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ON ENDOMORPHISM RING OF H-INVARIANT MODULES*

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0. ABSTRACT

The relationships between submodules of a module and ideals of the endomorphism ring of a module had been studied in [1]. For a submodule L of a moudle M, the set I^L of all endomorphisms whose images are contained in L is a left ideal of the endomorphism ring End(M) and for a submodule N of M, the set I_N of all endomorphisms whose kernels contain N is a right ideal of End(M).

In this paper, author defines an H-invariant module and proves that every submodule of an H-invariant module is the image and kernel of unique endomorphisms. Every ideal $I^L(I_N)$ of the endomorphism ring End(M) when M is H-invariant is a left(respectively, right) principal ideal of End(M). From the above results, if a module M is H-invariant then each left, right, or both sided ideal I of End(M) is an intersection of a left, right, or both sided principal ideal and I itself appropriately. If M is an H-invariant module then the ACC on the set of all left ideals of type I^L implies the ACC on M. Also if the set of all right ideals of type I_L has DCC, then H-invariant module M satisfies ACC. If the set of all left ideals of type I^L satisfies DCC, then H-invariant module M satisfies DCC.

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If the set of all right ideals of type I_N satisfies ACC then H-invariant module M satisfies DCC. Therefore for an H-invariant module M, if the endomorphism ring End(M) is left Noetherian, then M satisfies ACC. And if End(M) is right Noetherian then M satisfies DCC. For an H-invariant module M, if End(M) is left Artinian then M satisfies DCC. Also if End(M) is right Artinian then M satisfies ACC.

1. INTRODUCTION

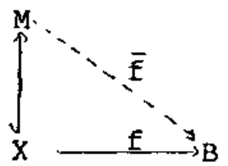
Every ring is assumed to be an associative ring with an identity and every module to be a left module over a ring.

For an element a of ring R, $^l(a)=Ra+Za$ means the left ideal generated by a. Also $^r(a)=aR+Za$ means the right ideal generated by a and $(a)={}^l(a)+{}^r(a)+RaR$ means the ideal generated by a in a ring R.
The ring of R-endomorphisms of a left R-module $_RM$, denoted by End $(_RM)$, will be written on the right side of $_RM$ as right operators on $_RM$, that is, $_RM_{End(_RM)}$ will be considered on this paper. For a submodule L of a left R-module $_RM$, the subset $\{f\varepsilon End(_RM)\,|\,Imf\leq L\}$ and the subset $\{f\varepsilon End(_RM)|L\leq kerf\}$ of the endomorphism ring $End(_RM)$ will be denoted by I^L, I_L respectively. Then I^L and I_L become to be a left and a right ideal of $End(_RM)$ respectively.

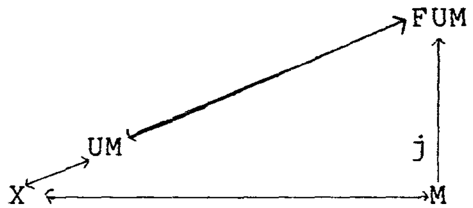
Especially, if L is a fully invariant submodule of a module M, then I^L and I_L turn out to be both sided ideals of End(M). Thus for two fully invariant submodules L, N of a module M, we have a both sided ideal $I^L_N=I^L\cap I_N$, which will be studied. Every right ideal I of End(M) is contained in the right ideal I_N where $N=\bigcap_{f\in I}kerf$.

Especially if such N is the kernel of an endomorphism, say g, then $I_N={}^r(g)\cap I$. Moreover if such N=kerg is fully invariant in M, then $I_N=(g)\cap I$ in End(M). A left R-moduel $_RM$ is said to be free if it is a sum of copies of R.

Theorem.(p57, [12]) Let $X=\{a_i | i \varepsilon A\}$ be a basis of a free module M. Given any module B and any function f : X→B there exists a unique homomorphism $\bar{f}$: M→B extending f.



