.3542 | 3 | .4125 | .6723 | 188.1098 | 1.1679 | .0393 |
| .0100 | .2500 | 4.6001 | -3.3542 | 5 | .4043 | .6948 | 174.7893 | 1.0852 | .0383 |
| .0100 | .2500 | 4.6001 | -3.3542 | 10 | .3992 | .7091 | 167.1865 | 1.0380 | .0376 |
| .0100 | .2500 | 4.6001 | -3.3542 | ∞ | .3944 | .7211 | 161.0681 | 1.0000 | .0370 |
| .0500 | .9900 | 2.9702 | -4.4974 | 2 | .3098 | .8540 | 15.5244 | 1.0571 | .1866 |
| .0500 | .9900 | 2.9702 | -4.4974 | 3 | .3147 | .8628 | 15.1038 | 1.0285 | .1904 |
| .0500 | .9900 | 2.9702 | -4.4974 | 5 | .3196 | .8671 | 14.8747 | 1.0129 | .1942 |
| .0500 | .9900 | 2.9702 | -4.4974 | 10 | .3230 | .8692 | 14.7524 | 1.0045 | .1969 |
| .0500 | .9900 | 2.9702 | -4.4974 | ∞ | .3241 | .8709 | 14.6858 | 1.0000 | .1977 |
| .0500 | .9000 | 2.9702 | -3.8042 | 2 | .3162 | .8546 | 19.4684 | 1.0668 | .1570 |
| .0500 | .9000 | 2.9702 | -3.8042 | 3 | .3063 | .8682 | 18.8325 | 1.0319 | .1517 |
| .0500 | .9000 | 2.9702 | -3.8042 | 5 | .3004 | .8754 | 18.5218 | 1.0149 | .1486 |
| .0500 | .9000 | 2.9702 | -3.8042 | 10 | .2964 | .8794 | 18.3611 | 1.0061 | .1465 |
| .0500 | .9000 | 2.9702 | -3.8042 | ∞ | .2928 | .8825 | 18.2500 | 1.0000 | .1447 |
| .0500 | .7000 | 2.9702 | -3.1558 | 2 | .2662 | .8542 | 24.6959 | 1.1076 | .1120 |
| .0500 | .7000 | 2.9702 | -3.1558 | 3 | .2515 | .8724 | 23.5071 | 1.0543 | .1072 |
| .0500 | .7000 | 2.9702 | -3.1558 | 5 | .2419 | .8825 | 22.8960 | 1.0269 | .1042 |
| .0500 | .7000 | 2.9702 | -3.1558 | 10 | .2358 | .8883 | 22.5578 | 1.0117 | .1023 |
| .0500 | .7000 | 2.9702 | -3.1558 | ∞ | .2303 | .8933 | 22.2973 | 1.0000 | .1007 |
| .0500 | .5000 | 2.9702 | -2.6037 | 2 | .1951 | .8504 | 29.5953 | 1.1506 | .0817 |
| .0500 | .5000 | 2.9702 | -2.6037 | 3 | .1786 | .8736 | 27.7353 | 1.0783 | .0784 |
| .0500 | .5000 | 2.9702 | -2.6037 | 5 | .1677 | .8869 | 26.7454 | 1.0398 | .0763 |
