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SHARP FUNCTION AND WEIGHTED L^p ESTIMATE FOR PSEUDO DIFFERENTIAL OPERATORS WITH REDUCED SYMBOLS

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In 1982, N. Miller [5] showed a weighted L^p boundedness theorem for pseudo differential operators with symbols in $S^0_{1,0}$. In this paper, we shall prove the pointwise estimates, in terms of the Fefferman, Stein sharp function and Hardy Littlewood maximal function, for pseudo differential operators with reduced symbols and show a weighted L^p-boundedness for pseudo differential operators with symbol in $S^m_{\rho,\delta}$, $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$, $\rho \neq 0$ and $m=(n+1)(\rho-1)$.

1. Introduction.

Let $a(x, \xi)$ be a sufficiently regular function defined on $R^n \times R^n$ The pseudo differential operator A with symbol $a(x, \xi)$ is defined on the Schwartz space $S(R^n)=S$ of rapidly decreasing and infinitely defferentible functions by the formula .

$$Au(x) = \int_{R^n} e^{2\pi i x \cdot \xi} a(x,\xi)\hat{u}(\xi)d\xi$$

where $\hat{u}(\xi)=\int_{R^n} e^{-2\pi i x \cdot \xi} u(x)dx$ is the Fourier transform of u

For w satisfying Muckenhoupt's A_p condition, $L^p(wdx)=L^p(R^n, wdx)$ is the space of all measurable functions f with $\|f\|_{p,w}=(\int_{R^n}|f(x)|^p w dx)^{1/p}$ $<\infty$

For a locally integrable function f,

(i) $f^{\#}(x)$ = the Fefferman Stein sharp function of f

$$=sup_{x\in Q}\frac{1}{|Q|}\int|f(y)-(f_Q)|dy$$

the supremum being taken over all cubes Q containing x and f_Q is the average value of f on the cube Q.

(ii) $M_r f(x)=\sup_Q\frac{1}{|Q|}\int_Q|f(y)|^r dy)^{1/r}$

the supremum being taken over all cubes containinig x.

(iii) $Mf(x)$ = the Hardy−Littlewood maximal function

$$=sup_Q\frac{1}{|Q|}\int_Q|f(y)|dy$$

the supremum being taken over all cubes Q containing x.

(iv) $f^*(x)$ = the dyadic maximal function of f

$$=\sup\frac{1}{|Q|}\int_Q|f(y)|dy$$

the supremum being taken over all dyadic cubes Q.

We shall say that a symbol $a(x,\xi)\varepsilon C^\infty(R^n\times R^n)$ is in the class $S^m_{\rho,\delta}$ if it satisfies the estimate

$$|(\frac{\partial}{\partial x})^\beta(\frac{\partial}{\partial\xi})^\alpha)a(x,\xi)|\leq C_{\alpha\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|} \tag{1.0}$$

for all multi-indices α and β.

Note that $S^{m_1}_{\rho_1\delta_1}\supset S^{m_2}_{\rho_2\delta_2}$ if $\rho_1\leq\rho_2$, $\delta_1\leq\delta_2$ and $m_1\geq m_2$.

We consider a symbol $a(x,\xi)$ which is represented as a sum of reduced symbols, plus another symbol vanishing on $|\xi|>1$ i.e.,

$$a(x,\xi)=a_0(x,\xi)+\sum_{k\in Z^n}V_k a_k(x,\xi) \tag{1.1}$$

The symbol $a_0(x, \xi)$ vanishes for $|\xi|>1$, and there exists a constant C_α such that $\|(\frac{\partial}{\partial\xi})^\alpha a_0(x, \xi)\|_\infty < C_\alpha$. The a_k's are reduced symbols which represented as

$$(1.2) \qquad a_k(x, \xi)=\sum_{j=1}^{\infty} b_{jk}(x)\psi(2^{-j}\xi)$$

where b_{jk} satisfies $\|b_{jk}\|_\infty \le C$, $\|\nabla_x b_{jk}\|_\infty \le C^{j\delta}$, $0\le\delta<1$, and $\psi\varepsilon C_0^\infty(R^n)$ is supported in $\{\frac{1}{3}\le|\xi|\le 1\}$, and $\sum_{k\varepsilon Z^n} V_k$ is a positive convergent series.