In a free module M with a basis X, for underlying set UM(forget addition and scalar multiplication) there is a free module with basis UM. Hence basis X is a subset of UM so that there is a unique homomorphism j : M→FUM extending the inclusion X→FUM since M is free



Hence we need a definition of an H-invariant module such that such j is an inclusion. A free module $_RM$ is said to be H-invariant if there is an inclusion R-homomorphism from $_RM$ into FUM

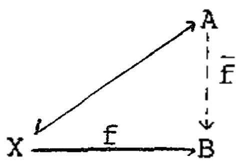
From this definition every submodule of an H-invariant module is an image of a unique endomorphism and a kernel of a unique endomorphism. Hence in an H-invariant module M, for each submodule L, I^L is a left principal ideal of End(M), I_L is a right principal ideal of End(M). Since every H-invariant module is projective (proposition 2, p82[9]) in an H-invariant module M, every epimorphism is left invertible in End (M). Thus if L is a small submodule of M, then the left ideal I^L is a small left ideal of End(M) and if N is a large submodule of an H-invariant module M, then the right ideal I_N is small in End(M), In an H-invariant module if I is a left ideal of End(M), then $I = l(f) \cap I$ for a unique endomorphism f such that $\sum_{g \in I} Img = Imf$. For a right idal I of End (M), we have $I = {}^r(f) \cap I$ for a unique endomorphism f such that $\bigcap_{g \in I} kerg = kerf$. Hence, since two sided ideal I has two fully invariant submodule $L = \sum_{g \in I} Img = Imh$ and $\bigcap_{g \in I} kerg = N = kerf$ for unique endomorphisms h, f in End(M) we have that $I = (h) \cap (f) \cap I$. For two fully invariant submodules L, N of an H-invariant module, we have a both sided ideal $I^L_N = (f) \cap (g)$, wherer L=Imf and N=kerg for unique endomorphisms f, g in End(M). In last section author invest these results to study the relationship between the ACC(ascending chain condition), DCC(descending chain condition)on H-invariant module left Noetherian, right Noetherian, right Artinian, left Artinian endomorphism ring. If $_RM$ satisfies ACC, then the set of all ideals of type I^L satisfies ACC and the set of ideals of type I_N satisfies DCC. If M satisfies DCC, then the set of all ideals of type I^L satsfies DCC and the set of all ideals of type I_N satisfies ACC. In an H-invariant module the partial converse holds. If the set of all ideals of type I^L satisfies ACC or the set of all ideals of type I_N satisfies DCC then H-invariant module M satisfies ACC. If End(M) is left Noetherian, then the set of all ideals of type I^Lsatisfies ACC, and hence H-invariant module M satisfies ACC. Consequently, if End(M) is left Noetherian, then H-invariant module M satisies ACC. The similar results are discussed in this paper.

1. H-INVARIANT MODULE

A left R-module $_RM$ is said to be free if it is a sum of copies of R.

For any set X, there exists a free module A having X as a basis.

Theorem 1.1. (p57[12]) Let $X=\{a_i | i\varepsilon I\}$ be a basis of a free module A. Given any module B and any function $f: X\rightarrow B$ there is a unique R-homorphism $\bar{f}: A\rightarrow B$ extending f.



Remark 1.2. In a free module $_RM$ with a basis X, let UM be the underlying set of M(forget addition and scalar multiplication), then we have the(up to isomorphism) free module FUM with basis UM. We know that basis X is a subset of UM, and UM is a subset of FUM. Thus the inclusion mapping $i: X\hookrightarrow FUM$ exists. If M is a free module, then we have a unique R-homomorphism $j: M\rightarrow FUM$. But such j need not to be an inclusion. hence we need the following definition.

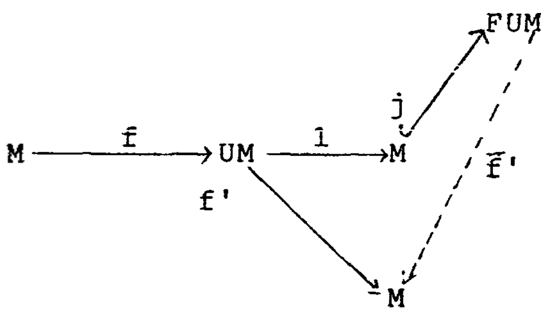
Definition. 1.3. A free R-module $_RM$ is said to be H-invariant if there is an inclusion R-homomorphism $j: M\rightarrow FUM$ where FUM is the free R-module generated by the underlying set UM of $_RM$

Remark 1.4. Since every free module is projective, every epimorphism is left invertible. Hence in an H-invariant module every epimorphism in the endomorphism ring is left invertible But not every free module need be injective and not every free module need be H-invariant, H-invariantness is not a sufficient condition to be injective.

