SHARP FUNCTION AND WEIGHTED $L^p$ ESTIMATE FOR PSEUDO DIFFERENTIAL OPERATORS WITH REDUCED SYMBOLS

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In 1982, N. Miller [5] showed a weighted $L^p$ boundedness theorem for pseudo differential operators with symbols in $S^m_{1,0}$. In this paper, we shall prove the pointwise estimates, in terms of the Fefferman, Stein sharp function and Hardy Littlewood maximal function, for pseudo differential operators with reduced symbols and show a weighted $L^p-$boundedness for pseudo differential operators with symbol in $S^m_{a,0}$, $0 \leq a \leq p \leq 1$, $\delta \neq 1$, $\rho \neq 0$ and $m = (a+1)(p-1)$.

1. Introduction.

Let $a(x, \xi)$ be a sufficiently regular function defined on $\mathbb{R}^n \times \mathbb{R}^n$. The pseudo differential operator $A$ with symbol $a(x, \xi)$ is defined on the Schwartz space $S(\mathbb{R}^n) = S$ of rapidly decreasing and infinitely differentiable functions by the formula.

$$Au(x) = \int_{\mathbb{R}^n} e^{2\pi i \langle x, \xi \rangle} \hat{a}(x, \xi) \hat{u}(\xi) \, d\xi$$

where $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \langle x, \xi \rangle} u(x) \, dx$ is the Fourier transform of $u$.

For $w$ satisfying Muckenhoupt's $A_p$ condition, $L^p(w \, dx) = L^p(\mathbb{R}^n, w \, dx)$ is
the space of all measurable functions $f$ with $\|f\|_{L^p} = (\int |f(x)|^p \, dx)^{1/p} < \infty$

For a locally integrable function $f$, $\mathcal{F}_f(x)$ is the Fefferman Stein sharp function of $f$

$$\mathcal{F}_f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q} |f(y) - f_0| \, dy$$

the supremum being taken over all cubes $Q$ containing $x$ and $f_0$ is the average value of $f$ on the cube $Q$.

(ii) $Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q} |f(y)| \, dy$

the supremum being taken over all cubes containing $x$.

(iii) $Mf(x) = \text{the Hardy-Littlewood maximal function}

\quad = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q} |f(y)| \, dy

\quad \text{the supremum being taken over all cubes } Q \text{ containing } x.$

(iv) $f^*(x) = \text{the dyadic maximal function of } f$

\quad = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q} |f(y)| \, dy

\quad \text{the supremum being taken over all dyadic cubes } Q.$

We shall say that a symbol $a(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is in the class $\mathcal{S}_\kappa^\infty$, if it satisfies the estimate

$$| (\frac{\partial}{\partial x})^\alpha (\frac{\partial}{\partial \xi})^\beta a(x, \xi)| \leq C_{\alpha \beta} (1 + |\xi|)^{m_1 + \kappa}$$

for all multi-indices $\alpha$ and $\beta$.

Note that $\mathcal{S}_{\kappa_1, \delta_1}^{m_1} \supseteq \mathcal{S}_{\kappa_2, \delta_2}^{m_2}$ if $\rho_1 \leq \rho_2$, $\delta_1 \leq \delta_2$ and $m_1 \geq m_2$.

We consider a symbol $a(x, \xi)$ which is represented as a sum of reduced symbols, plus another symbol vanishing on $|\xi| > 1$ i.e.,

$$a(x, \xi) = a_0(x, \xi) + \sum_{\omega \in \mathbb{Z}^n} V_\omega a_\omega(x, \xi)$$
The symbol \( a_0(x, \xi) \) vanishes for \( |\xi| > 1 \), and there exists a constant \( C_0 \) such that \( \| (\partial_\xi^j a_0(x, \xi)) \|_{\infty} < C_0 \). The \( a_j \)'s are reduced symbols which represent

\[
a_j(x, \xi) = \sum_{i=1}^n b_{ij}(x) \psi(2^{-i} |\xi|)
\]

where \( b_{ij} \) satisfies \( \| b_{ij} \|_{\infty} \leq C_i \), \( \| \nabla b_{ij} \|_{\infty} \leq C_i \), \( 0 \leq \delta < 1 \), and \( \psi \in C_0(\mathbb{R}^n) \) is supported in \( \{ \frac{1}{3} \leq |\xi| \leq 1 \} \), and \( \sum_{\omega \in \Lambda} \nu_\omega \) is a positive convergent series.

