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Conditions for Pettis integrability

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1. Introduction

In this paper, the long-standing question of whether $f : (\Omega, \Sigma, \mu) \to X$ is Pettis integrable or not is discussed provided f is bounded weakly measurable. One way in which Pettis integrability can fail, is through lack of proper boundedness. The first example of a bounded weakly measurable function which is not Pettis integrable was the example of R. S. Phillips (3) in 1940. Of late, Pallares-vera (2) analyzed the Pettis integrability of weakly continuous functions defined on a completely regular space and taking values in a Banach space. He proved that the set of Baire measures with respect to which such functions are universally Pettis integrable is precisely the space of Grothendieck measures introduced by Wheeler (6).

The purpose of this paper is to characterize the conditions of Pettis integrability of a bounded weakly measurable function defined on a finite measure space using the continuity of its Stonian transform. We use some results on $\hat{f}$ to derive conditions for Pettis integrability expressed in terms of f and Ω alone. It leads to conditions for Pettis integrability expressed in terms of X^* and X^{***}. We obtain the equivalence conditions of both f and $\hat{f}$ are Pettis integrable provided that (Ω, Σ, μ) is a finite measure space. Especially we obtain that f and $\hat{f}$ are both Pettis integrable if and only if there is a subset M of S, $\mu(M) = 0$ such that $x^{***}\hat{f}(s) = \widehat{x^{***}f}(s)$ for all $x^{***} \varepsilon X^{***}$ and all $s \varepsilon S\text{-}M$.

2. Preliminaries

Let S denote the Stone representation space of the complete Boolean algebra $\Sigma/\mu^{-1}(0)$, where (Ω, Σ, μ) is a finite measure space. Thus S is compact, hyperstonian, and has the countable chain condition on open sets. Let for $A \varepsilon \Sigma$, $[A]$ denote its equivalence class and a the corresponding clopen subsets of S, that is a is the clopen counterpart of $(A+\mu^{-1})(0)$ in S. We denote the identification of $L^\infty(\Omega, \Sigma, \mu)$ with C (S), a Banach algebra with a uniform norm, by $[h] \to \hat{h}$, a norm isometry for a scalar function h. By (5, Thm 4.6), μ on $\Sigma/\mu^{-1}(0)$ has a unique Borel representation $\bar{\mu}$ on S of the functional $\phi(\hat{h}) = \int_\Omega h\, d\mu$ on C(S). Thus if $[h]$ ε $L^\infty(\Omega, \Sigma, \mu)$, then $\int_a \hat{h}\, d\bar{\mu} = \int_A h\, d\mu$ for all $A\varepsilon\Sigma$ and for all scalar function h. We shall replace $\bar{\mu}$ by μ and write $\int_a \hat{h}\, d\mu$ for integrals over subset a of S and thus also write $\int_a \hat{h}\, d\mu = \int_A h\, d\mu$. As long as expressions involve integration or measure, A and a, and $[h]$ and $\hat{h}$ are indistinguishable. Similarly the natural isometry between $L^1(\Omega, \mu)$ and $L^1(S, \mu)$ are denoted by $h \to \hat{h}$ for a scalar function h.

Define now the Stonian transform $\hat{f} : S \to (X^{**}, \sigma(X^{**}, X^*))$, the second (topological) dual of a Banach space X, which is given the $\sigma(X^{**}, X^*)$ – topology, of a bounded weakly measurable function $f : (\Omega, \Sigma, \mu) \to X$ by $\langle \hat{f}(s), x^* \rangle = \widehat{x^* f}(s)$ for all $x^* \varepsilon X$. and for all $s\varepsilon S$. In this paper, for each finite subset A of X and each $\varepsilon>0$, $H(A,\varepsilon) = \{x^* f : \|x^*\| \leq 1,\ |x^*(x)| < \varepsilon, \forall x\varepsilon A, \text{ and } \int_E x^* f \geq n > 0\}$.