But we have the following result, every monomorphism is right invertible in endomorphism ring of an H-invariant module.

Theorem 1.5. In an H-invariant module $_RM$, every monomorphism in End($_RM$) is invertible.

Proof. Let f be any monomorphism in End($_RM$) then consider a diagram



As a set map f has a right inverse f'such that $ff'=1$ For such f' there exists a unique R-homomorphism $\bar{f}'$ extending f' so we have an R-endomorphism $j\bar{f}'$: M→M such that $fj\bar{f}'=1$. Hence f is right invertible in End($_RM$). Since j is an inclusion R-homomorphism from H-invariantness of $_RM$.

Theorem 1.6. If L is a small submodule of an H-invariant module $_RM$, then the left ideal I^L is small in End($_RM$).

Proof. This easily follows from the similar way of proof Theorem 4.4 in [17] and using that every epimorphism is right invertible.

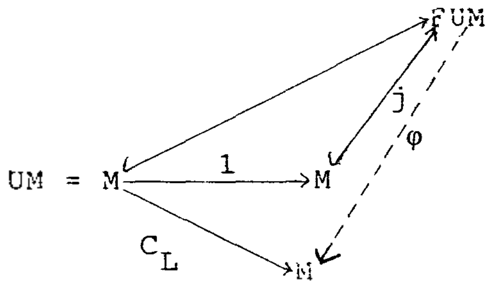
Theorem 1.7. If N is a large submodule of an H-invariant ${}_RM$, then the right ideal I_N is small in End$({}_RM)$.

Proof. This easily follows from the same way in proof of Theorem 5.4 in [17] and using Theorem 1.5.

Theorem 1.8. For any submodule L of an H-invariant module ${}_RM$, there is a unique endomorphism $f \varepsilon \mathrm{End}({}_RM)$ such that $L = \mathrm{Im}f$

Proof. Define C_L : $UM = M \rightarrow M$ by $xC_L = \begin{cases} x & \text{if } x \varepsilon L \\ 0, & \text{otherwise.} \end{cases}$

Then we have a diagram in which there is a unique R-homomorphism φ : $FUM \rightarrow M$ such that $C_L = 1j\varphi = j\varphi$(note the composition map $j\varphi$ is an R-homomorphism). Hence $\mathrm{Im}C_L = L = \mathrm{Im}j\varphi$. Once we had regarded an H-invariant module ${}_RM$ as a submodule of a free module FUM then the composition $j\varphi$ is unique. Hence $L = \mathrm{Im}j\varphi$. $f = j\varphi$ is the required one.



Theorem 1.9. For any submodule L of an H-invariant module ${}_RM$, the left ideal I^L is principal in End$({}_RM)$.

Proof. Since every free module is projective (Proposition) 2, p82[9]) and since $L = \text{Im}f$ for a unique $f \varepsilon \text{End}({}_RM)$ by Theorem 1.8. $I^L = {}^l(f)$ by Theorem 4.7 in [17]

Corollary 1.10. For any fully invariant submodule L of an H-invariant module ${}_RM$, the both sided ideal I^L is principal in $\text{End}({}_RM)$.

Proof. By Corollary 4.8 in [17] and Theorem 1.10 $I^L = (f)$ for a unique endomorphism.

Theorem 1.11. Let I be a left ideal of $\text{End}({}_RM)$ for an H-invariant module ${}_RM$, $I = {}^l(f) \cap I$ for a unique endomorphism f.

Proof. Let $L = \sum_{i \in I} \text{Im}i$. Then there is a unique endomorphism f such that $L = \text{Im}f$ by Theorem 1.8. From (1) 1.1, in [17] $I \leq I^L$ and from Theorem 1.10, $I^L = {}^l(f)$, we have $I = {}^l(f) \cap I$.