2. Main Results.

In order to prove our first theorem, we need the following Lemma.

**Lemma 2.1** Let \( \psi \in C_0(\mathbb{R}^n) \) with \( \text{supp} \ \psi \subset \{ \frac{1}{3} \leq |\xi| \leq 3 \} \). If \( t \geq 0 \), then there is a constant \( C_t \geq 0 \) such that the following inequalities hold.

(i) \( |y| \left| \int_{\mathbb{R}^n} \psi(2^{-i} |\xi|) e^{2\pi i \xi \cdot y} \ d\xi \right| \leq C_t 2^{(i-1)} \)

(ii) \( |y| \left| \int_{\mathbb{R}^n} \psi(2^{-i} |\xi|) e^{2\pi i \xi \cdot y} \ d\xi \right| \leq C_t 2^{(i+1-\delta)} \)

where \( \xi_i \) is the \( i \)-th coordinate of \( \xi \).

Above lemma is a slight modification of Lemma 2.9 in N.Miller [5].

**Theorem 2.2** Let \( A \) be a pseudo differential operator with symbol \( a(x, \xi) \) satisfying (1.1) and (1.2), then there is a constant \( c > 0 \) such that the pointwise estimate

\[
(Au)^{(r)}(x) \leq CMu(x) \text{ for all } x \in \mathbb{R}^n, u \in S(\mathbb{R}^n)
\]

holds.

**Proof.** The proof follows the lines of the argument in Theorem 2.8
of Miller [5]. Given \(x^o \in \mathbb{R}^n\), we let \(Q\) be a cube containing \(x^o\) with center \(x'\) and diameter \(d\). Let \(\tau \in C_c(\mathbb{R}^n)\) satisfy \(0 \leq \tau(x) \leq 1\), be 1 when \(|x - x'| \leq 2d\), and vanish when \(|x - x'| > 3d\). Then for \(u \in S(\mathbb{R}^n)\),

\[
\frac{1}{|Q|} \int_Q |Au(x) - (Au)_Q| \, dx \\
\leq \frac{1}{|Q|} \int_Q |Au(x) - (Au)_Q| \, dx + \frac{1}{|Q|} \int_Q |A((1-\tau)u)(x) - (A((1-\tau)u)_Q| \, dx.
\]

Let \(Q'\) be the cube centered at \(x\), with sides of length \(7d\) parallel to those of \(Q\). We can dominate the first term in the inequality above because \(A\) is bounded on \(L^2(\mathbb{R}^n)\).

\[
(2.1) \quad \frac{2}{|Q|} \int_Q |\tau u| \, dx \leq C \left( \frac{1}{|Q|} \int_Q |\tau u| \, dx \right)^{1/2} \\
\leq C \left( \frac{1}{|Q|} \int_{Q'} |\tau u| \, dx \right)^{1/2} \\
\leq C \left( \frac{1}{|Q'|} \int_{Q} |u| \, dx \right)^{1/2} \\
\leq CMu(x^o)
\]

To deal with second term, we simplify notation, writing \(u\) for \((1-\tau)u\), and we assume that \(u\) has supported in the set \(\{x : |x-x'| \geq 2d\}\). We must estimate the quantity \(\frac{1}{|Q|} \int_Q |Au(x) - (Au)_Q| \, dx\).

