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COMPLEX HOLOMORPHIC LINE BUNDLES AND PSEUDOCONVEXITY

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1. Introduction

In this paper we investigate the relation between a holomorphic line bundle over a complex n-torus T^n and a Chern class. And, we introduce the theorem that allows us to refer to the Picard variety of T^n as a group of weakly pseudoconvex manifolds, so that the Picard variety of T^n is considered as an important tool for the research on a weakly pseudoconvex manifold.

2. The holomorphic line bundle on a complex n-torus

Let $C(O)$ be the sheaf of germs of continuous (holomorphic) functions on T^n and $C^*(O^*)$ the sheaf of germs of nonvanishing continuous (holomorphic) functions on T^n. Since $O \subset C$ and $O^* \subset C^*$. We have the following commutative, exact diagram :

$$\begin{array}{ccccccccc} 0 & \to & Z & \to & O & \xrightarrow{\exp} & O^* & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & Z & \to & C & \xrightarrow{\exp} & C^* & \to & 0. \end{array}$$

This yields the commutative, exact diagram :

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$$\begin{array}{l}
0 \to H^0(T^n,Z) \to H^0(T^n,\mathcal{O}) \to H^0(T^n,\mathcal{O}^*) \xrightarrow{\delta^*} \\
\quad \to H^1(T^n,Z) \to H^1(T^n,\mathcal{O}) \to H^1(T^n,\mathcal{O}^*) \xrightarrow{\delta^*} H^2(T^n,Z) \to \cdots \\
\qquad\quad \| \qquad\qquad\quad \downarrow \qquad\qquad\quad \downarrow \qquad\qquad\qquad \| \\
\quad \to H^1(T^n,Z) \to H^1(T^n,\mathcal{C}) \to H^1(T^n,\mathcal{C}^*) \xrightarrow{\delta^*} H^2(T^n,Z) \to \cdots.
\end{array}$$

Here the verticle maps are induced by the natural inclusions of $\mathcal{O}$ in $\mathcal{C}$, $\mathcal{O}^*$ in $\mathcal{C}^*$ and the identity map of Z [3].

Definition 2.1[5]. For $E \varepsilon H^1(T^n,\mathcal{C}^*)$ we call $C_1(E) = \delta^*(E)$ the first Chern class of E.

Lemma 2.2. Let $\Phi : H^1(T^n,\mathcal{O}^*) \to H^1(T^n,\mathcal{C}^*)$ be the canonical map induced by the natural inclusion of $\mathcal{O}^*$ in $\mathcal{C}^*$. Then $E \varepsilon H^1(T^n,\mathcal{O}^*)$ is a trivial complex holomorphic line bundle over T^n if and only if $\Phi(E) = 1$.

Lemma 2.3. $\Phi(E) = 1$ if and only if $C_1(E) = 0$.

By Lemma 2.2 and Lemma 2.3, we have the following theorem.

Theorem 2.4. $E \varepsilon H^1(T^n,\mathcal{O}^*)$ is a trivial complex holomorphic line bundle over T^n if and only if $C_1(E) = 0$.

3. Weakly pseudoconvex manifolds

Proposition 3.1[2]. Let Ω be a domain of C^n and let $f \varepsilon C^0(\Omega,R)$. The function f is pluriharmonic if and only if f is locally the real part of a holomorphic function.

Let P be the sheaf of germs of C^∞ pluriharmonic functions on T^n. Consider D as the sheaf of germs of constant functions with values in $\{z \varepsilon C : |z| = 1\}$. We define $L : \mathcal{O}^* \to P$ by $L(f)(x) = \log |f(x)|$, $x \varepsilon U$,

$f \varepsilon H(U)$. For $g \varepsilon P$, by Proposition 3.1, there is $f \varepsilon H(U)$ such that g is the real part of f on a simply connected subset U. $\exp(f) \varepsilon O^*$ and $L(\exp(f(x))) = \log | \exp(f(x)) | = \log(\exp(g(x))) = g(x)$ on U so that L is surjective. Since $\text{Ker } L = \{f \varepsilon O^* : L(f) = 0\} = \{z \varepsilon C : | z | = 1\} = D$, we get a (short) exact sequence of sheaves on $T^n : 0 \to D \to O^* \to P \to 0$. We denote by $L^* : H^1(T^n, O^*) \to H^1(T^n, P)$ the homomorphism induced by $L : O^* \to P$. Since T^n is compact, $H^0(T^n, O^*) \cong C^*$ and $H^0(T^n, P) \cong R$ so that $H^0(T^n, O^*) \to H^0(T^n, P)$ is surjective. Hence we have a (long) exact sequence :

$$\cdots \to H^0(T^n, O^*) \to H^0(T^n, P) \to H^1(T^n, D)$$
$$\to H^1(T^n, O^*) \xrightarrow{L^*} H^1(T^n, P) \to \cdots$$

Thus $H^1(T^n, D) \cong \text{Ker } L^* \subset H^1(T^n, O^*)$.

Lemma 3.2[4]. $\text{Ker } L^* = \{E \varepsilon H^1(T^n, O^*) : C_1(E) = 0\}$.

Theorem 3.3. Every trivial complex holomorphic line bundle on T^n is a weakly pseudoconvex manifold.

Proof. Let E be a trivial holomorphic line bundle on T^n. By lemma 3.2, $E \varepsilon \text{Ker } L^* = H^1(T^n, D) \subset H^1(T^n, O^*)$. Hence there is $(\theta_{jk}) \varepsilon Z^1(U, D)$ such that $\{\theta_{jk}\}$ are transition functions of the line bundle $\pi : E \to T^n$. Then there are biholomorphic functions $\theta_j : \pi^{-1}(U_j) \to U_j \times C$ defined by $\theta_j \circ \theta^{-1}_k (x, z_k) = (x, z_j)$ if and only if $\theta_{jk}(z_k) = z_j$ for $x \varepsilon U_j \cap U_k$. For $a \varepsilon E$, $\theta_j(a) = (\pi(a), z_j(a))$ if $a \varepsilon \pi^{-1}(U_j)$. We define $\Phi : E \to R$ by $\Phi(a) = | z_j(a) |^2$ if $a \varepsilon \pi^{-1}(U_j)$. Then $E_c = \{a \varepsilon E : \Phi(a) < c\} \subset\subset E$ for each $c \varepsilon R$ and the Levi form $L(\Phi) = dz_j d\bar{z}_j$ is everywhere positive semi-definite. Hence E is a weakly pseudoconvex manifold.

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