| .0500 | .5000 | 2.9702 | -2.6037 | 10 | .1609 | .8950 | 26.1779 | 1.0173 | .0750 |
| .0500 | .5000 | 2.9702 | -2.6037 | ∞ | .1545 | .9019 | 25.7209 | 1.0000 | .0738 |
| .0500 | .2500 | 2.9702 | -1.7243 | 2 | .0000 | .8196 | 35.7528 | 1.2405 | .0500 |
| .0500 | .2500 | 2.9702 | -1.7243 | 3 | .0000 | .8592 | 33.4040 | 1.1275 | .0500 |
| .0500 | .2500 | 2.9702 | -1.7243 | 5 | .0000 | .8834 | 31.5831 | 1.0660 | .0500 |
| .0500 | .2500 | 2.9702 | -1.7243 | 10 | .0000 | .8988 | 30.5147 | 1.0300 | .0500 |
| .0500 | .2500 | 2.9702 | -1.7243 | ∞ | .0000 | .9124 | 29.6273 | 1.0000 | .0500 |
| .1000 | .9900 | 2.2504 | -3.7775 | 2 | .0871 | .9598 | 9.6556 | 1.0391 | .1362 |
| .1000 | .9900 | 2.2504 | -3.7775 | 3 | .1125 | .9513 | 9.5141 | 1.0239 | .1489 |
| .1000 | .9900 | 2.2504 | -3.7775 | 5 | .1307 | .9458 | 9.4067 | 1.0123 | .1585 |
| .1000 | .9900 | 2.2504 | -3.7775 | 10 | .1419 | .9427 | 9.3366 | 1.0048 | .1648 |
| .1000 | .9900 | 2.2504 | -3.7775 | ∞ | .1489 | .9408 | 9.2921 | 1.0000 | .1688 |
| .1000 | .9000 | 2.2504 | -3.0844 | 2 | .0463 | .9827 | 9.9516 | 1.0028 | .1144 |
| .1000 | .9000 | 2.2504 | -3.0844 | 3 | .0520 | .9811 | 9.9379 | 1.0014 | .1164 |
| .1000 | .9000 | 2.2504 | -3.0844 | 5 | .0554 | .9801 | 9.9295 | 1.0006 | .1175 |
| .1000 | .9000 | 2.2504 | -3.0844 | 10 | .0570 | .9797 | 9.9255 | 1.0001 | .1181 |
| .1000 | .9000 | 2.2504 | -3.0844 | ∞ | .0576 | .9795 | 9.9240 | 1.0000 | .1183 |
| .1000 | .7000 | 2.2504 | -2.4360 | 2 | .0000 | 1.0000 | 10.0100 | 1.0002 | .1000 |
| .1000 | .7000 | 2.2504 | -2.4360 | 3 | .0000 | 1.0000 | 10.0095 | 1.0002 | .1000 |
| .1000 | .7000 | 2.2504 | -2.4360 | 5 | .0000 | 1.0000 | 10.0088 | 1.0001 | .1000 |
| .1000 | .7000 | 2.2504 | -2.4360 | 10 | .0000 | 1.0000 | 10.0083 | 1.0001 | .1000 |
| .1000 | .7000 | 2.2504 | -2.4360 | ∞ | .0000 | 1.0000 | 10.0076 | 1.0000 | .1000 |