We begin by decomposing the symbol \(a(x, \xi)\) into the sum of reduced symbols and a symbol vanishing on \(|\xi| > 1\).

\[
Au(x) = \int_{\mathbb{R}^n} \hat{a}(\xi) a(x, \xi) e^{2\pi i \xi \cdot x} \, d\xi \\
= \int_{\mathbb{R}^n} \hat{a}(\xi) a_0(x, \xi) e^{2\pi i \xi \cdot x} \, d\xi
\]
We consider the first term of the last inequality. $B$ is a pseudo differential operator whose symbol is $a_0(x, \xi)$; the $\xi$-support of this symbol is contained in the set $\{\xi : |\xi| \leq 1\}$ and $a_0(x, \xi)$ has the property that $|(\frac{\partial}{\partial \xi})a_0(x, \xi)| \leq C_0$. Hence we have

$$(Bu)^s(x^*) \leq CMu(x^*)$$ for $r > 1$. Now

$$(Au)^s(x^*) \leq (Bu)^s(x^*) + \sum_{k \in \mathbb{Z}} V_k(Au)^s(x^*)$$

Therefore the next task is to examine the operator $A_k$. For every $k \in \mathbb{Z}$

$$A_ku(x) := \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} \sum_{j=1}^n b_k(x) \psi(2^{-j} \xi) e^{i \xi \cdot \gamma} d\xi dy$$

$$= \sum_{j=1}^n \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} b_k(x) \psi(2^{-j} \xi) e^{i \xi \cdot \gamma} d\xi dy$$

$$= \sum_{j=1}^n A_ku(x).$$

We now estimate $(A_ku)^s(x^*)$

$$- \frac{1}{|Q|} \int_Q |A_ku(x) - (A_ku)_Q| \ dx$$

(2.2)

$$= - \frac{1}{|Q|} \int_Q \left| \frac{1}{|Q|} \int_Q \left| A_ku(x) - \sum_{j=1}^n [A_ku(x) - A_ku(z)] \right| \ dx \right| \ dx$$

$$= \frac{1}{|Q|} \int_Q \left| \frac{1}{|Q|} \int_Q \left| \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} \psi(2^{-j} \xi) (b_k(x) e^{i \xi \cdot \gamma} - b_k(z) e^{i \xi \cdot \gamma}) d\xi dy \right| \ dx \right| \ dx$$

$$- b_k(z) e^{i \xi \cdot \gamma} l|d\xi dy| dz \ dx$$

To estimate this last quantity, we distinguish two cases.
Case 1. $2d \geq 1$. Then (2.2) is dominated by

$$2\sum_{k=1}^{\infty} \frac{1}{|Q|} \int_{Q} \int_{|x-y| \leq 2^{-d}} |u(y)| \int_{\mathbb{R}^n} b_{n}(x) \psi(2^{-s} \xi) e^{2\pi i (x-y) \cdot \xi} \, d\xi \, dy \, dx$$

$$\leq C \sum_{n=1}^{\infty} \int_{Q} \frac{2^{2n}}{|Q_n|} \int_{|x-y| \leq 2^{-d}} |u(y)| \frac{1}{|x-y|^{s+1}} \, |x-y|^{-s} \, dy \, dx$$

where $Q_n$ is the cube with center $x'$, sides parallel to those of $Q$ and radius $2^{n+d}$.

The last term is bounded by

$$C \sum_{n=1}^{\infty} 2^{2n} \frac{2^{2n} |Q_n|}{|Q|} \int_{Q_n} |u(y)| \, dy$$

(By Lemma 2.1, with $t = n + 1$, and $\|b_n\|_{L^1} \leq C$)

$$\leq C (2d)^{-1} M u(x')$$

Case 2. $2d < 1$. In this case we write the last term of (2.2).

$$- \frac{1}{|Q|} \int_{Q} \frac{1}{|Q|} \int_{Q} \sum_{n=1}^{\infty} \int_{|x-y| \leq 2^{-d}} |u(y)| \int_{\mathbb{R}^n} \psi(2^{-s} \xi) \, d\xi \, dy \, dx$$

$$b_{n}(x) e^{2\pi i (x-y) \cdot \xi} - b_{n}(x) e^{2\pi i (y-x) \cdot \xi} = b_{n}(x) e^{2\pi i (x-y) \cdot \xi}$$