3. Conditions for Pettis integrability expressed in terms of f and Ω.

Let $\langle F,G \rangle$ be a duality. For any subset H of F, $H^0 = \{y\ \varepsilon\ G : \langle x,y \rangle \leq 1, \forall x\ \varepsilon\ H\}$ is a subset of G, called the polar set of F. Let $\cdot$ and $\circ$ denote polarity with respect to $\langle X,X^* \rangle$ and $\langle X^*,X^{**} \rangle$ respectively.

The following result, called the bipolar theorem, is a consequence of the Hahn-Banach theorem and is an indispensable tool in working with dualities.

Lemma 2.1. (Bipolar theorem (4)). Let $\langle F,G\rangle$ be a duality. For any subset $M \subset F$, the bipolar of M is the $\sigma(F,G)$ – closed, convex hull of $M \cup \{0\}$.

Lemma 2.2. Let $h \in L^\infty(\Omega, \Sigma, \mu)$. For each s in S, $\hat{h}(s) = \lim_{b \searrow s} \operatorname{ess\,sup}_{w \in B} h(w) = \lim_{b \searrow s} (1/\mu(B)) \int_B h \, d\mu$, where the limit is taken through the neighborhood filter of clopen sets b in S corresponding to $B \in \Sigma$, containing s.

Lemma 2.3. Let f be a bounded and weakly measurable function from a finite measure space (Ω, Σ, μ) to a Banach space X. If a is a non-empty clopen subset of S, then
$\{\hat{f}(a)\}^0 = \{(1/\mu(B)) \cdot x_B^{**} : B \subset A,\ \mu(B)>0\}^0$ in X^*.

Proof. Let $x^* \in \{\hat{f}(a)\}^0$, then $\forall\ \hat{f}(s) \in \hat{f}(a)$, $< \hat{f}(s), x^* > \leq 1$. Let $B \subset A$ and $\mu B>0$.

$$
\begin{aligned}
\text{Hence } x^*((1/\mu(B))x_B^{**}) &= (1/\mu(B))\ x^*(x_B^{**}) \\
&= (1/\mu(B))\langle x_B^{**}, x^*\rangle \\
&= (1/\mu(B)) \int_B x^*f \, d\mu \\
&= (1/\mu(b)) \int_b x^*\hat{f} d\mu \leq 1.
\end{aligned}
$$

Thus $x^* \in \{(1/\mu(B))x_B^{**} : B \subset A,\ \mu B>0\}$.
conversely, if $x^*((1/\mu(B))x_B^{**}) \leq 1$ for all $B \subset A$, $\mu(B)>0$, then for a fixed $s \in a$,

$$
\begin{aligned}
x^*\hat{f}(s) &= \lim_{b \downarrow s, b \subset a} (1/\mu(b)) \int_b x^*\hat{f} \, d\mu \\
&= \lim_{b \downarrow s, b \subset a} (1/\mu(B)) \int_B x^*f \leq 1.
\end{aligned}
$$

Hence $x^* \varepsilon \{\hat{f}(a)\}^0$.

Lemma 2.4. Let $\tilde{X} = \{x^{**} : x^{**}$ is in the weak* closure of a countable subset of X$\}$. Then f is Pettis integrable if and only if $(\overline{co}\hat{f}(a)) \cap \tilde{X} \neq \phi$ for every non-empty clopen subset a of S.

Proof. Suppose f is not Pettis integrable over Ω.

Then the Dunford integral x_Ω^{**} is not $\sigma(X^{**},X^*)$-contiuous on $\{x^* : \| x^* \| \leq 1\}$. Hence $\eta>0$ such that each $H(A,\varepsilon)$ is non-empty. Choosing y^* such that $\int y^* f \geq \eta \geq 0$. Since $y^*\hat{f}$ is continuous, there exists a nonempty clopen set a in S with $y^*\hat{f}(s) > \eta/2$ for all $s \varepsilon a$.

Choose $\tilde{X} \varepsilon (\overline{co}\,\hat{f}(a)) \cap \tilde{X}$, so that $y^*(\tilde{x}) \geq \eta/2$. Let $C = \{x_n\}$ be a sequence in X, containing $\tilde{x}$ containing $\tilde{x}$ in its weak*-closure. Choose z^* such that $z^*(\tilde{x}) = 0$(since $z^* \mid c = 0$), and $z^*(\tilde{x}) \geq \eta/2$ (since $z^*\hat{f} = y^*\hat{f}$), a contradiction. A similar proof holds if f is not Pettis integrable over some measurable subset E of Ω.

