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PROPERTIES OF POSITIVE DERIVATIONS ON ORDERED STRONGLY REGULAR NEAR-RINGS

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1. Introduction

G. Pilz(10) in 1972 has defined the concept of ordered near-ring and then studied direct sums of ordered near-rings.

A near-ring N is called (partially) ordered if its additive group is (partially) ordered, so that, we can speak of positive elements in the sense that an element a of the additive group of N is called positive if $a \geq o$, and negative if $a \leq o$, and if, in addition, the product of positive elements is positive, i.e., if it follows from $a \geq o$, $b \geq o$ that $ab \geq o$. If $a > o$, $b > o$ implies $ab > o$ then we call N strictly ordered. Examples of strictly ordered near rings are the polynomial near-rings of all $\sum_{i=1}^{n} a_i x^i \, (a_n \neq o)$ with addition and substitution of polynomials and coefficient from an ordered ring. $\sum_{i=1}^{n} a_i x^i$ is then defined to be greater than o if a_n is greater than o. Ordered near-ring N is called linear if its additive group is linearly ordered.

It follows from this definition that the trivial (partial) ordering of the additive group of an arbitrary near-ring is a trivial (partial) ordering of the near-ring itself.

D.J.Hansen (3), in 1984, studied positive derivations on ordered strongly regular rings, in this paper, we will be show that an ordered near-ring$(N, +, \cdot, \leq)$ which is strongly regular and has the additional property that $a^2 \geq o$ for each $a \varepsilon N$ can not have nontrivial positive derivations.

An ordered near-ring which is also integral, is called an ordered integal near-ring and which is also a near-field is called an ordered near-field.

In an ordered near-ring, $P = P(N) = \{a \varepsilon N \mid a \geq o\}$ is called the positive cone of N. Every partial ordering of a near-ring N is determined by P : $a \leq b$ if and only if $b - a \varepsilon P$.

Near-ring N is called strongly regular if for any $a \varepsilon N$ there exists an element x in N such that $a = xa^2$.

We will investigate some properties of ordered near-rings (§2) and positive derivations on it(§3).

2. Some properties of ordered near-rings

Remark 2.1. The condition on the product of positive elements in our definition is obviously equivalent to the fact that if $a \leq b$ and $c \geq o$, then $ac \leq bc$. Now clearly, it follows form $a \geq o$ that $-a \leq o$: We only have to add -a to both sides of the first inequality ; conversely, it follows from $a \leq o$ that $-a \geq o$. Hence we have the following inequalities :

$$\text{if } a \leq o,\ b \geq o, \text{ then } ab \leq o \text{ and } b(-a) \geq o.$$
$$\text{if } a \geq o,\ b \leq o, \text{ then } ba \leq o \text{ and } a(-b) \geq o.$$
$$\text{if } a \leq o,\ b \leq o, \text{ then } a(-b) \leq o \text{ and } b(-a) \leq o.$$

From the definition of an ordered near-rings we derive the following equivalent concept.

Lemma 2.2. Let N be a near-ring. If $N = (N, +, \cdot, \leq)$ is an ordered near-ring with positive cone P, then P has the following properties :

(1) P is a subseminear-ring of N

(2) If $a\varepsilon P$ and $-a\varepsilon P$, then $a = o$

(3) P is an additive normal subset of N, that is, $P + a = a + P$ for all $a\varepsilon N$.

Conversely, suppose N has a subset P satisfying these conditions. If we define $a \leq b$ to mean that $b - a\varepsilon P$, then N becomes an ordered near-ring with positive cone P.

Proof. suppose, first, that N is an ordered near-ring with positive cone P. Let $a,b\varepsilon P$. Then $a \geq o$, $b \geq o$ implies $a + b \geq o$ and $ab \geq o$, that is, P is a subseminear-ring of N. If $a\varepsilon P$ and $-a\varepsilon P$ then $a \geq o$ and $-a \geq o$, i.e., $a \geq o$ and $a \leq o$ by symmetry property of partial ordering, $a = o$. Finally, Let $a\varepsilon N$ and $x\varepsilon P$, then from $x \geq o$, $-a + x + a \geq o$, and $a + x - a \geq o$. This yields part(3)

Conversely, suppose that a subset P of N has the properties (1)~(3), and define $a \leq b$ to mean that $b - a\varepsilon P$. So that, by (3), we also have

$$-a + b = -a + (b - a) + a\varepsilon P.$$

this is a partial ordering of N :

$$a \leq a, \text{ because, by (1), } a - a = o\varepsilon P.$$

If $a \leq b$ and $b \leq a$, i.e., $b - a\varepsilon P$ and $a - b = -(b - a)\varepsilon P$ then by (2) $b - a = o$, i.e., $a = b$.