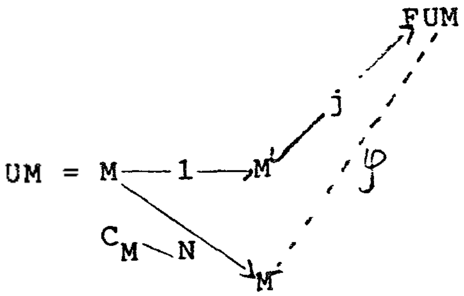
Corollary 1.12. Let I be a both sided ideal of $\text{End}({}_RM)$ for an H-invariant module ${}_RM$. Then $I = (f) \cap I$ for a unique $f \varepsilon \text{End}({}_RM)$.

Proof. By Remark 2.3. in[17]$L = \sum_{i \in I} \text{Im}i$ is a fully invariant submodule of an H-invariant module ${}_RM$. By Corollary 1.10. and similar computation in Corollary 1.11. we have $I = (f) \cap I$ for a unique endomorphism f.

Theorem 1.13. For any submodule N of an H-invariant module ${}_RM$, there is a unique endomorphism $g \varepsilon \text{End}({}_RM)$ such that $\ker g = N$.

Proof. Define $C_{M-N} : UM = M \to M$ by $xc_{M-N} = x$, if $x \varepsilon M - N$
0, otherwise.

Then we have a diagam, where $M - N = \{x \varepsilon M \mid x \notin N\}$,



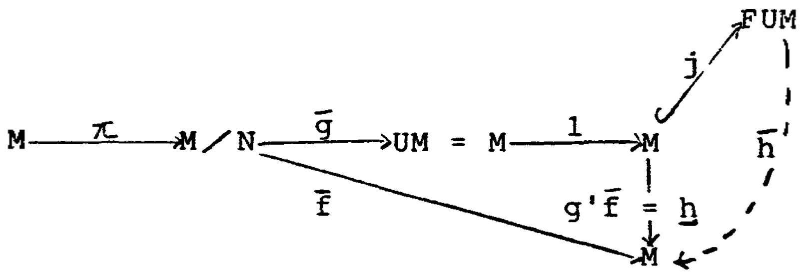
in which there is a unique R-homomorphism φ : FUM→M such that $C_{M-N} = 1j\varphi = j\varphi$(here, the composition $j\varphi$ is an R-homomorphism). Hence $\{x \varepsilon M \mid x c_{M-N} = 0\} = \ker j\varphi = N$. The composition $g = j\varphi$ is the required endomorphism.

Theorem 1.14. For any submodule N of an H-invariant module $_RM$, the right ideal I_N is a right principal ideal of End($_RM$).

Proof. By Theorem 1.13, there is a unique endomorphism g such that N=kerg. If $f \varepsilon I_N$ is given arbitrarily, then we claim that f=gh for some endomorphism h : M→M.

Let f, g : M/N→M be defined by (x+N)f=xf and (x+N)g=xg for all x+N in M/N. Then for the projection π : M→M/N, we have $\pi f = f$ and $\pi g = g$ Since N=kerg≤ker f which implies that f and g are well defined on M/N. Considering a diagram where g′is denoted by a right inverse map of a 1-to-1 map g since as a set map, every one-to-one map has a right inverse.

And let $\underline{h} = g'f$, then $f = g\underline{h} = gg'f$ and for such $\underline{h}$ there is a unique R-



homomorphism h : FUM→M extending h. As a map πf=f=πg1jh=gh when h=1jh=gh when h=1jh=jh is taken.

Therefore f=gh. Hence $I_N={}^r(g)$ which is a right principle ideal of End($_RM$).

Corollary 1.15. For any fully invariant submodule N of an H-invariant module $_RM$, the both sided I_N is principal in End($_RM$).

Proof. By the above theorem, $I_N=(g)$ for a unique endomorphism g∈End($_RM$).

Theorem 1.16. Let I be any right ideal of End($_RM$) for an H-invariant module $_RM$. Then $I={}^r(g)\cap I$ for a unique endomorphism g ∈ End($_RM$).

Proof. Let $N=\bigcap_{i\in I}$ keri. Then there exists a unique endomorphism g∈ End($_RM$) such that N=kerg, by Theorem 6.11.