Conversely, by the Hahn-Banach theorem

$(1/\mu(A))x_A^{**} \varepsilon\ \overline{co}\ \hat{f}(a)$ Hence Pettis integrability yields a non-empty intersection.

Theorem 2.5. If $f : (\Omega, \Sigma, \mu) \to X$ is bounded and weakly measurable, (Ω, Σ, μ) is a finite measure space, and X is a Banach space, then the following are equivalent.

(a) f is Pettis integrable.

(b) $\{\hat{f}(a)\}^0 = \{x^* \varepsilon X^* \mid \forall\ y \varepsilon \hat{f}(a), \langle y,x^* \rangle \leq 1\}$ is $\sigma(X^*,X)$-closed in X^*.

(c) $\hat{f}(a)^\perp = \{x^* \varepsilon X^* \mid \forall\ y \varepsilon \hat{f}(a), \langle y,x^* \rangle = 0\}$ is $\sigma(X^*,X)$-closed in X^*

(d) $\{x^* \varepsilon X^* : \forall\ y \varepsilon \hat{f}(a), \langle y,x^* \rangle \geq 0\}$ is $\sigma(X^*,X)$-closed in X^*.

(e) $\hat{f}(S)^\perp : \sigma(X^*,X)$-closed in X^*.

(f) $\{x^* \varepsilon X^* : x^*f \leq 1$ a.e. on $A \varepsilon \Sigma\}$ is $\sigma(X^*,X)$-closed in X^*.

(g) $\{x^* \varepsilon X^* : x^*f = 0$ a.e. on $A \varepsilon \Sigma\}$ is $\sigma(X^*,X)$-closed in X^*.

(h) $\{x^* \varepsilon X^* : x^*f \geq$ a.e. on $A \varepsilon \Sigma\}$ is $\sigma(X^*,X)$-closed in X^*.

(i) $\{x^* \varepsilon X^* : x^*f = 0$ a.e. on $\Omega\}$ is $\sigma(X^*,X)$-closed in X^*.

for all clopen sets a in S and for all corresponding A in Σ.

Proof. (a) → (b) Suppose (x^*_α) is a sequence in $\{f(a)\}^0$ converging x^* over X. If $x^*\hat{f}>1$, then by continuity there exists clopen set a such that $x^*\hat{f}(a)-1 \geq \alpha>0$(or $\leq \alpha < 0$). Hence $(x^*\overline{co}\hat{f}(a)) \geq 1+\alpha$. If f is Pettis integrable then $\overline{co}\,\hat{f}(a)\cap X$ is non-empty. Hence there exists x in $\overline{co}\,\hat{f}(a)\cap X$, thus $\langle x^*,x\rangle>1$. But $\langle x^*_\alpha, x\rangle \leq 1$ for all α. Hence $\langle x^*,x\rangle \leq 1$. This is a contradiction. Hence$\{f(a)\}^0$ is $\sigma(X^*,X)$-closed in X^*.

(b) → (a) Assume that $C = \hat{f}(a)^0$ is $\sigma(X^*,X)$-closed in X^*. Let $D = C$; then $D = C^0\cap X = \hat{f}(a)^{00}\cap X = \overline{co}(\hat{f}(a)\cup\{0\})\cap X$, by the bipolar theorem ; here $\overline{co}$ denotes the $\sigma(X^{**},X^*)$-closed convex hull. Since $\hat{f}(a)$ is $\sigma(X^{**},X^*)$-compact, $\overline{co}\,(\hat{f}(a)\cup\{0\}) = \{\lambda x^{**} : 0 \leq \lambda \leq 1, x^{**}\varepsilon\,\overline{co}\,\hat{f}(a)\}$.

(i) If $D = \{0\}$, then $D^\cdot = \{0\}^\cdot = X^*$ and $D^\cdot = C$, since C is $\sigma(X^*,X)$-closed, and hence $C = \{0\}^\cdot$ and so $\hat{f}(a) \subset \hat{f}(a)^{00} = C^0 = \{0\}$. Thus $\overline{co}\,\hat{f}(a)\cap X\neq\phi$.

