PROPERTIES OF POSITIVE DERIVATIONS ON ORDERED STRONGLY REGULAR NEAR-RINGS

Yong-Uk Cho

1. Introduction

G. Pilz(10) in 1972 has defined the concept of ordered near-ring and then studied direct sums of ordered near-rings.

A near-ring \( N \) is called (partially) ordered if its additive group is (partially) ordered, so that, we can speak of positive elements in the sense that an element \( a \) of the additive group of \( N \) is called positive if \( a \geq o \), and negative if \( a \leq o \), and if, in addition, the product of positive elements is positive, i.e., if it follows from \( a \geq o \), \( b \geq o \) that \( ab \geq o \). If \( a>o \), \( b>o \) implies \( ab>o \) then we call \( N \) strictly ordered. Examples of strictly ordered near rings are the polynomial near-rings of all \( \sum a_i x_i (a_i \neq o) \) with addition and substitution of polynomials and coefficient from an ordered ring. \( \sum a_i x_i \) is then defined to be greater than \( o \) if \( a_i \) is greater than \( o \). Ordered near-ring \( N \) is called linear if its additive group is linearly ordered.

It follows from this definition that the trivial (partial) ordering of the additive group of an arbitrary near-ring is a trivial (partial) ordering of the near-ring itself.

D.J. Hansen (3), in 1984, studied positive derivations on ordered strongly regular rings, in this paper, we will be show that an ordered near-
ring(N, +, ·, ≤) which is strongly regular and has the additional property that \(a^2 \geq 0\) for each \(a \in N\) can not have nontrivial positive derivations.

An ordered near-ring which is also integral, is called an ordered integral near-ring and which is also a near-field is called an ordered near-field.

In an ordered near-ring, \(P = P(N) = \{a \in N \mid a \geq 0\}\) is called the positive cone of \(N\). Every partial ordering of a near-ring \(N\) is determined by \(P : a \leq b\) if and only if \(b - a \in P\).

Near-ring \(N\) is called strongly regular if for any \(a \in N\) there exists an element \(x\) in \(N\) such that \(a = xa^2\).

We will investigate some properties of ordered near-rings (§2) and positive derivations on it (§3).

2. Some properties of ordered near-rings

Remark 2.1. The condition on the product of positive elements in our definition is obviously equivalent to the fact that if \(a \leq b\) and \(c \geq 0\), then \(ac \leq bc\). Now clearly, it follows from \(a \geq 0\) that \(-a \leq 0\). We only have to add \(-a\) to both sides of the first inequality; conversely, it follows from \(a \leq 0\) that \(-a \geq 0\). Hence we have the following inequalities:

- if \(a \leq 0\), \(b \geq 0\), then \(ab \leq 0\) and \(b(-a) \leq 0\).
- if \(a \geq 0\), \(b \leq 0\), then \(ba \leq 0\) and \(a(-b) \geq 0\).
- if \(a \leq 0\), \(b \leq 0\), then \(a(-b) \leq 0\) and \(b(-a) \leq 0\).

From the definition of an ordered near-rings we derive the following equivalent concept.

Lemma 2.2. Let \(N\) be a near-ring. If \(N = (N, +, \cdot, \leq)\) is an ordered near-ring with positive cone \(P\), then \(P\) has the following proper-
PROPERTIES OF POSITIVE DERIVATIONS

185

ties:

(1) \( P \) is a subseminear-ring of \( N \)
(2) If \( a \in P \) and \(-a \in P \), then \( a = 0 \)
(3) \( P \) is an additive normal subset of \( N \), that is, \( P + a = a + P \) for all \( a \in N \).

Conversely, suppose \( N \) has a subset \( P \) satisfying these conditions. If we define \( a \leq b \) to mean that \( b - a \in P \), then \( N \) becomes an ordered near-ring with positive cone \( P \).