By (1) 2.2 in [17] $I\leq I_N$ and by Theorem 1.14, $I_N={}^r(g)$ so we have $I=I_N\cap I={}^r(g)\cap I$.

Theorem 1.17 In an H-invariant module $_RM$, if I is a both sided ideal of End($_RM$), then there exist f, g∈End($_RM$) such that $I=(f)\cap(g)\cap I$, uniquely.

Proof. From Remarks 1.4 and 2.3 in[17], every both sided ideal I has two fully invariant submodules $L=\sum_{i\in I} \mathrm{Im}\,i$ and $N=\bigcap_{i\in I} \ker i$. Now we have two unique endomorphisms f, g ε End($_R$M) such that L=Imf and N=kerg by Theorem 1.8 and 1.13. Thus $I=I_N^L\cap I=I^L\cap I_N\cap I=(f)\cap(g)\cap I$ is followed.

Corollary 1.18. Let $_R$M be an H-invariant module. Then for two fully invariant submodules L, N the both sided ideal $I_N^L=(f)\cap(g)$ for unique endomorphisms f, g in End($_R$M).

Proof. By Theorems 1.8 and 1.13, L=Imf and N=kerg for unique endomorphisms f, g, in End($_R$M). Hence by Corollaries 1.10 and 1.15, $I_N^L=I^L\cap I_N=(f)\cap(g)$.

Corollary 1.19. Let $_R$M be an H-invariant module and let I be a subset of End($_R$M). Then we have the following

(1) if I is a left ideal, then $I={}^l(f)\cap(g)\cap I$

(2) if I is a right ideal, then $I=(f)\cap^r(g)\cap I$

where $L=\sum_{i\in I} \mathrm{Im}\,i=\mathrm{Im}f$ and $N=\bigcap_{i\in I} \ker i=\ker g$ for unique endomorphisms f, gεEnd($_R$M).

Proof. By Theorems 1.8 and 1.13, the existences of f, g in End($_R$M) are guaranteed such that L=Imf and N=kerg.

By Remark 1.4 in[17] for a left ideal I, N is a fully invariant submodule. Hence we have (1) $I={}^l(f)\cap(g)\cap I$. similarly for a right sided ideal I, L is a fully invariant submodule of $_R$M hence we can conclude that $I=(f)\cap^r(g)\cap I$.

2. MODULE WITH CHAIN CONDITION (ACC/DCC)

A module M is said to satisfy the ascending chain condition(ACC) on submodules(or to be Noetherian) if every chain $L_1 \leq L_2 \leq L_3 \leq \cdots\cdots$ of submodules of M, there is an integer n such that $L_i = L_n$ for all $i \geq n$. [7]. A module M is said to satisfy the descending chain condition(DCC) on submodules (or to be Artinian) if for every chain $N_1 \geq N_2 \geq N_3 \geq \cdots\cdots$ of submodules of M, there is an integer m such that $N_i = N_m$ for all $i \geq m$.

A ring R is left(resp. right) Noetherian if R satisfies ACC on left(resp. right) ideals. R is said to be Notherian if R is both left and right Noetherian. A ring R is left(resp. right) Artinian if R satisfies the DCC on left(resp. right) ideals. R is said to be Artinian if R is both left and right Artinian.

But it is still hard to say that ACC on a module M is possible to imply ACC on the endomorphism ring End($_RM$). For certain module, namely an H-invariant module, the converse holds, in other words, if M is H-invariant then ACC on End(M) implies ACC on submodules of M. We are going to prove this gradually.

Lemma 2.1 In an H-invariant module, if L and L' are distinct submodules of M, then the left ideals I^L and $I^{L'}$ are distinct in End(M).

Proof. In an H-invariant module M, by Theorem 1.8, L=Imf and L' =Img for unique endomorphisms f, g$\in$End(M). Suppose the left ideals I^L and $I^{L'}$ are equal. Then $f \in I^L = I^{L'}$ says that $L = \mathrm{Im} f \leq L'$ thus we have $L \leq L'$. similar argument says that $L' \leq L$. Hence L=L' which is contradicted.

Lemma 2.2 In an H-invariant module M, two distinct submodules N, N'have distinct right ideals I_N $I_{N'}$, respectively.