If $a \leq b$ and $b \leq c$, i.e., $b - a\varepsilon P$ and $c - a\varepsilon P$, then

$$c - a = (c - b) + (b - a)\varepsilon P, \text{ i.e., } a \leq c.$$

Next, if $a \leq b$, i.e., $b - a\varepsilon P$ then $(b + x) - (a + x)\varepsilon P$, that is, $a + x \leq b + x$ and by (3), $(x + b) - (x + a) = (x + b) - a - x = x + (b - a) - x\varepsilon P$ for all $x\varepsilon N$, i.e., $x + a \leq x + b$.

Finally, if $a \geq o$, $b \geq o$, i.e., $a\varepsilon P$, $b\varepsilon P$ by (1), $ab\,\varepsilon\, P$, i.e., $ab \geq o$.

We will show a weaker condition of ordered near-ring as following : A near-ring N is said to be a right-ordered near-ring if a partial ordering is given for its elements such that it follows from $a \leq b$ that $a + x \leq b + x$ for all $x \varepsilon N$ and if $a \geq o$, $b \geq o$, implies $ab \geq o$. Again, the positive cone $P = \{x \varepsilon N \mid x \geq o\}$ defines the ordering. For example of right orderability, a positive cone of an ordered near-ring is right ordered, because for any near-ring, only right distributive laws hold.

Proposition 2.3. Let N be a near-ring. If N is a right ordered near-ring with positive cone P, then P has the following properties:

(1) P is subseminear-ring of N

(2) If $a \varepsilon P$ and $-a \varepsilon P$ then $a = o$.

Conversely, suppose N has a subset satisfying these conditions. If we define $a \leq b$ to mean that $b - a \varepsilon P$, then N becomes a right ordered near-ring with positive cone P.

Proof. This is, of course, similar to the result of Lemma 2.2, and we merely indicate the difference here. If $a, b \varepsilon P$ then $a \geq o$, $b \geq o$ implies that $a + b \geq o + b = b \geq o$ so that $a + b \varepsilon P$, $ab \geq o$ by definition of right orderability, i.e., $ab \varepsilon P$. This yields (1), (2) follows from Lemma 2.2.

Conversely, if P satisfies (1) and (2), as in the proof of Lemma 2.2, $\leq$ is a partial ordering on N, and product of any two positive elements is positive. Finally, if $a \leq b$ and $x \varepsilon N$, then $b - a \varepsilon P$ so that $(b + x) - (a + x) = b + x - x - a = b - a \varepsilon P$ Therefore $a + x \leq b + x$

Let us consider the extension properies of this near-rings.

Proposition 2.4. Let I be an ideal of a near-ring N. If I and N/I are both right ordered near-rings then so is N.

Proof. Let $\pi : N \to N/I$ be the natural near-ring homomorphism. Now we are given $P(I)$ and $P(N/I)$, and we define $P(N)$ by

$$P(N) = \{x \varepsilon N \mid \pi(x) \varepsilon P(N/I) \text{ or } x \varepsilon P(I)\}$$

Then P(N) is subseminear-ring of N and if aεP(N) and -a ε P(N) then a = o, thus by proposition 2.3. N is a right ordered near-ring.

Proposition 2.5. An ordering of a near-ring N determined by a subset P of N with the properties (1)~(3) of Lemma 2.2. is linear if and only if the following additional condition holds :

(4) For every aεN, either aεP or -aεP.

Proof. It suffices to show that the linear property of ordered near-ring is equivalent to the condition (4) by Lemma 2.2.

If N is linearly ordered and the element a is not positive, then a<o so that $1<a^{-1}$, that is, a^{-1} is positive.

Suppose, conversely, a subset P of N satisfies the condition (1)~(4) and that a, bεN If b − aεP we are done. But if b − a $\notin$ P then by (4), -(b − a) = a − bεP, i.e., b $\leq$ a.

Proposition 2.6. In any linearly ordered near-field N, all squares of non-zero elements are positive.

Proof. Suppose that a($\neq$o)εN by proposition 2.5,(4) either aεP or -a εP. Since P is closed under multiplication, $a^2 = (-a)^2 \varepsilon P$ in either case, as asserted.

It is a corollary that the identity $1 = 1^2 \varepsilon N$ is always positive, while − 1 is never positive.

Theorem 2.7. Any lineredly ordered near-field N is an integral near-ring of characteristic o.

Proof. First, suppose that a$\neq$o, b$\neq$o were zero divisors, with ab = o. Then ($\pm$a)($\pm$b) = o, but, by the linearity of proposition 2.5, one of $\pm$a and one of $\pm$b is in P, hence some one of the four products ($\pm$a)($\pm$b) is in P, say $a\varepsilon P$, $-b\varepsilon P$ then $a(-b) = -ab>o$, a contradiction to $ab = o$. Hence N is integral.

Second, since $1\varepsilon P$, it follows by repeated application of part (1) of Lemma 2.2 that $1, 1 + 1, 1 + 1 + 1, \cdots$ are different positive elements of N, and hence can not be o. Therefore the characteristic of N is o.