2. Main Results.

In order to prove our first Theorem, we need the following Lemma.

Lemma 2.1 *Let $\psi\varepsilon C_0^\infty(R^n)$ with supp $\psi\subset\{\frac{1}{3}\le|\xi|\le 3\}$. If $t\ge 0$, then there is a constant $C_t\ge 0$ such that the following inequalities hold.*
(i) $|y|^t\left|\int_{R^n}\psi(2^{-j}\xi)e^{2\pi iy\cdot\xi}d\xi\right|\le C_t\, 2^{j(n-t)}$ and
(ii) $|y|^t\left|\int_{R^n}\psi(2^{-j}\xi)\xi_\ell e^{2\pi iy\cdot\xi}d\xi\right|\le C_t\, 2^{j(n+1-t)}$
where ξ_ℓ is the ℓ-th coordinate of ξ.

Above lemma is a slight modification of Lemma 2.9 in N.Miller [5].

Theorem 2.2 *Let A be a pseudo differential operator with symbol $a(x, \xi)$ satisfying (1.1) and (1.2), then there is a constant $c>0$ such that the pointwise estimate*

$$(Au)^\#(x^0)\le CM_2u(x^0) \text{ for all } x^0\varepsilon R^n,\ u\varepsilon S(R^n)$$

holds

Proof. The proof follows the lines of the argument in Theorem 2.8

of Miller [5]. Given $x^0 \varepsilon R^n$, we let Q be a cube containing x^0 with center x' and diameter d. Let $\tau \varepsilon C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \tau(x) \le 1$, be 1 when $|x-x'| \le 2d$, and vanish when $|x-x'| \ge 3d$. Then for $u \varepsilon S(R^n)$,

$$\frac{1}{|Q|}\int_Q |Au(x)-(Au)_Q|\,dx$$

$$\le \frac{2}{|Q|}\int_Q |A(\tau u)|\,dx + \frac{1}{|Q|}\int_Q |A((1-\tau)u)(x)-(A((1-\tau)u)_Q|\,dx.$$

Let Q' be the cube centered at x', with sides of length $7d$ parallel to those of Q. We can dominate the first term in the inequality above because A is bounded on $L^2(R^n)$.

$$(2.1)\qquad \frac{2}{|Q|}\int_Q |A(\tau u)|\,dx \le C\Big(\frac{1}{|Q|}\int_Q |A(\tau u))|^2 dx\Big)^{\frac{1}{2}}$$

$$\le C\Big(\frac{1}{|Q|}\int_{R^n} |\tau u|^2 dx\Big)^{\frac{1}{2}}$$

$$\le C\Big(\frac{1}{|Q'|}\int_{Q'} |u|^2 dx\Big)^{\frac{1}{2}}$$

$$\le CM_2u(x^0)$$

To deal with second term, we simplify notation, writing u for $(1-\tau)u$, and we assume that u has supported in the set $\{x : |x-x'| \ge 2d\}$. We must estimate the quantity $\frac{1}{|Q|}\int_Q |Au(x)-(Au)_Q|\,dx$.

We begin by decomposing the symbol $a(x, \xi)$ into the sum of reduced symbols and a symbol vanishing on $|\xi|>1$.

$$Au(x) = \int_{R^n} \hat{u}(\xi)a(x, \xi)e^{2\pi ix\cdot\xi}d\xi$$

$$= \int_{R^n} \hat{u}(\xi)a_0(x, \xi)e^{2\pi ix\cdot\xi}d\xi$$

$$+\int_{R^n} u(y) \int_{R^n} \sum_{k \in Z^n} V_k a_k(x, \xi) e^{2\pi i (x-y) \cdot \xi} d\xi dy$$

$$= B_u(x) + \sum_{k \in Z^n} V_k A_k u(x)$$

We consider the first term of the last inequality. B is a pseudo differential operator whose symbol is $a_0(x, \xi)$; the ξ-support of this symbol is contained in the set $\{\xi : |\xi| \leq 1\}$ and $a_0(x, \xi)$ has the property that $|(\frac{\partial}{\partial \xi})^\alpha a_0(x, \xi)| \leq C_\alpha$. Hence we have