(ii) If D properly contains 0, then $\{\lambda x^{**} : 0<\lambda \leq 1, x^{**} \varepsilon\,\overline{co}\,\hat{f}(a)\}\cap X\neq\phi$, and so $\overline{co}\,\hat{f}(a)\cap X$ is not empty.

Thus f is Pettis integrable.

(c) → (a) Assume f is not Pettis integrable.

Choose $y^* \varepsilon X^*$ sucht that $\| y^* \| \leq 1$ and $y^*\hat{f} \varepsilon H(A,\varepsilon)$ for each finite subset A of X and each $\varepsilon>0$, then

$T = \{y^* + \hat{f}(S)^\perp\}\cap\{x^* : \| x^* \| \leq 1\}$ is $\sigma(X^*,X)$-closed in X^*. Choose $z^* \varepsilon X^*$ for each countable subset C of X such that $z^* \mid C = 0$ and $z^*f = y^*f$, then z^*-y^* belongs to $\hat{f}(S)^\perp$, so z^* belongs to T. Since T is $\sigma(X^*,X)$-closed, This implies that $O \varepsilon T$.

This is a contradiction, since y^*f is not identically zero.

Theorem 2.6. Let (Ω, Σ, μ) be a finite measure space, X a Banach space, and f a bounded weakly measurable function on (Ω, Σ, μ) to X, then the following are equivalent.

(a) f is weakly equivalent to a strongly measurable function

(b) there is a subset M of S, $\mu(M) = 0$, such that $x^{***}\hat{f}(s) = \widehat{x^{***}f}(s)$ for all $x^{***} \varepsilon X^{***}$ and all s ε S-M

(c) $f : \Omega \to X$ and $\hat{f} : S \to X^{**}$ are both Pettis integrable. In this case, (p) $\int_A f\, d\mu$ equals to (p) $\int_a \hat{f}\, d\mu$ for each A ε Σ, and $a = A + \mu^{-1}(0) \varepsilon S$.

Proof. (a) → (b) By hypothesis, there is a subset M of S, $\mu(M) = 0$. such that $\hat{f}(S\text{-}M) \subset X$. Fix x^{***} and s ε S-M. Let $x^* = x^{***} \mid X$ then $x^{***}\hat{f}(s) = x^*\hat{f}(s) = \widehat{x^*f(s)} = \widehat{x^{***}f(s)}$

(b) → (a) Let $X^{\perp} = \{x^{***} : x^{***} \mid X = 0\}$, then $X^{***} = X^* \oplus X^{\perp}$. If $x^{***} \varepsilon X^{\perp}$, then $x^{***}f(w) = 0$ for all w ε Ω so $\widehat{x^{***}f}(s) = 0$ for all s in S. Then $x^{***}(\hat{f}(s)) = 0$ for all a in S-M, and so $\hat{f}(S\text{-}M) \subset X^{\perp\perp} = X$. Hence f is strongly measurable, thus f is weakly equivalent to a strongly measurable function.

(b) → (c) Let (x^*_α) be a bounded set in X^*. If $x^*_\alpha f \le 1$ a.e. on Ω and (x^*_α) is $\sigma(X^*,X)$-convergent to x^*, then each $x^*_\alpha f \le 1$ on S. Let x^{***} be a $\sigma(X^{***},X^{**})$-cluster point of (x^*_α), then $x^{***}\hat{f} \le 1$ on S, and so $\widehat{x^{***}f} \le 1$ a.e. on S, hence everywhere on S by continuity.

Now $x^{***} \mid X = x^*$, so we get $x^* f \le 1$ a.e. on Ω. Thus f is Pettis integrable, by theorem 2.5. Since $\int_a x^{***}\hat{f}\, d\mu = \int_a \widehat{x^{***}f}\, d\mu = \int_A x^{***}f\, d\mu = \int_A x^* f d\mu\ \ \varepsilon\ X \subset X^{**}$. Hence f is Pettis integrable.