Proof. Suppose, first, that \( N \) is an ordered near-ring with positive cone \( P \). Let \( a, b \in P \). Then \( a \geq 0, b \geq 0 \) implies \( a + b \geq 0 \) and \( ab \geq 0 \), that is, \( P \) is a subseminear-ring of \( N \). If \( a \in P \) and \(-a \in P \) then \( a \geq 0 \) and \(-a \geq 0 \), i.e., \( a \geq 0 \) and \( a \leq 0 \) by symmetry property of partial ordering, \( a = 0 \).

Finally, Let \( a \in N \) and \( x \in P \), then from \( x \geq 0, -a + x + a \geq 0 \), and \( a + x - a \geq 0 \). This yields part (3).

Conversely, suppose that a subset \( P \) of \( N \) has the properties (1)−(3), and define \( a \leq b \) to mean that \( b - a \in P \). So that, by (3), we also have

\[
-a + b = -a + (b - a) + aP.
\]

This is a partial ordering of \( N \):

\[
a \leq a, \text{ because, by (1), } a - a = 0 \in P.
\]

If \( a \leq b \) and \( b \leq a \), i.e., \( b - a \in P \) and \( a - b = -(b - a) \in P \) then by (2)

\[
b - a = 0, \text{ i.e., } a = b.
\]

If \( a \leq b \) and \( b \leq c \), i.e., \( b - a \in P \) and \( c - a \in P \), then

\[
c - a = (c - b) + (b - a) \in P, \text{ i.e., } a \leq c.
\]

Next, if \( a \leq b \), i.e., \( b - a \in P \) then \( (b + x) - (a + x) \in P \), that is, \( a + x \leq b + x \) and by (3), \( (x + b) - (x + a) = (x + b) - a - x = x + (b - a) - x \in P \) for all \( x \in N \), i.e., \( x + a \leq x + b \).

Finally, if \( a \geq 0, b \geq 0 \), i.e., \( a \in P \), \( b \in P \) by (1), \( ab \in P \), i.e., \( ab \geq 0 \).

We will show a weaker condition of ordered near-ring as following: A near-ring \( N \) is said to be a right-ordered near-ring if a partial ordering
is given for its elements such that it follows from \( a \leq b \) that \( a + x \leq b + x \) for all \( x \in N \) and if \( a \geq 0, b \geq 0 \), implies \( ab \geq 0 \). Again, the positive cone \( P = \{ x \in N \mid x \geq 0 \} \) defines the ordering. For example of right orderability, a positive cone of an ordered near-ring is right ordered, because for any near-ring, only right distributive laws hold.

Proposition 2.3. Let \( N \) be a near-ring. If \( N \) is a right ordered near-ring with positive cone \( P \), then \( P \) has the following properties:

1. \( P \) is subsemiring of \( N \)
2. If \( aP \) and \( -aP \) then \( a = 0 \).

Conversely, suppose \( N \) has a subset satisfying these conditions. If we define \( a \leq b \) to mean that \( b - aP \), then \( N \) becomes a right ordered near-ring with positive cone \( P \).

Proof. This is, of course, similar to the result of Lemma 2.2, and we merely indicate the difference here. If \( a, b \in P \) then \( a \geq 0, b \geq 0 \) implies that \( a + b \geq 0 + b = b \geq 0 \) so that \( a + bcP, ab \geq 0 \) by definition of right orderability, i.e., \( ab \in P \). This yields (1), (2) follows from Lemma 2.2.

Conversely, if \( P \) satisfies (1) and (2), as in the proof of Lemma 2.2, \( \leq \) is a partial ordering on \( N \), and product of any two positive elements is positive. Finally, if \( a \leq b \) and \( x \in N \), then \( b - aP \) so that \( (b + x) - (a + x) = b + x - x - a = b - aP \). Therefore \( a + x \leq b + x \).

Let us consider the extension properties of this near-rings.