$$- b_{n}(x) e^{2\pi i (y-x) \cdot \xi} \, d\xi \, dy \, dx$$

$$\leq - \frac{1}{|Q|} \int_{Q} \frac{1}{|Q|} \int_{Q} \sum_{n=1}^{\infty} \int_{|x-y| \leq 2^{-d}} |u(y)| \int_{\mathbb{R}^n} \psi(2^{-s} \xi) \, d\xi \, dy \, dx$$

$$= - \frac{1}{|Q|} \int_{Q} \frac{1}{|Q|} \int_{Q} \sum_{n=1}^{\infty} \int_{|x-y| \leq 2^{-d}} |u(y)| \int_{\mathbb{R}^n} \psi(2^{-s} \xi) \, d\xi \, dy \, dx$$

$$b_{n}(x) e^{2\pi i (x-y) \cdot \xi} - e^{2\pi i (y-x) \cdot \xi} \, b_{n}(x) \, d\xi \, dy \, dx$$

$$\leq - \frac{1}{|Q|} \int_{Q} \frac{1}{|Q|} \int_{Q} \sum_{n=1}^{\infty} \int_{|x-y| \leq 2^{-d}} |u(y)| \int_{\mathbb{R}^n} \psi(2^{-s} \xi) \, d\xi \, dy \, dx$$

$$b_{n}(x) e^{2\pi i (x-y) \cdot \xi} - e^{2\pi i (y-x) \cdot \xi} \, b_{n}(x) \, d\xi \, dy \, dx$$
\[ + \frac{1}{Q} \int_Q \frac{1}{Q} \int_Q \sum_{m \leq 1} \int_{m \leq |x-y| < 2^m \cdot 1} |u(y)| \int_{2^m} \psi(2^{-1} u) \]

\[ [b^\#(x) - b^\#(z)] e^{2^{m+1} - 1} dy \, dz dx \]

\[ = A + B \]

First to deal with \( A \),

\[ e^{2^{m+1} - 1} - e^{2^m} \]

\[ = \sum_{i=1}^n (x_i - z_i) \int_0^1 \frac{\partial}{\partial x_i} e^{2^{m+1} - 1} \, dt \]

where \( x(t) = tx + (1-t)z \), \( x_i \) : the \( i \)-th coordinate of \( x \).

Note that (i) \( |x_i - z_i| \leq d \) since both \( x \) and \( z \) are in \( Q \), and (ii) if \( 2^m d \leq |y - y'| \leq 2^{m+1} d \), then \( 2^{m-1} d \leq |x(t) - y| \leq 2^{m+1} d \) since \( x(t) \in Q \).

\( A \) is dominated by

\[ C \sup_{x \in Q} \sum_{m \geq 1} \frac{1}{Q} \int_Q \frac{1}{Q} \int_Q \int_{m \leq |x-y| < 2^m \cdot 1} \frac{|u(y)|}{y-x} \]

\[ \sum_{i=1}^n |x_i - z_i| \int_0^1 |x(t) - y|^{2^m} \]

\[ |\int_{2^m} \psi(2^{-1} u) \frac{\partial}{\partial x_i} e^{\psi(2^{-1} u) - 1} \, d\xi | \, dt \, dy \, dz \, b^\#(x) \, dx \]

\[ \leq C \sum_{m \geq 1} 2^{m+3} (2^m d)^{-n-2} (2^{m+1} d) 2^{m+1} \int_{Q_m} \frac{1}{Q_m} \int_{Q_m} |u(y)| \, dy \]

(Lemma 2.1, \( t = n + 1/2 \), \( \| b^\# \| \leq C \)
and \( Q_m \) is the cube with centered \( x \), sides parallel to those of \( Q \)
and radius \( 2^{m+1}d \)

\[
\leq C \sum_{n=1}^{\infty} 2^{-n} d^{1/2} 2^n M u(x^0)
\]

\[
\leq C d^{1/2} 2^n M u(x^0)
\]

Next we estimate \( B \)

\[
B \leq \frac{1}{|Q|} \int_Q \frac{1}{|Q|} \int_Q \sum_{n=1}^{\infty} \int_{Q_m} 2^{n+1} d^{1/2} 2^n M u(y) \left| \frac{y - x}{z - y} \right|^{n+1} dy d\xi d\eta dx d\mu
\]

\[
|b_{n}(x) - b_{m}(z)| d\eta dx
\]

\[
\leq C \sum_{n=1}^{\infty} (2^{n+1} d)^{1/2} 2^n (2^n d)^{-\frac{1}{2}} (-n+1) 2^{-t} |Q_m| \int_{Q_m} u(y) dy
\]