(c) → (b) If both f and $\hat{f}$ are Pettis integrable (implying that $\hat{f}$ is X^{***}-measurable into X^{**}), (p)$\int_A f\, d\mu$ = (p)$\int_a \hat{f}\, d\mu$. If not, choosing an x^* which separates (p) $\int_A f\, d\mu$ and (p) $\int_a \hat{f}\, d\mu$ leads to a contradiction. Thus, fixing x^{***} and letting a be clopen set in S,

$\int_a x^{***}\hat{f}\, d\mu = x_a^{****}(x^{***}) = x^{***}(x_a^{****}) = x^{***}((p)\int_a \hat{f} d\mu) =$

$x^{***}((p)\int_A f\,d\mu) = \int_a x^{***}f\,d\mu = \int_A \widehat{x^{***}f}\,d\mu$. Hence $x^{***}\hat{f} = \widehat{x^{***}f}$ a.e. on S.

Lemma 2.7. Let (Ω, Σ, μ) be a finite measure space, X a Banach space, and f a bounded weakly measurable function from (Ω, Σ, μ) to X. Consider

$T : X^* \to L^1(\Omega, \mu)$ defined by $T(x^*) = x^*f$ and

$\hat{T} : X^* \to L^1(S,\mu)$ defined by $\hat{T}(x^*) = x^*\hat{f}$, then

(1) $\widehat{T(x^*)} = \hat{T}(x^*)$

(2) $\hat{T}^*(\chi_a) = T^*(\chi_A)$, χ_a and χ_A are the characteristic functions.

Proof. (1) $\widehat{T(x^*)} = \widehat{x^*f} = x^*\hat{f} = \hat{T}(x^*)$

(2) $\langle \hat{T}^*(\chi_a), x^*\rangle = \langle \chi_a, \hat{T}(x^*)\rangle = \langle \chi_a, x^*\hat{f}\rangle = \int_a x^*\hat{f} = \int_A x^*f = \langle \chi_A, x^*f\rangle = \langle \chi_A, Tx^*\rangle = \langle T^*(\chi_A), x^*\rangle$ for all $x^* \varepsilon X^*$. Hence $\hat{T}^*(\chi_a) = T^*(\chi_A)$.

Theorem 2.8. Let f be a bounded and weakly measurable function from a finite measure space (Ω, Σ, μ) to a Banach space X. Then the following are equivalent.

(a) f is Pettis integrable

(b) $T^{**}(X^{\perp}) = 0$

(c) $\hat{T}^{**}(x^{***}) = \widehat{x^{***}f}$ a.e. on S.

Proof. (a) → (b) Assume that f is Pettis integrable, then f is weak*-to weak-continuous. Hence T is a weakly compact operator (1), so the range of T^{**} is in $L^1(\Omega, \mu)$. Then if $A \varepsilon \Sigma$ and $x^{***} \varepsilon X^{\perp}$, $\langle T^{**}(x^{***}), \chi_A\rangle = \langle x^{***}, T^*(\chi_A)\rangle = \langle x^{***}, (p)\int_A f\,d\mu\rangle = 0$.

Hence $T^{**}(X^{\perp}) = 0$.

(b) → (c) (i) In case $x^{***} \varepsilon X^{\perp}$, then $\langle \hat{T}^{**}(x^{***}), \chi_a\rangle = \langle x^{***}, \hat{T}^*(\chi_a)\rangle = \langle x^{***}, T^*(\chi_A)\rangle = \langle T^{**}(x^{***}), \chi_A\rangle = \langle 0, \chi_A\rangle = 0$.

Hence $\hat{T}^{**}(x^{***}) = 0$. On the other hand, if $x^{***} \varepsilon X^{\perp}$, then $x^{***} \mid X = 0$. Hence $x^{***}f = 0$ for all x^{***} such that $x^{***} \mid X = 0$. Thus $\hat{T}^{**}(x^{***}) = \widehat{x^{***}f}$ a.e. on S.