Proposition 2.4. Let \( I \) be an ideal of a near-ring \( N \). If \( I \) and \( N/I \) are both right ordered near-rings then so is \( N \).

Proof. Let \( \pi : N \to N/I \) be the natural near-ring homomorphism. Now we are given \( P(I) \) and \( P(N/I) \), and we define \( P(N) \) by

\[
P(N) = \{ x \in N \mid \pi(x)eP(N/I) \text{ or } xeP(I) \}
\]
Then $P(N)$ is subsemnear-ring of $N$ and if $a\in P(N)$ and $-a \in P(N)$ then $a = 0$, thus by proposition 2.3. $N$ is a right ordered near-ring.

**Proposition 2.5.** An ordering of a near-ring $N$ determined by a subset $P$ of $N$ with the properties $(1)-(3)$ of Lemma 2.2. is linear if and only if the following additional condition holds:

(4) For every $a \in N$, either $a \in P$ or $-a \in P$.

**Proof.** It suffices to show that the linear property of ordered near-ring is equivalent to the condition (4) by Lemma 2.2.

If $N$ is linearly ordered and the element $a$ is not positive, then $a < 0$ so that $1 < a^{-1}$, that is, $a^{-1}$ is positive.

Suppose, conversely, a subset $P$ of $N$ satisfies the condition $(1)-(4)$ and that $a, b \in N$. If $b \not< a \in P$ we are done. But if $b - a \not\in P$ then by (4), $-(b - a) = a - b \in P$, i.e., $b \leq a$.

**Proposition 2.6.** In any linearly ordered near-field $N$, all squares of non-zero elements are positive.

**Proof.** Suppose that $a(\neq 0) \in N$ by proposition 2.5, (4) either $a \in P$ or $-a \in P$. Since $P$ is closed under multiplication, $a^2 = (-a)^2 \in P$ in either case, as asserted.

It is a corollary that the identity $1 = 1 \in P$ is always positive, while $-1$ is never positive.

**Theorem 2.7.** Any linearly ordered near-field $N$ is an integral near-ring of characteristic 0.

**Proof.** First, suppose that $a \not= 0, b \not= 0$ were zero divisors, with $ab = 0$. Then $(\pm a)(\pm b) = 0$, but, by the linearity of proposition 2.5, one of $\pm a$ and one of $\pm b$ is in $P$, hence some one of the four products $(\pm a)(\pm b)$
is in P, say aeP, -beP then a(-b) = -ab > 0, a contradiction to ab = o. Hence N is integral.

Second, since 1eP, it follows by repeated application of part (1) of Lemma 2.2 that 1, 1 + 1, 1 + 1 + 1, ... are different positive elements of N, and hence can not be o. Therefore the characteristic of N is o.

3. Positive derivations on ordered strongly regular near-rings

Now, We will introduce a positive derivation on ordered near-rings and investigate that an ordered near-ring which is strongly regular and has the additional property that a ≥ o for each aeN can not have non-trivial positive derivations.

Definition 3.1. The statement that \( \delta \) is a positive derivation on an ordered near-ring N mean that \( \delta \) is a map from N into N such that:

1. \( \delta(a + b) = \delta(a) + \delta(b) \) for each a,beN
2. \( \delta(ab) = \delta(a)b + a\delta(b) \) for each a,beN
3. \( \delta(a) \geq o \) for each aeN, with a ≥ o.

Lemma 3.2. Let N be a strongly regular near-ring. If for any a,b in N with ab = o, then \((ba)^n = bo\), for all positive integer n.

Lemma 3.3. For any strongly regular near-ring N, if a, b in N with ab = o and \(a^n = ao\), for any positive integer \(n > 1\), then a = o. In this case, in N is zero-symmetric, then N is reduced.