Optimal ALT plans under continuous inspection are determined by results of Meeker and Nelson(1976), Nelson and Meeker(1978) and Escobar and Meeker(1986).

General conclusions may be drawn from the tables.

(1) One of main conclusion is that $V_{q,k}$ is not sensitive to $k$ and this insensitivity becomes more apparent as $q$ decreases and $k \geq 3$.

This implies that in designing ALT plans under intermittent inspection for estimating a low quantile at the design stress, the large number of inspections must be not necessarily, which is a reason that experimenters choose intermittent inspection rather than continuous inspection by reducing inspection cost and testing efforts.

(2) A striking phenomenon is that the optimal stress level($s_1^*$) and the optimal proportion of test units allocated to the low stress($\pi^*$) are rather stable over $k \geq 3$ for all cases considered.

(3) Incidently as quantile is close to $P_d$ and $P_h \leq 0.9$, $\pi^*$ is more stable. As $P_d$ increase and/or $P_h$ decreases, $s_1^*$ gets close to the design stress level and $\pi^*$ increases for all quantiles. In particular, when $q \approx P_d$ and $P_h \leq 10 \cdot P_d$, $s_1^* \approx 0$ and $\pi^* \approx 1$, which almost no need for an ALT.

(4) As number of inspections increase, $\pi^*$ doesn't decrease and $s_1^*$ almost nondecreases so that information of test results at the low stress can be gained a little more when $q > P_d$.

## EXAMPLE

Meeker and Hahn(1985) gave data for an adhesive-bonded power element in an ALT that estimated the relationship between failure time and temperature.

In this example, 300 items were available for testing between 50℃ and 120℃. Censoring time was 6 months and it was anticipated that probability of failure at 50℃ and 120℃ would be about 0.1% and 90%.

Tenth percentile at the design stress(50℃) need to be estimated with three inspections. For example, $s_1^* = 0.6868$ and $\pi^* = 0.6715$ from Table 2. Thus,

$$x_1^* = (120-50)s_1^* + 50 = 98℃$$

$$B_1/\sigma = \frac{-7.7413}{(120-50)} = -0.1106$$

$$B_0/\sigma = (6.9073 - 0.1106) + \ln 180 = 25.499$$

$$n_1 = 300 \cdot 0.6715 \approx 201$$

$$n_2 = 99$$

Inspection times at the high and low stresses can be determined by $P_h$, $P_l$, in Table 2, Eqs.

(3.5) and (3.6) and $\sigma$ which may be estimated by graphical methods such as hazard or probability plots or techniques described in Mann et al. (1974), chapter 5 using preliminary test at any overstress as the high stress etc. and historical data.

## 4. Sensitivity Analysis and Small Sample Study

To determine an optimal ALT plan, required inputs are guessed values for failure probability $P_d$ and $P_h$ until censoring time at the design and high stresses on behalf of unknown $b_0$, $b_1$ and $\sigma$.

Chernoff(1962) called this situation "locally optimal" and recommended sensitivity analysis. Let $\tilde{P}_d$ and $\tilde{P}_h$ be the guessed values of $P_d$ and $P_h$ respectively. For $\tilde{P}_d$ and $\tilde{P}_h$, false optimal $\tilde{s}_1^*$ and $\tilde{\pi}^*$ can be determined as $s_1^*$, $\pi^*$.

In Table 3−5, $V_{q,\ k}(\tilde{s}_1^*,\ \tilde{\pi}^*)/V_{q,\ k}(s_1^*,\ \pi^*)$ to asses the sensitivity of the test plans are listed for various cases of true $P_d$, $P_h$ with $k=3$, $q=0.01$, $0.1$ and $k=10$, $q=0.1$. This sensitivity means the relative amount of increase in $V_{q,\ k}$ due to uncertainties involved in $P_d$ and $P_h$.

From Table 3−5, I make an observation that moderate deviations from $P_d$ and $P_h$ (approximately −70∼+200% from $P_d$ and −10∼+30% from $P_h$) are appreciably tolerable in terms of the asymptotic variance of a quantile at the design stress in sense that results of sensitivity is mostly not greater than 15%, although the detailed tabulations are not given here by limitation of space.

Table 4 and 5 show that the sensitivity values are fairly stable over number of inspection for given $P_d$ and $P_h$.

Values of ratio less than 1 in Tables can be occured because inspection times at two stress levels determined by $P_d$ and $P_h$ are different from them by $\tilde{P}_d$ and $\tilde{P}_h$ and these times by true $P_d$ and $P_h$ are not optimal inspection times to minimize $V_{q,\ k}$ in the design of ALT plans.

It is a matter of great practical interest to asses the property of optimal ALT plans obtained using the method of maximum likelihood and asymptotic variance of the estimator.