(By Mean Value Theorem, \( \| b_{n}(k) - b_{m}(z) \|_\infty \leq C d \| \nabla b_{n} \|_\infty \)

and \( \| \nabla b_{n} \|_\infty \leq C 2^k \) and Lemma 2.1, \( t=n+1 \).

\( Q_m \) is above \( Q_n \)

\[
\leq C \sum_{n=1}^{\infty} 2^{-m} 2^{(s-1)} 2^{-t} |Q_n| \int_{Q_n} u(y) dy
\]

\[
\leq C 2^{(s-1)} M u(x^0)
\]

Putting the two cases together, we have shown that if \( Q \) is any cube containing \( x^0 \), then

\[
\frac{1}{|Q|} \int_Q \sum_{i=1}^{\infty} A_i u(x) - (\sum_{i=0}^{\infty} A_i u)_Q \ dx
\]
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\[
\leq C \sum_{j \in \mathbb{Z}} \left| \frac{1}{|Q|} \int_Q |A_{\omega}u(x) - (A_{\omega})_Q| \, dx \right|
\leq C \left( \sum_{j \in \mathbb{Z}} (2^j)^{-1} + \sum_{j \in \mathbb{Z}} (d^{2m}2^j + 2^{m-1}) \right) \mathcal{M}u(x^0)
\leq C \mathcal{M}u(x^0)
\]

Thus we have

\[
\left( \sum_{j \in \mathbb{Z}} A_{\omega}u \right)^*(x^0) \leq C \mathcal{M}u(x^0)
\]

Going back to our original notation, and summarizing we have shown that if $Q$ is any cube containing $x^0$ then

\[
\frac{1}{|Q|} \int_Q |(Au(x)) - (Au)_Q| \, dx
\leq (A(u))^{*}(x^0) + (B((1-\tau)u)^*)^{*}(x^0) + \sum_{j \in \mathbb{Z}} V_j(A_{\omega}u)^*(x^0)
\leq C_1 M_{\omega u}(x^0) + C_2 M_{\omega u}(x^0) + C_3 \sum_{j \in \mathbb{Z}} V_j \mathcal{M}u(x^0) \quad \text{for } r \geq 1
\leq C \mathcal{M}u(x^0) \quad \text{for } q \geq 2
\]

When we take the supremum of the left side over all cubes containing $x^0$, we obtain the inequality

\[
(\mathcal{M}u)^*(x^0) \leq C \mathcal{M}u(x^0), \quad \text{Q.E.D.}
\]

We now wish to show the sharp function estimate for general symbol $a(x, \xi) \in S^m_{\rho, \delta}$, $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$, $\rho \neq 0$, and $m = (n+1)(\rho - 1)$. Take a function $\lambda(\xi) \in C^\infty(R^n)$ satisfying $\text{supp } \lambda \subset \{ \frac{1}{3} \leq |\xi| \leq 1 \}$, $\sum_{j=-\infty}^{\infty} \lambda(2^{-j}\xi) = 1$ $(\xi \neq 0)$. And let $\varphi(\xi) = 1 - \sum_{j=-\infty}^{\infty} \lambda(2^{-j}\xi)$. Put then $a(x, \xi)$
= \phi(\xi) a(x, \xi) and \( a_j(x, \xi) = \lambda(2^j \xi) a(x, \xi) \), \( j = 1, 2, \ldots \). Then \( a_j(x, \xi) \) satisfy the condition (1.0) uniformly in \( j \) and \( a_0(x, \xi) \) has the \( \xi \)-compact support.