Proof. Assume the conditions hold. Then \( a = xa^2 = aa = x^{-1}a^r = x^{-1}ao \) for some x\(eN \) so that \( ao = a^n = aa^{n-1} = x^{-1}ao = a \). Thus we have \( a = ao = ao \) = ab = o.
Corollary 3.4. (G. Mason (6)). Let $N$ be a zero-symmetric near-ring. If for any $a, b$ in $N$ with $ab = o$, then $ba = o$ and $N$ is reduced.

Lemma 3.5. Every strongly regular near-ring is regular. Moreover if $N$ is strongly regular such that for $a, x$ in $N$, $a = xa^2$, then $ax = xa$.

Proof. Let $N$ be strongly regular. Then for any $a \in N$, $a = xa^2$ for some $x$ in $N$, so that $(a - axa) = o$, by Lemma 3.2, $a(a - axa) = ao$. It follows that $(a - axa)^2 = ao - axao = (a - axa)o$ by Lemma 3.3, $a = axa$. Hence $N$ is regular.

Next, since $(ax - xa)^2 = axo - xao = (ax - xa)o$. Therefore $ax = xa$.

Before proving the main theorem, we will prove the following, hereafter we may assume that $N$ is zero-symmetric.

Theorem 3.6. Let $(N, +, \cdot, \leq)$ be an ordered strongly regular near-ring such that $a^2 \geq o$ for each $a \in N$. If $\delta$ is a positive derivation defined on $N$ and $a \in N$, with $a \geq o$, then $\delta(a) = o$.

Proof. Suppose $N$ is ordered strongly regular and $a$ is any element of $N$ with $a \geq o$, then there exists an element $x$ in $N$ such that $a = xa^2$. By Lemma 3.5, we have $ax = xa$ and $a = axa$. Applying $\delta$ for $a = axa$, $\delta(a) = \delta(axa) = \delta(a)xa + a\delta(xa)$. Multiplying on the right side of this equation by $a$, $\delta(a)a = \delta(a)xa + a\delta(xa)a$. This implies that $a\delta(xa)a = o$. Hence $(a\delta(xa))^2 = a\delta(xa)a\delta(xa) = o$. By Lemma 3.3, $N$ is reduced, so that $a\delta(xa) = o$. It follows that $\delta(a) = \delta(a)xa$.

Next, Since $ax = xa$, $\delta(xa) = \delta(ax) = \delta(a)x + a\delta(x)$. Multiplying on the right side by $a$, $\delta(xa)a = \delta(a)xa + a\delta(x)a$. Obviously, $\delta(xa)a = o$ by using Lemma 3.3, and the equality $a\delta(xa)a = o$. Thus, $\delta(a)xa = -a\delta(x)a$, that is, $\delta(a) = -a\delta(x)a$. Since $N$ is ordered, $x^2a \geq o$, and since $\delta$ is positive, $\delta(x^2a) \geq o$, namely, $\delta(x)xa + x\delta(xa) \geq o$. Multiplying on the right
by \(a(a \geq o)\), we obtained that
\[
\delta(x)xa^2 + x\delta(x)a = \delta(x)a + o = \delta(x)a \geq o.
\]
Since, \(a \geq o\), \(a\delta(a) = -\delta(a) \geq o\), by Remark 2.1, \(\delta(a) \leq o\). Therefore \(\delta(a) = o\).

**Theorem 3.7.** Let \((N, +, \cdot, \leq)\) be an ordered strongly regular near-ring such that \(a^2 \geq o\) for each \(a \in N\), then \(\delta(a) = o\), for each \(a \in N\).

**Proof.** Left to the reader, using Theorem 3.6.

**Corollary 3.8.** If \((F, +, \cdot, \leq)\) is a strongly regular ordered near-field and \(\delta\) is a positive derivation defined on \(F\), then \(\delta(x) = o\), for any \(x \in F\).

**Proof.** It is straight-forward from proposition 2.6.

**REFERENCES**

27(1985), 49~52

Department of Mathematics
Pusan Women's University
Pusan 607-082
Korea