**Table 3. Sensitivities of $V_{q,\ k}$ when false $P_d=0.01$, false $P_h=0.9$, $k=3$ and $q=0.01$**

| true $P_d$ \ true $P_h$ | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
|---|---|---|---|---|---|
| 0.003 | 1.0843 | 1.1112 | 1.1374 | 1.1436 | 1.1027 |
| 0.005 | 1.0209 | 1.0341 | 1.0446 | 1.0416 | 0.9964 |
| 0.01 | 1.0075 | 1.0027 | 1.0000 | 0.9962 | 0.9700 |
| 0.02 | 1.0448 | 1.0352 | 1.0352 | 1.0373 | 1.0462 |
| 0.03 | 1.0824 | 1.0774 | 1.0896 | 1.1091 | 1.1253 |

**Table 4. Sensitivities of $V_{q,k}$ when false $P_d=0.01$, false $P_h=0.9$, $k=3$ and $q=0.1$**

| true $P_h$ / true $P_d$ | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
|---|---|---|---|---|---|
| 0.003 | 1.0191 | 1.0437 | 1.0843 | 1.1229 | 1.2031 |
| 0.005 | 0.9868 | 1.0035 | 1.0316 | 1.0576 | 1.1058 |
| 0.01 | 0.9803 | 0.9864 | 1.0000 | 1.0131 | 1.0340 |
| 0.02 | 1.0556 | 1.0481 | 1.0483 | 1.0539 | 1.0689 |
| 0.03 | 1.1770 | 1.1554 | 1.1469 | 1.1446 | 1.1671 |

**Table 5. Sensitivities of $V_{q,k}$ when false $P_d=0.01$, false $P_h=0.9$, $k=10$ and $q=0.1$**

| true $P_h$ / true $P_d$ | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
|---|---|---|---|---|---|
| 0.003 | 1.0476 | 1.0631 | 1.0842 | 1.0997 | 1.1193 |
| 0.005 | 1.0100 | 1.0187 | 1.0311 | 1.0399 | 1.0489 |
| 0.01 | 0.9989 | 0.9980 | 1.0000 | 1.0023 | 1.0048 |
| 0.02 | 1.0712 | 1.0566 | 1.0469 | 1.0447 | 1.0493 |
| 0.03 | 1.1892 | 1.1594 | 1.1378 | 1.1316 | 1.1387 |

Especially, when deal with field data, size of sample is often small or moderate. Thus whether small or moderate size of sample is applied to optimal ALT plans by large sample criterion, Monte Carlo study has been conducted.

The sample and parameter values included in this study is outlined in Table 6.

Analyses are based on 1000 replications for each of configurations. All data generation, parameter estimation and analyses are carry out using FORTRAN program and it is available from the author.

**Table 6. Monte Carlo Configuration**

| $P_d$ | $P_h$ | $k$ | quantile | sample size |
|---|---|---|---|---|
| 0.01 | 0.9 | 3 | 0.01, 0.1 | 40, 100, 200 |
| 0.01 | 0.9 | 5, ∞ | 0.1 | 40, 100, 200 |
| 0.01* | 0.9 | 3 | 0.1 | 40, 100, 200 |

*) a standard plan of the mid−low stress level, equal sample size at two stresses under equally spaced inspection scheme.

The quality of maximum likelihood estimates by optimal design of ALT has been measured by sample average, sample standard deviation, $T$ - statistic, mean squared error (MSE) for $b_0$, $b_1$, $\sigma$ and quantiles at the design stress.

The corresponding formulas related to $b_0$ are :

$$\bar{b}_0 = \sum_{j=1}^{r} \hat{b}_0^{(j)} / r,$$

$$sd(b_0) = \sum_{j=1}^{r} (\hat{b}_0^{(j)} - \bar{b}_0)^2 / r$$

$$T(b_0) = \sqrt{r}\,(\bar{b}_0 - b_0) / sd(b_0)$$

$$\mathrm{MSE}(b_0) = (\bar{b}_0 - b_0)^2 + \{sd(b_0)\}^2 \qquad (4.1)$$

$r$ : number of replications that converge by scoring method
$\hat{b}_0^{(j)}$ : estimate of $b_0$ at $j$ th replication.