**Lemma 2.3.** Let \( a(x, \xi) \) and \( a_j(x, \xi) \) be above. Let \( \psi(\xi) \in C_0(\mathbb{R}^n) \) be such that \( \psi = \lambda \) and \( \text{supp} \psi \subset \{|\xi| < 1/3|, | \xi | \leq 1|\}. \) Then there exist functions \( a_n, j \in \mathbb{N}, k \in \mathbb{Z}^n \), such that

\[
\begin{align*}
(i) & \quad \| \psi a_j \| \leq C \\
(ii) & \quad \| \nabla a_j \| \leq C J \psi
\end{align*}
\]

\[
(iii) \quad a_j(x, \xi) = \sum_{k \in \mathbb{Z}^n} (1 + |k| j)^{-n+1} a_k(x) \psi(2^{-j} \xi)
\]

**Proof.** It is similar that of [2].

Lemma 2.3 says that above symbol \( a(x, \xi) \) can be represented as a sum of reduced symbols plus another symbol vanishing on \( |\xi| > 1 \). That is:

\[
a(x, \xi) = a_0(x, \xi) + \sum_{j=1}^\infty \sum_{k \in \mathbb{Z}^n} (1 + |k| j)^{-n+1} a_k(x) \psi(2^{-j} \xi)
\]

where \( a_0(x, \xi) = \sum_{j=1}^\infty a_k(x) \psi(2^{-j} \xi) \) is a reduced symbol and the double series converges absolutely.

**Theorem 2.4.** Let \( A \) be a pseudo differential operator with symbol \( a \in S_\rho^{\delta, \rho} \) \( 0 \leq \delta \leq \rho \leq 1, \delta \neq 1, \rho \neq 0, m = (n + 1)(\rho - 1) \). Then there is a constant \( C > 0 \) such that the pointwise estimate,

\[
(Aw)^\gamma(x^o) \leq CM\rho u(x^o) \text{ for all } x^o \in \mathbb{R}^n, \ u \in S(\mathbb{R}^n).
\]
and $1<r<\infty$ holds.

Furthermore, for $1<p<\infty$ and $\nu \in A_\nu$, above $A$ has a bounded extension to all of $L^p(\mathbb{R}^n, wd\nu)$. 

**Proof.** By Lemma 2.3, $a(x, \xi)$ can be represented as a sum of reduced symbols, plus another symbol vanishing on $|\xi| > 1$. Put $b_{\nu}(x) = e^{2\pi i \langle x \rangle} a_{\nu}(x)$ and put $V_{\nu} = (1 + |k|^2)^{-\frac{1}{2}}$, then $A$ satisfies the condition of [Theorem 2.2]. By [Theorem 2.2] and [Theorem 5.3], $A$ is bounded on $L^p(\mathbb{R}^n)$, $p>2$. We consider $A^*$, the adjoint of $A$. It is also a pseudo differential operator with symbol in $S^{0,0}_{\nu,\nu}$. [4], [6]. Thus $A^*$ is bounded on $L^p(\mathbb{R}^n)$, $2<p<\infty$, by [Theorem 2.2] and [Theorem 5.3]. Hence $A$ is bounded on $L^p(\mathbb{R}^n)$, $1<p<2$, and consequently on $L^p(\mathbb{R}^n)$ as well, by interpolation. This means that $A$ is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$.

By replacing $M_{\nu}(x)$ by $M_{\nu}(x)$, $1<r<\infty$, in (2.1), (2.3) and (2.4) of the proof of [Theorem 2.2], we get the desired result because $M_{\nu}(x^\nu) \leq M_{\nu}(x^\nu)$, $1<r<\infty$.

We next go to the proof of the second assertion. For $ueS(\mathbb{R}^n)$ and $weA_\nu$,

$$
\|Au\|_{L^p} \leq \| (Au)^* \|_{L^p} \leq C \| (Au)^* \|_{L^p} \leq C \| Mu \|_{L^p} \text{ if } 1<r<\infty
$$

Since $Au \in S \subset L^p(\mathbb{R}^n wd\nu) \cap L^p(\mathbb{R}^n)$, we can apply Lemma 2.7 in Miller [4] to prove the second inequality. So we can now extend $A$ to a bounded operator on $L^p(\mathbb{R}^n, wd\nu)$ because $S(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, wd\nu)$. 
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