3. Positive derivations on ordered strongly regular near-rings

Now, We will introduce a positive derivation on ordered near-rings and investigate that an ordered near-ring which is strongly regular and has the additional property that $a \geq o$ for each $a\varepsilon N$ can not have nontrivial positive derivations.

Definition 3.1. The statement that δ is a positive derivation on an ordered near-ring N mean that δ is a map from N into N such that :

(1) $\delta(a + b) = \delta(a) + \delta(b)$ for each $a,b\varepsilon N$

(2) $\delta(ab) = \delta(a)b + a\delta(b)$ for each $a,b\varepsilon N$

(3) $\delta(a) \geq o$ for each $a\varepsilon N$, with $a \geq o$.

Lemma 3.2. Let N be a strongly regular near-ring. If for any a,b in N with $ab = o$, then $(ba)^n = bo$, for all positive integer n.

Lemma 3.3. For any strongly regular near-ring N, if a, b in N with $ab = o$ and $a^n = ao$, for any positive integer $n>1$, then $a = o$. In this case, in N is zero-symmetric, then N is reduced.

Proof. Assume the conditions hold. Then $a = xa^2 = aa = x^{n-1}a^n = x^{n-1}ao$ for some $x\varepsilon N$ so that $ao = a^n = aa^{n-1} = x^{n-1}ao = a$. Thus we have $a = ao = aob = ab = o$.

Corollary 3.4.(G.Mason(6)). Let N be a zero-symmetric near-ring. If for any a,b in N with $ab = o$, then $ba = o$ and N is reduced.

Lemma 3.5. Every strongly regular near-ring is regular Moreover if N is strongly regular such that for a,x in N, $a = xa^2$, then $ax = xa$.

Proof. Let N be strongly regular. Then for any $a \varepsilon N$, $a = xa^2$ for some x in N, so that $(a - axa)a = o$, by Lemma 3.2, $a(a - axa) = ao$. It follows that $(a - axa)^2 = ao - axao = (a - axa)o$ by Lemma 3.3, $a = axa$. Hence N is regular.

Next, since $(ax - xa)^2 = axo - xao = (ax - xa)o$. Therefore $ax = xa$.

Before proving the main theorem, we will prove the following, here after we may assume that N is zero-symmetric.

Theorem 3.6. Let $(N, +, \cdot, \leq)$ be on ordered strongly regular near-ring such that $a^2 \geq o$ for each $a\varepsilon N$. If δ is a positive derivation defined on N and $a\varepsilon N$, with $a \geq o$, then $\delta(a) = o$.

Proof. Suppose N is ordered strongly regular and a is any element of N with $a \geq o$, then there exists an element x in N such that $a = xa^2$. By Lemma 3.5, we have $ax = xa$ and $a = axa$. Applying δ for $a = axa$, $\delta(a) = \delta(axa) = \delta(a)xa + a\delta(xa)$. Multiplying on the right side of this equation by a, $\delta(a)a = \delta(a)xa^2 + a\delta(xa)a$. This implies that $a\delta(xa)a = o$. Hence $[a\delta(xa))^2 = a\delta(xa)a\delta(xa) = o$. By Lemma 3.3, N is reduced, so that $a\delta(xa) = o$. It follows that $\delta(a) = \delta(a)xa$.

Next, Since $ax = xa$, $\delta(xa) = \delta(ax) = \delta(a)x + a\delta(x)$. Multiplying on the right side by a, $\delta(xa)a = \delta(a)xa + a\delta(x)a$. Obviously, $\delta(xa)a = o$ by using Lemma 3.3, and the equality $a\delta(xa)a = o$. Thus, $\delta(a)xa = -a\delta(x)a$, that is, $\delta(a) = -a\delta(x)a$. Since N is ordered, $x^2a \geq o$, and simce δ is positive, $\delta(x^2a) \geq o$, namely, $\delta(x)xa + x\delta(xa) \geq o$. Multiplying on the right by $a(a \geq o)$, we obtained that $\delta(x)xa^2 + x\delta(xa)a = \delta(x)a + o = \delta(x)a \geq o$. Since, $a \geq o$, $a\delta(a) = -\delta(a) \geq o$, by Remark 2.1, $\delta(a) \leq o$. Therefore $\delta(a) = o$.

Theorem 3.7. Let $(N, +, \cdot, \leq)$ be an ordered strongly regular near-ring such that $a^2 \geq o$ for each $a \varepsilon N$, then $\delta(a) = o$, for each $a \varepsilon N$.

Proof. Left to the reader, using Theorem 3.6.

Corollary 3.8. If $(F, +, \cdot, \leq)$ is a strongly regular ordered near-field and δ is a positive derivation defined on F, then $\delta(x) = o$, for any $x \varepsilon F$.

Proof. It is straight-for-ward from proposition 2.6.

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