In similar way, those values for other parameters can be computed. Monte Carlo results are summarized in Table 7 for estimates of $b_0$, $b_1$, $\sigma$ and some quantiles at the design stress.

I draw some conclusions on properties of optimal ALT plans developed by large sample criterion for small and moderate size of sample.

(1) For moderate sample size ($N=100$, 200), optimal ALT plans produce overall satisfactory estimation of parameters. However, for small sample size ($N=40$), the estimates produce the substantially large biases and mean squared errors.

(2) Results show that estimates of $b_0$ and quantiles at the design stress are almost positively biased and estimates of $b_1$ and $\sigma$ are almost negatively biased.

(3) The increase of a sample size has a strong favorable effect on the quality of estimates.

(4) Finally, simulation results show that major contribution of the mean squared error is due to the sample variance.

Table 8 contains results from comparision of estimates of 0.1th quantile at the design stress by optimal ALT plans($k=3$, 5, $\infty$) and a standard plan($k=3$), assuming that $\sigma=1$, with mid-low stress level($\xi_1=0.5$) and equal sample size at two stress levels under equally spaced inspection scheme.

Inspection of Table 8 indicates that large sample criterion can be applied to moderate size of sample and estimates of 0.1th quantile at the design stress are a little positively skewed for all plans. The usefulness of large sample approximation can be checked by comparing sample standard deviation with asymptotic values, i.e., differences are 10% low under optimal ALT plans with $k=3$, 5.

It appears that scoring method is quite satisfactory for censored data, but it may be necessary to replace it Newton-Raphson method with concave reparameterization by Burridge(1980) for grouped data owing to lower proportion of convergence by scoring method when number of inspections are small.

Also, optimal ALT plans are better than a standard plan and plans with small number of inspections can be comparable with plans under continuous inspection. But for small sample size, optimal ALT plans with $k=3$, 5, $\infty$ are resulted in a poor quality of estimates.

## 5. Conclusion

In this paper, asymptotically optimal accelerated life test plans for Weibull distributed lifetimes under type-I censoring and intermittent inspection whose advantages lie in the simplicity obtained in the collection and handling of the data are developed for various cases to be applied practically in field.

**Table 7. Monte Carlo Results by Optimal ALT Plans with $P_d=0.01$, $P_h=0.9$ and $k=3$**

| parameter | * quantile | true parameter/$\sigma$ | sample size | sample mean /$\sigma$ | sample std. deviation /$\sigma$ | $T$-statistic | MSE /$\sigma^2$ | parameter | quantile | true parameter/$\sigma$ | sample size | sample mean /$\sigma$ | sample std. deviation /$\sigma$ | $T$-statistic | MSE /$\sigma^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $b_0$ | 0.01 | 4.6001 | 40 | 4.8897 | 2.5736 | 3.1897 | 6.7073 | $\sigma$ | 0.01 | 1 | 40 | 1.0187 | 0.5069 | 1.0495 | 0.2571 |
| | | | 100 | 4.6394 | 1.2145 | 0.9457 | 1.4765 | | | | 100 | 0.9898 | 0.2448 | -1.2135 | 0.0606 |
| | | | 200 | 4.6453 | 0.8571 | 1.5184 | 0.7367 | | | | 200 | 0.9952 | 0.1724 | -0.7993 | 0.0298 |
| | 0.1 | 4.6001 | 40 | 4.8530 | 1.3689 | 3.6555 | 3.9406 | | 0.1 | 1 | 40 | 1.0190 | 0.3764 | 1.4284 | 0.1421 |
| | | | 100 | 4.6500 | 0.9960 | 1.4213 | 0.9945 | | | | 100 | 0.9938 | 0.1834 | -0.3558 | 0.0337 |
| | | | 200 | 4.6449 | 0.7257 | 1.7694 | 0.5286 | | | | 200 | 0.9997 | 0.1246 | -0.0593 | 0.0155 |
| $b_1$ | 0.01 | -5.4342 | 40 | -5.7373 | 2.7605 | -3.7293 | 7.7524 | $y_q$ | 0.01 | 0 | 40 | 0.2037 | 1.2235 | 4.7209 | 1.5383 |
| | | | 100 | -5.5133 | 1.4032 | -1.6509 | 1.9751 | | | | 100 | 0.0861 | 0.7077 | 3.5621 | 0.5082 |
| | | | 200 | -5.4962 | 0.9656 | -1.8500 | 0.9363 | | | | 200 | 0.0671 | 0.5119 | 3.7807 | 0.2666 |
| | 0.1 | -5.4342 | 40 | -5.7306 | 2.1070 | -4.0044 | 4.5273 | | 0.1 | 2.3498 | 40 | 2.5593 | 1.4080 | 4.2465 | 2.0265 |
| | | | 100 | -5.4999 | 1.1094 | -1.6832 | 1.2352 | | | | 100 | 2.4135 | 0.7559 | 2.3947 | 0.5754 |
| | | | 200 | -5.4774 | 0.8004 | -1.5501 | 0.6426 | | | | 200 | 2.3951 | 0.5650 | 2.3020 | 0.3213 |