Proof. In an H-invariant module, by Theorem 1.13, for submodules N, N'of a module M, there are unique endomorohisms f, g such that N= kerf, N'=kerg. Suppose that the right ideals I_N and $I_{N'}$ are equal. Then $f \varepsilon I_N = I_{N'}$ implies that $N' \leq$ kerf=N and $g \varepsilon I_{N'} = I_N$ implies that $N \leq$ kerg =N'Thus N=N'

Remark 2.3. In an H-invariant module M, if $L \neq L'$, then $I^L \neq I^{L'}$, and also if $N \neq N'$ then $I_N \neq I_{N'}$.

Note 2.4. Without H-invariantness of a module M, the above Lemma 2.1 and 2.2 don't have to have these properties. For an example let M =R the set of all reals, Q the set of all rationals, and Z the set of all integers. Then R is not an H-invariant(since R is not free) module over Z and $I^Q = I^Z = 0$ even though $Z \leq Q$. And $I_Q = I_Z = 0$.

Theorem 2.5 Let M be an H-invariant module and the set $\{I^L | L \leq M\}$ satisfy the ACC. then M satisfies ACC.

Proof. Let $L_1 \leq L_2 \leq L_3 \cdots\cdots$ be any ascending chain of submodules of M. Then we have an ascending chain $I^{L_1} \leq I^{L_2} \leq I^{L_3} \leq \cdots$ of the set $\{I^L | L \leq M\}$

By the hypothesis, the set $\{ I^L | L \leq M \}$ satisfies ACC, hence there is an integer n such that $I^{L_i} = I^{L_n}$ for all $i \geq n$. By Lemma 2.1, $L_i = L_n$ for all $i \geq n$ Thus theorem has been proved

Theorem 2.6. Let M be an H-invariant module and the set $\{I_N | N \leq M\}$ satisfy the DCC. Then M satisfies ACC.

Proof. Let $N_1 \leq N_2 \leq N_3 \leq \cdots\cdots$ be any ascending chain of submodules of

M. Then we have a descending chain
$I_{N1} \geq I_{N2} \geq I_{N3} \geq \cdots\cdots$ of the set $\{I_N | N \leq M\}$ which satisfies DCC by hypothesis and hence there is an integer n such that $I_{N_i} = I_{N_n}$ for all $i \geq n$
By Lemma 2.2, $N_i = N_n$ for all $i \geq n$.

Corollary 2.7. Let M be an H-invariant module and the set $\{I^L | L \leq M\}$ satiafy DCC. Then M satisfies DCC.

Proof. It is proved by a similar argument of Theorem 2.5.

Corollary 2.8. Let M be an H-invariant module and the set $\{I_N | N \leq M\}$ satisfy the ACC. Then M satifies DCC.

Theorem 2.9. Let M be an H-invariant module and End(M) be left Notherian. Then M satisfies ACC.

Proof. Since End(M) is left Noetherian, the set $\{I^L | L \leq M\}$ satisfies ACC. By Theorem 2.5, M satisfies ACC.

Theorem 2.10. Let M be an H-invariant module and End(M) be right Noehterian. Then M satisfies DCC.

Proof. Since End(M) is a right Noetherian ring, the set $\{I_N | N \leq M\}$ satisfies ACC. Thence M satisfies DCC by Corollary 2.8.

Corollary 2.11. Let M be an H-invariant module and End(M) be Noetherian. Then M satisfies ACC and DCC.

Theorem 2.12. Let M be an H-invariant module and End(M) be left Artinian. Then M satifies DCC.

Proof. Since End(M) is left Artinian, the set $\{I^L | L \leq M\}$ satisfies DCC. Then by Corollary 2.7, M satisfies DCC.

Theorem 2.13. Let M be an H-invariant module and End(M) be right Artinian. Then M satisfies ACC.

Proof. Since End(M) is right Artinian, the set $\{I_N | N \leq M\}$ satisfies DCC. Then by Theorem 2.6, M satisfies ACC.

Corollary 2.14. Let M be an H-invariant module and End(M) be Artinian. Then M satisfies ACC and DCC.

Proof. Form the above 2.12 and 2.13, it follows immediateiy.

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