$(Bu)^\#(x^0) \leq CM_r u(x^0)$ for $r>1$. Now

$$(Au)^\#(x^0) \leq (Bu)^\#(x^0) + \sum_{k \in Z^n} V_k (A_k u)^\#(x^0)$$

Therefore the next task is to examine the operator A_k. For every $k \in Z^n$

$$A_k u(x) = \int_{R^n} u(y) \int_{R^n} \sum_{j=1}^{\infty} b_{jk}(x) \psi(2^{-j}\xi) e^{2\pi i (x-y) \cdot \xi} d\xi dy$$

$$= \sum_{j=1}^{\infty} \int_{R^n} u(y) \int_{R^n} b_{jk}(x) \psi(2^{-j}\xi) e^{2\pi i (x-y) \cdot \xi} d\xi dy$$

$$= \sum_{j=1}^{\infty} A_{jk} u(x).$$

We now estimate $(A_k u)^\#(x^0)$

$$\frac{1}{|Q|} \int_Q |A_{jk} u(x) - (A_{jk} u)_Q| dx$$

$$(2.2) \qquad = \frac{1}{|Q|} \int_Q \left| \frac{1}{|Q|} \{A_{jk} u(x) - A_{jk} u(z)\} dz \right| dx$$

$$= \frac{1}{|Q|} \int_Q \left| \frac{1}{|Q|} \int_Q \left\{ \int_{R^n} u(y) \int_{R^n} \psi(2^{-j}\xi) [b_{jk}(x) e^{2\pi i (x-y) \cdot \xi} \right.\right.$$

$$\left.\left. - b_{jk}(z) e^{2\pi i (z-y) \cdot \xi}] d\xi dy \right\} dz \right| dx$$

To estimate this last quantity, we distinguish two cases.

Case 1 $2^j d \geq 1$. Then(2.2) is dominated by

$$2\sum_{m=1}^{\infty} \frac{1}{|Q|}\int_Q \int_{2^m d \leq |y-x'| \leq 2^{m+1}d} |u(y)| \left| \int_{R^n} b_{jk}(x)\psi(2^{-j}\xi)e^{2\pi i(x-y)\cdot\xi}d\xi \right| dydx$$

$$\leq C\sum_{m=1}^{\infty}\int_Q \frac{2^{nm}}{|Q_m|} \int_{2^m d \leq |y-x| \leq 2^{m+1}d} \frac{|u(y)|}{|x-y|^{n+1}} |x-y|^{n+1}$$

$$\left| \int_{R^n}\psi(2^{-j}\xi)e^{2\pi i(x-y)\cdot\xi}d\xi \right| dy\, |b_{jk}(x)|\, dx$$

(Q_m is the cube with center x', sides parallel to thoese of Q and radius $2^{m+1}d$)

The last term is bounded by

$$C\sum_{m=1}^{\infty} d^n 2^{n\,m}(2^m d)^{-(n+1)}2^{-j} \frac{1}{|Q_m|}\int_{Q_m} |u(y)|\, dy$$

(By Lemma 2.1, with $t=n+1$, and $\|b_{jk}\|_{\infty} \leq C$)

$$\leq C(2^j d)^{-1}Mu(x^0)$$

Case 2. $2^j d<1$. In this case we write the last term of (2.2).