(ii) If x^{***} is an arbitrary member of $X^{***} = X^{*} \oplus X^{\perp}$, and $x^{***} = x^{*} + x^{\perp}$ be the canonical decomposition, then $\hat{T}^{**}(x^{***}) = \hat{T}^{**}(x^{*} + x^{\perp}) = \hat{T}^{**}(x^{*}) + \hat{T}^{**}(x^{\perp}) = \hat{T}^{**}(x^{*}) + 0 = \hat{T}^{**}(x^{*})$. Hence $\langle \hat{T}^{**}(x^{***}), \chi_a \rangle = \langle \hat{T}^{**}(x^{*}), \chi_A \rangle = \langle x^{*}, \hat{T}^{*}(\chi_a) \rangle = \langle \hat{T}(x^{*}), \chi_a \rangle = \langle x^{*}\hat{f}, \chi_a \rangle = \langle \widehat{x^{*}f}, \chi_a \rangle = \langle \widehat{x^{***}f}, \chi_a \rangle$ for all clopen set a in S. Hence $\hat{T}^{**}(x^{***}) = \widehat{x^{***}f}$ a.e. on S.

(c) → (a) Suppose that f is not Pettis integrable. that, $\chi_A^{**} \notin X$. Choose $x^{***} \varepsilon X^{\perp}$, $x^{***}(\chi_A^{**}) = 1$ Then $\hat{T}^{**}(x^{***}) = \widehat{x^{***}f} = 0$ a.e. on S. Hence

$1 = \langle x^{***}, \chi_A^{**} \rangle = \langle x^{***}, \hat{T}^{*}(\chi_a) \rangle = \langle \hat{T}^{**}(x^{***}), \chi_a \rangle = 0.$

This is a contradiction.

Theorem 2.9. Let (Ω, Σ, μ) be a finite measure space, X a Banach space, and f a bounded weakly measurable function on (Ω, Σ, μ) to X.

Then each of the following implies next.

(a) f is weakly equivalent to a strongly measurable function.

(b) $x^{***}\hat{f}(s) = \widehat{x^{***}f}(s)$ a.e. on S for each x^{***}.

(c) f is Pettis integrable.

(d) $\hat{f}(S)^{\perp}$ is $\sigma(X^{*},X)$-closed in X^{*}

Theorem 2.10. Let $f : (\Omega, \Sigma, \mu) \to X$ be a bounded weakly measurable function on a $\{0, 1\}$-measure space.

Then the following are equivalent.

(a) f is weakly equivalent to a strongly measurable function.

(b) $x^{***}\hat{f}(s) = \widehat{x^{***}f}(s)$ a.e. on S for each x^{***}

(c) f is Pettis integrable

(d) $\hat{f}(S)^{\perp}$ is $\sigma(X^{*}, X)$-closed in X^{*}.

Proof. (a) → (b) → (c) → (d) hold for any finite measure space, by the preceding results.

(d) → (a) Suppose that (Ω, Σ, μ) is a $\{0,1\}$-valued measure space. Then the measure algebra $\Sigma/\mu^{-1}(0)$ is $\{[\phi],[\Omega]\}$, and the Stone space S consists of a single point $s_0 = \{[\Omega]\}$. A bounded function $f : (\Omega, \Sigma, \mu)$ to X is weakly measurable if and only if x^*f has a constant value $c(x^*)$ a.e. for each $x^* \in X^*$. The range of the Dunford integral contains (at most) 2 points, x_Ω^{**} and 0. Also, $\hat{f}(s_0)(x^*) = \widehat{x^*f}(s_0) = \lim \operatorname{ess\,sup}_{w \in \Omega} x^*f(w) = c(x^*) = \int_\Omega x^*f \, d\mu = x_\Omega^{**}(x^*)$. Thus $\hat{f}(S)$ is the singleton $\{x_\Omega^{**}\}$.

Since ${x_\Omega^{**}}^{-1}(0) = \{x_\Omega^{**}\}^\perp$, $\hat{f}(S)^{-1}(0) = \{x_\Omega^{**}\}^\perp$. Hence ${x_\Omega^{**}}^{-1}(0)$ is $\sigma(X^*,X)$-closed, so x_Ω^{**} is $\sigma(X^*,X)$-contiuous on X^*. Thus f is weakly equivalent to a constant function.

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