*) the quantile at the design stress level estimated by an optimal ALT plan

**Table 8. Comparision of the Estimates of 0.1th Quantile at the Design Stress by Some ALT Plans with $P_d=0.01$, $P_h=0.9$ and True Parameter$=2.3498\sigma$**

| plan | sample size | sample mean $/\sigma$ | sample std. deviation $/\sigma$ | $T-$ statistic $/\sigma$ | MSE $/\sigma^2$ | asymptotic standard deviation $/\sigma$ |
|---|---|---|---|---|---|---|
| optimal ALT plan under equal probability inspection( $k=3$) | 40 | 2.5599 | 1.4080 | 4.2465 | 2.0265 | 1.1528 |
| | 100 | 2.4135 | 0.7559 | 2.3947 | 0.5754 | 0.7291 |
| | 200 | 2.3951 | 0.5650 | 2.3020 | 0.3213 | 0.5155 |
| optimal ALT plan under equal probability inspection( $k=5$) | 40 | 2.6558 | 1.4694 | 6.0674 | 2.2526 | 1.1209 |
| | 100 | 2.4647 | 0.7735 | 4.3172 | 0.6209 | 0.7146 |
| | 200 | 2.4087 | 0.5133 | 2.3654 | 0.2721 | 0.5052 |
| optimal ALT plan under continuous inspection | 40 | 2.4726 | 1.5672 | 2.4465 | 2.4712 | 1.1092 |
| | 100 | 2.4287 | 0.7311 | 3.3621 | 0.5408 | 0.7015 |
| | 200 | 2.3902 | 0.4908 | 1.9379 | 0.2418 | 0.4960 |
| *a standard plan under equally spaced inspection( $k=3$) | 40 | 2.6414 | 1.7474 | 4.4188 | 3.1385 | 1.3361 |
| | 100 | 2.4718 | 0.9494 | 3.4693 | 0.9162 | 0.3450 |
| | 200 | 2.3844 | 0.5759 | 1.6153 | 0.3329 | 0.5975 |

*) an ALT plan tested at a mid−low stress and high stress levels with equal sample sizes assuming that $\sigma=1$

Computational results indicate that the large number of inspections at overstress levels are not necessary and effect of guess estimates for parameters would expect to be fairly tolerable for range of parameter values considered.

Results for small sample study on optimal plans using large sample criterion show that the effect of large sample approximation is not substantial for moderate size of sample, but is considerable for small size of sample.

Finally, future research in this area will extend more complicated problems in which other probability models, inspection schemes and censoring types as type−II or progressively will be considered.

Also, because optimal plan is less robust to deviations from assumed model(see Meeker(1984) and Meeker and Hahn(1985)), compromise accelerated life test plans with subexperiments more than two stress levels may be recommended.

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