$$\frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \sum_{m=1}^{\infty} \int_{2^m d \leq |y-x| \leq 2^{m+1}d} |u(y)| \left| \int_{Rn}\psi(2^{-j}\xi) \right.$$

$$[b_{jk}(x)e^{2\pi i(x-y)\cdot\xi}-b_{jk}(x)e^{2\pi i(z-y)\cdot\xi}+b_{jk}(x)e^{2\pi i(z-y)\cdot\xi}$$

$$\left. -b_{jk}(z)e^{2\pi i(z-y)\cdot\xi}]d\xi \right| dydzdx$$

$$\leq \frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \sum_{m=1}^{\infty} \int_{2^m d \leq |y-x'| \leq 2^{m+1}d} |u(y)| \left| \int_{R^n}\psi(2^{-j}\xi) \right.$$

$$\left. [e^{2\pi i(x-y)\cdot\xi}-e^{2\pi i(z-y)\cdot\xi}]\cdot b_{jk}(x)d\xi \right| dydzdx$$

$$+\frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \sum_{m=1}^{\infty} \int_{2^m d\le |y-x'|\le 2^{m+1}d} |u(y)| \, \Big|\int_{R^n} \psi(2^{-j}\xi)$$

$$[b_{jk}(x)-b_{jk}(z)]e^{2\pi i(x-y)\cdot\xi}dy \Big| dydzdx$$

$$=A+B$$

First to deal with A,

$$e^{2\pi i(x-y)\cdot\xi}-e^{2\pi i(z-y)\cdot\xi}$$

$$=\sum_{\ell=1}^{n}(x_\ell-z_\ell)\int_0^1 \frac{\partial}{\partial x_\ell}e^{2\pi i(x(t)-y)\cdot\xi}dt$$

where $x(t)=tx+(1-t)z$, x_ℓ : the ℓ-th coordinate of x.

Note that (i) $|x_\ell-z_\ell|\le d$ since both x and z are in Q, and (ii) if $2^m d\le |y-x'|\le 2^{m+1}d$, then $2^{m-1}d\le |x(t)-y|\le 2^{m+2}d$ since $x(t)\varepsilon Q$.

A is dominated by

$$C\sup_{x\in Q}\sum_{m=1}^{\infty}\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\int_{2^m d\le |y-x|\le 2^{m+1}d}\frac{|u(y)|}{|y-x|^{n+1/2}}$$

$$\sum_{\ell=1}^{n}|x_\ell-z_\ell|\cdot\int_0^1 |x(t)-y|^{n+1/2}$$

$$\Big|\int_{R^n}\psi(2^{-j}\xi)\frac{\partial}{\partial x_\ell}\partial^{2\pi i(x(t)-y)\cdot\xi}d\xi\Big| dtdydz\,|b_{jk}(x)|\,dx$$

$$\le C\sum_{m=1}^{\infty}2^{(m+1)n}(2^m d)^{-n-1/2}(nd)2^{j/2}d^n\frac{1}{|Q_m|}\int Q_m|u(y)|\,dy$$

(Lemma 2.1, $t=n+1/2$, $\|b_{jk}\|_\infty\le C$

and Q_m is the cube with centered x', sides parallel to those of Q

and radius $2^{m+1}d$)

$$\leq C\sum_{m=1}^{\infty} 2^{-\frac{m}{2}} d^{1/2} 2^{j/2} Mu(x^0)$$

$$\leq Cd^{1/2} 2^{j/2} Mu(x^0)$$

Next we estimate B

$$B \leq \frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \sum_{m=1}^{\infty} \int_{2^m d \leq |y-x^0| \leq 2^{m+1}d} \frac{|u(y)|}{|z-y|^{n+1}}$$

$$|z-y|^{n+1} \int_{R^n} \xi(2^{-j}\xi) e^{2\pi i(z-y)\cdot\xi} d\xi dy$$

$$|b_{jk}(x) - b_{jk}(z)|\, dz dx$$

$$\leq C\sum_{m=1}^{\infty} (2^{m+1}d)^n (2^m d)^{-(n-1)} 2^{-j} \frac{1}{|Q_m|} \int_{Q_m} |u(y)|\, dy$$

