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Optimality and Duality for Vector Optimization Problems

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1. Introduction

M.A. Hanson [3] defined the invex function which is a generalization of the convex function, proved that the Kuhn-Tucker conditions are sufficient for a solution of the scalar optimization problem concerning invex functions and establisbed Wolfe [10] duality theorems for this problem. R.N. Kaul and S. Kaur [5] established optimality criteria in the scalar optimization problem involving pseudo-invex functions and guasi-invex functions. Very recently, R.R. Egudo [2] showed that Wolfe type duality theorems hold for the vector optimization problem which consists of ρ-convex functions which defined by J.P. Vial [8]. D.S. Kim and G.M. Lee [6] established that Wolfe type duality theorems hold for the vector optimization problem which consists of invex functions.

In this paper, we prove that the Kuhn-Tucker optimality conditions are sufficient for an efficient solution of the vector optimization problem which consists of pseudoinvex functions and establish Wolfe type duality theorems for this problem.

2. Definitions

Throughout this paper, we use the following conventions ; Let R^p be a p-dimensionl Euclidean space, $x=(x_1,\cdots,x_p)^t \varepsilon R^p$, and $y=(y_1,\cdots,y_p)^t \varepsilon R^p$.

1. $x \leqq y$ if and only if $x_i \leqq y_i,\ i=1,\cdots,p$.
2. $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.
3. $x \nleq y$ is the negation of $x \leq y$.

Definition 2.1. A differentiable function $h : R^n \to R$ is pseudo-invex with respect to η if and only if there exists a vector valued function η defined on $R^n \times R^n$ such that for all $x, u \varepsilon R^n$

$$\nabla h(u)\eta(x, u) \geqq 0 \text{ implies } h(x) \geqq h(u).$$

Definition 2.2. A differentiable function $h : R^n \to R$ is quasi-invex with respect to η if and only if there exists a vector valued function η defined on $R^n \times R^n$ such that for all $x, u \varepsilon R^n$,

$$h(x) \leqq h(u) \text{ implies } \nabla h(u)\eta(x,u) \leqq 0$$

We consider the vector optimization problem ;

(P) Miminize $f(x)$
subject to $x \varepsilon X = \{x \varepsilon R^n : g(x) \leqq 0\}$,

where $f : R^n \to R^p$ and $g : R^n \to R^m$ are differentiable functions.

In relation to (P), we consider the following vector optimization problem.

The Wolfe vector dual of (P) [10] ;

(D) Maximize $f(x)+y^t g(x)e$
subject to $(x, \lambda, y) \varepsilon Y = \{(x, \lambda, y) : \nabla\lambda^t f(x)+\nabla y^t g(x) = 0,\ y \geqq 0,\ \lambda \varepsilon \Lambda^+\}$,
where $e=(1,\cdots,1)^t \varepsilon R^p$ and $\Lambda^+ = \{r \varepsilon R^p : r>0,\ r^t e=1\}$.

Optimization in (P) and (D) means obtaining efficient solutions for the corresponding problems. T. Weir [9] first considered the above dual problems of (P).

Definition 1.3. 1. A point $\bar{x} \varepsilon X$ is an efficient solution for (P) if and only if for any $x \varepsilon X$, $f(x) \not\leq f(\bar{x})$.

2. A point $(\bar{x}, \bar{\lambda}, \bar{y}) \varepsilon Y$ is an efficient solution for (D) if and only if for any$(x, \lambda, y) \varepsilon Y$,

$$f(\bar{x}) + \bar{y}^t g(\bar{x}) e \not\leq f(x) + y^t g(x) e.$$

3. Optimality Conditions

Now, we consider optimality conditions for an efficient solution for (P).

Theorem 3.1. Suppose that there exist $\lambda > 0$, $\lambda \varepsilon R^r_+$, $y_I \varepsilon R^l$ such that

(1) $\lambda^t f + y^t_I g_I$ is pseudo-invex w.r.t. η,

(2) for all $x \varepsilon X$, $[\nabla \lambda f(\bar{x}) + \nabla y^t_I g_I(\bar{x})]\eta(x, \bar{x}) \geqq 0$,

(3) $g(\bar{x}) \leqq 0$,

where $I = \{i : g_i(\bar{x}) = 0\}$.

Then, $\bar{x}$ is an efficient solution for (P).

Proof. Suppose that $\bar{x}$ is not an efficient solution for (P). Then there exists $x^* \varepsilon R^n$ such that

$$f(x^*) - f(\bar{x}) \leq 0 \text{ and } g(x^*) \leqq 0.$$

Hence we have

$$f(x^*) - f(\bar{x}) \leq 0 \text{ and } g_I(x^*) - g_I(\bar{x}) \leqq 0.$$

Thus, we have

$$\lambda^t f(x^*) + y^t_I g_I(x^*) < \lambda^t f(\bar{x}) + y^t_I g_I(\bar{x}).$$

By the definition of pseudo-invexity, we have

$$[\nabla \lambda^t f(\bar{x}) + \nabla y^t_I g_I(\bar{x})] \eta (x^*, \bar{x}) < 0,$$

which contradicts the assumption (2).