(By Mean Value Theorem, $\| b_{jk}(k) - b_{jk}(z) \|_\infty \leq Cd \| \nabla b_{jk} \|_\infty$ and $\| \nabla b_{jk} \|_\infty \leq C2^{j\delta}$ and Lemma 2.1, $t = n+1$.
Q_m is above Q_m)

$$\leq C\sum_{m=1}^{\infty} 2^{-m} 2^{j(\delta-1)} \frac{1}{|Q_m|} \int_{Q_m} |u(y)|\, dy$$

$$\leq C2^{j(\delta-1)} Mu(x^0)$$

Putting the two cases together, we have shown that if Q is any cube containing x^0, then

$$\frac{1}{|Q|}\int_Q \left| \sum_{j=1}^{\infty} A_{jk} u(x) - \left(\sum_{j=0}^{\infty} A_j u\right)_Q \right| dx$$

$$\leq C\sum_{j=0}^{\infty} \frac{1}{|Q|} \int_Q |A_{jk}u(x)-(A_{jk})_Q| dx$$

$$\leq C\{ \sum_{2^j d \geq 1} (2^{jd})^{-1} + \sum_{2^j d \leq 1} (d^{1/2}2^{j/2}+2^{j(\delta-1)})\} Mu(x^0)$$

$$\leq CMu(x^0)$$

Thus we have

$$(\sum_{j=0}^{\infty} A_{jk}u)^{\#}(x^0) \leq CMu(x^0)$$

Going back to our original notation, and summarizing we have shown that if Q is any cube containing x^0 then

$$\frac{1}{|Q|}\int_Q |(Au(x)-(Au)_Q| dx$$

(2.3) $$\leq (A(\tau u))^{\#}(x^0)+(B((1-\tau)u))^{\#}(x^0)+\sum_{k\varepsilon Z^n} V_k(A_k u)^{\#}(x^0)$$

$$\leq C_1M_2u(x^0)+C_2M_ru(x^0)+C_3\sum_{k\varepsilon Z^n} V_kMu(x^0) \text{ for } r\geq 1$$

$$\leq CM_qu(x^0) \text{ for } q\geq 2$$

When we take the supremum of the left side over all cubes containing x^0, we obtain the inequality

(2.4) $$(Au)^{\#}(x^0) \leq CM_2u(x^0), \text{ Q.E.D.}$$

We now wish to show the sharp function estimate for general symbol $a(x, \xi)$ ε $S^m_{\rho,\delta}$, $0\leq \delta \leq \rho \leq 1$, $\delta\neq 1$, $\rho\neq 0$, and $m=(n+1)(\rho-1)$. Take a function $\lambda(\xi)$ ε $C_0^\infty(R^n)$ satisfying supp $\lambda \subset \{\frac{1}{3} \leq |\xi| \leq 1\}$, $\sum_{j=-\infty}^{\infty} \lambda(2^{-j}\xi)=1$ $(\xi\neq 0)$. And put $\varphi(\xi)=1-\sum_{j=1}^{\infty}\lambda(2^{-j}\xi)$. Put then $a_0(x, \xi)$

$=\varphi(\xi)a(x,\ \xi)$ and $a_j(x,\ \xi)=\lambda(2^{-j}\xi)a(x,\ \xi)$, $j=1,\ 2,\cdots$. Then $a_j(x,\ \xi)$ satisfy the condition (1.0) uniformly in j and $a_0(x,\ \xi)$ has the ξ−compact support.

Lemma 2.3. *Let $a(x,\ \xi)$ and $a_j(x,\ \xi)$ be above. Let $\psi(\xi)\ \varepsilon\ C_0^\infty(R^n)$ be such that $\psi\lambda=\lambda$ and supp $\psi\ \subset\{\frac{1}{3}\ \leq\ |\xi|\ \leq 1\}$, Then there exist functions a_{jk}, $j\varepsilon N$, $k\varepsilon Z^n$, such that*

(i) $\|a_{jk}\|_\infty\leq C$
(ii) $\|\nabla a_{jk}\|_\infty\leq C2^{j\delta}$
(iii) $a_j(x,\ \xi)=\sum_{k\varepsilon Z^n}(1+|k|^2)^{-\frac{n+1}{2}}a_{jk}(x)e^{2\pi ik\xi 2^{-j}}\psi(2^{-j}\xi)$

Proof. It is similar that of in [2].

Lemma 2.3 says that above symbol $a(x,\ \xi)$ can be represented as a sum of reduced symbols plus another symbol vanishing on $|\xi|>1$. That is :

$$a(x,\ \xi)=a_0(x,\ \xi)+\sum_{j=1}^{\infty}\sum_{k\varepsilon Z^n}(1+|k|^2)^{-\frac{n+1}{2}}a_{jk}(x)e^{2\pi ik\xi 2^{-j}}\psi(2^{-j}\xi)$$
$$=a_0(x,\ \xi)+\sum_{k\varepsilon Z^n}^{\infty}(1+|k|^2)^{-\frac{n+1}{2}}a_k(x,\ \xi)$$

where $a_k(x,\ \xi)=\sum_{j=1}^{\infty}a_{jk}(x)e^{2\pi ik\xi 2^{-j}}\psi(2^{-j}\xi)$ is a reduced symbol and the double series converges absolutely.

Theorem 2.4. *Let A be a pseudo differential operator with symbol $a\ \varepsilon\ S^m_{\rho,\delta}$, $0\ \leq\ \delta\ \leq\ \rho\ \leq 1$, $\delta\neq 1$ $\rho\neq 0$, $m=(n+1)(\rho-1)$. Then there is a constant $C>0$ such that the pointwise estimate,*
$(Au)^{\#}(x^0)\leq CMru(x^0)$ for all $x^0\varepsilon R^n$, $u\varepsilon S(R^n)$

and $1<r<\infty$, *holds.*

Furthermore, for $1<p<\infty$ *and* $w \varepsilon A_p$, *above* A *has a bounded extension to all of* $L^p(R^n, wdx)$.

Rroof. By Lemma 2.3, $a(x, \xi)$ can be represented as a sum of reduced symbols, plus another symbol vanishing on $|\xi|>1$. Put $b_{jk}(x)=e^{2\pi i x \cdot \xi 2^{-j}}$ $a_{jk}(x)$ and put $V_k=(1+|k|^2)^{-\frac{n+1}{2}}$, then A satisfies the condition of [Theorem 2.2]. By [Theorem 2.2] and [Theorem 5.[3]], A is bounded on $L^p(R^n)$, $p>2$. We consider A^*, the adjoint of A. It is also a pseudo differential operator with symbol in $S^m_{\rho,\delta}$. [4], [6]. Thus A^* is bounded on $L^p(R^n)$, $2<p<\infty$, by [Theorem 2.2] and [Theorem 5. [3]]. Hence A is bounded on $L^p(R^n)$, $1<p<2$, and consequently on $L^2(R^n)$ as well, by interpolation. This means that A is bounded on $L^p(R^n)$ for $1<p<\infty$.

By replacing $M_2u(x)$ by $M_ru(x)$, $1<r<\infty$, in (2.1), (2.3) and (2.4) of the proof of [Theorem 2.2], we get the desired result because $Mu(x^0) \leq M_ru(x^0)$, $1<r<\infty$.

We next go to the proof of the second assertion.
For $u \varepsilon S(R^n)$ and $w \varepsilon A_p$,

$$\begin{aligned} \| Au \|_{p,w} &\leq \| (Au)^* \|_{p,w} \leq C \| (Au)^\# \|_{p,w} \\ &\leq C \| (M_r u \|_{p,w} \text{ if } 1<r<\infty \\ &\leq C \| u \|_{p,w} \text{ if } 1<r<p \end{aligned}$$

Since $Au \varepsilon S \subset L^p(R^n, wdx) \cap L^1(R^n)$, we can apply Lemma 2.7 in Miller [4] to prove the second inequality. So we can now extend A to a bounded operator on $L^p(R^n, wdx)$ because $S(R^n)$ is dense in $L^p(R^n, wdx)$.

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