Lemma 3.1 [1]. $\bar{x}$ is an efficient solution for (P) if and only if $\bar{x}$ is a solution for the following scalar optimization problems ;

(P_i) Minimize $f_i(x)$
subject to $f_j(x) \leqq f_j(\bar{x})$ for all $j \neq i$, $g(x) \leqq 0$,

for each $i=1,\cdots,p$.

From Lemma 3.1, we can prove the following theorem by the method similar to the proof in Theorem 3.4 of [4].

Theorem 3.2. If $\bar{x}$ is an efficient solution for (P) and if we assume that $\bar{x}$ satisfies a constraint qualification ([7]) for (P_i), $i=1,\cdots,p$, then there exist $\bar{\lambda} \varepsilon \Lambda^+$ and $\bar{y} \geqq 0$, $\bar{y} \varepsilon R^m$ sucht that $\nabla \bar{\lambda}^t f(\bar{x}) + \nabla \bar{y}^t g(\bar{x}) = 0$ and $\bar{y}^t g(\bar{x}) = 0$.

4. Duality Theorems

Now we establish duality theorems for (P) and (D).

Theorem 4.1. If, for all $x \varepsilon X$ and $(u, \lambda, y) \varepsilon Y$, (1) $f_i + y^t g$ is pseudo-invex w.r.t. η for all i : or (2) $\lambda^t f + y^t g$ is pseudo-invex w.r.t. η, then

$f(x) \nleq f(u) + y^t g(u) e$.

Proof. (1) Let $x \varepsilon X$ and $(u. \lambda, y) \varepsilon Y$. Suppose that $f(x) \leq f(u) + y^t g(u) e$.

Then for some i, $f_i(x) < f_i(u) + y^t g(u)$ and for all j, $j \neq i$, $f_j(x) \leqq f_j(u) + y^t g(u)$.

Since $f_i + y^t g$ pseudo-invex w.r.t. η and $f_i(x) + y^t g(x) < f_i(u) + y^t g(u)$, we have

$[\nabla f_i(u) + \nabla y^t g(u)] \eta (x, u) < 0$.

Since $f_i + y^t g$ is quasi-invex w.r.t. η and $f_i(x) + y^t g(x) \geqq f_i(u) + y^t g(u)$, we have

$$[\nabla f_i(u) + \nabla y^t g(u)]\ \eta\ (x, u) \leq 0.$$

Thus,

$$[\nabla \lambda^t f(u) + \nabla y^t g(u)]\ \eta\ (x, u) < 0.$$

This is a contradiction.

(2) Let $x \varepsilon X$ and $(x, \lambda, y) \varepsilon Y$. Suppose that $f(x) \leq f(u) + y^t g(u) e$. Then

$$f(x) + y^t g(x) e \leq f(u) + y^t g(u) e.$$

Moreover,

$$\lambda^t f(x) + y^t g(x) < \lambda^t f(u) + y^t g(u).$$

By the pseudo-invexity of $\lambda^t f + y^t g$, we have

$$[\nabla \lambda^t f(u) + \nabla y^t g(u)]\ \eta\ (x, u) < 0.$$

This is a contradiction.

Theorem 4.2 Let $\bar{x}$ is an efficient solution for (P) and assume that $\bar{x}$ satisfies a constraint ([7]) for (P_i), $i = 1, \cdots, p$. Then there exist $\bar{\lambda}$ and $\bar{y}$ sucht that $(\bar{x}, \bar{\lambda}, \bar{y}) \varepsilon Y$ and the objective values of (P) and (D) are equal. If, also, (1) $f_i + \bar{y}^t g$ is pseudo-invex w.r.t. η for all i or (2) $\bar{\lambda}^t f + \bar{y}^t g$ is pseudo-invex w.r.t. η, then $(\bar{x}, \bar{\lambda}, \bar{y})$ is an efficient solution for (D).

Proof. By Theorem 3.2, there exist $\bar{\lambda} \varepsilon \Lambda^+$ and $\bar{y} \geq 0$, $\bar{y} \varepsilon R^m$ such that $\nabla \bar{\lambda} f(\bar{x}) + \nabla \bar{y}^t g(\bar{x}) = o$ and $\bar{y}^t g(\bar{x}) = 0$. Thus $(\bar{x}, \bar{\lambda}, \bar{y}) \varepsilon Y$. Since $\bar{y}^t g(\bar{x}) = 0$, the objective values of (P) and (D) are equal. Suppose that $(\bar{x}, \bar{\lambda}, \bar{y})$ is not an efficient solution for (D). Then there exists $(u^*, \lambda^*, y^*) \varepsilon Y$ such that

$f(\bar{x})+\bar{y}^t g(\bar{x})e \leq f(u^*)+y^{*t}g(u^*)e.$

By the similar method of Theorem 4.1, we can prove that

$[\nabla\lambda^{*t}f(u^*)+\nabla y^{*t}g(u^*)]\ \eta\ (\bar{x}\ , u^*) < 0.$

This is a contradiction.

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