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A NOTE ON LOCALLY PRODUCT KAEHLERIAN METRICS WITH VANISHING CONFORMAL CURVATURE TENSOR FIELD

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§0. Introduction

In 1949, S.Bochner([1]) has introduced "Bochner curvature tensor" on a Kaehlerian manifold analogous to the Weyl conformal curvature tensor on a Riemannian manifold. However, we have not know the exact meaning of his tensor yet.

In 1989, H.Kitahara, K.Matsuo and J.S.Pak([2],[3]) defined a new tensor field on a Hermitian manifold which is conformally invariant and studied some properties of the new tensor field. They called this new tensor field "Conformal curvature tensor field."

In 1970, S. Tachibana and R.C. Liu([4]) studied locally product Kaehlerian metrics with vanishing Bochner curvature tensor.

The purpose of the present paper is to study Kaehlerian metrics with vanishing conformal curvature tensor field.

In section 1, we recall the conformal curvature tensor field on a Kaehlerian manifold. In section 2, we obtain the theorem for Kaehlerian metrics with vanishing conformal curvature tensor field.

§1. Preliminaries

We use throughout this paper the systems of indices as follows :

$A, B, C, D, \ldots = 1, 2, \ldots, 2n;$
$a, b, c, d, \ldots = 1, 2, \ldots, n;$
$a^* = n + a.$

The summation covention will be used with respect to those systems of indices.

Consider a complex n-dimensional Kaehlerian manifold M^n with metric

$$(1.1) \qquad ds^2 = g_{AB}\, dz^A\, dz^B,$$

where $\{z^a\}$ is a local complex coordinate and $z^{a^*} = \bar{z}^a$ is a conjugate of z^a.

As the metric is Kaehlerian, we have

$$(1.2) \qquad g_{ab} = g_{a^*b^*} = 0,\ g_{bd^*} = g_{b^*a} = \overline{g}_{a^*b} = \overline{g}_{ba^*},$$

and (1.1) becomes

$$ds^2 = 2g_{ab^*} dz^a dz^{b^*},$$

Moreover, the Christoffel symbols Γ^A_{BC} vanish except

$$(1.3) \qquad \Gamma^a_{bc} = g^{ad^*} \frac{\partial g_{bd^*}}{\partial z^c}$$

and their conjugates. As to the curvature tensor $R^A{}_{BCD}$, only the components of the form $R^a{}_{bcd^*}$, $R^a{}_{bc^*d}$ and their conjugates can be different from zero, and

$$(1.4) \qquad R^a{}_{bcd^*} = \frac{\partial \Gamma^a_{bc}}{\partial z^{d^*}}$$

hold good. The Ricci tensor $R_{AB} = -g^{CD} R_{ACDB}$ satisfies

$$R_{ab} = R_{a^*b^*} = 0,$$

(1.5)

$$R_{ab^*} = -g^{c^*d} R_{ac^*db^*},$$

and the scalar curvature $R = g^{AB} R_{AB}$ is $R = 2g^{ab^*}R_{ab^*}$.

A Kaehlerian manifold is called a space of constant holomorphic curvature if its curvature tensor satisfies

$$R_{ab^*cd^*} = -\frac{1}{2n(n+1)} R(g_{ab^*}g_{cd^*} + g_{ad^*}g_{cb^*}).$$

A conformal curvature tensor field B_0 on a Kaehlerian manifold is given by([2],[3])

(1.6)

$$B_{0,ab^*cd^*} = R_{ab^*cd^*} + \frac{1}{n}(g_{ab^*}R_{cd^*} + R_{ab^*}g_{cd^*})$$
$$-\frac{n+2}{2n^2(n+1)}R\, g_{ab^*}g_{cd^*} + \frac{1}{2n(n+1)}R\, g_{ad^*}g_{cb^*}$$

H. Kitahara, K. Matsuo and J.S. Pak([2],[3]) proved recently the following

Theorem A. Let M be a Kaehlerian manifold of complex dimension $n(n \geq 3)$. Then M is of constant holomorphic sectional curvature if and only if B_0 vanishes everywhere.

§2. Locally product Kaehlerian metrics

In this section we shall admit the following ranges of indices keeping the notation in §1.

$i, j, k, l, \ldots = 1, 2, \ldots, p$;

$i^* = i + n$;

$u, v, x, y, \ldots = p + 1, p + 2, \ldots, n$;

$$x^* = x + n.$$

Consider a Kaehlerian metric (1.1) of the form

(2.1) $$ds^2 = ds_1^2 + ds_2^2,$$

where $ds_1^2 = 2g_{ij^*}dz^i dz^{j^*}$ and $ds_2^2 = 2g_{xy^*}dz^x dz^{y^*}$ are Kaehlerian metrics of dimensions p and n-p, respectively.

For a metric of this type, we have

(2.2) $$R_{ij^*xy^*} = 0.$$

Now we assume that the conformal curvature tensor field B_0 with respect to the metric of the form (2.1) vanishes. Then from (1.6) and (2.2), it follows that

(2.3) $$R_{ij^*} = \frac{1}{2}\left\{\frac{n+2}{n(n+1)}R - \frac{1}{n-p}R_2\right\} g_{ij^*},$$

and

(2.4) $$R_{xy^*} = \frac{1}{2}\left\{\frac{n+2}{n(n+1)}R - \frac{1}{p}R_1\right\} g_{xy^*},$$

where R_1 and R_2 denotes the scalar curvature of ds_1 and ds_2 respectively.

From (2.3) or (2.4), we have

(2.5) $$\frac{n+2}{n(n+1)}R - \frac{1}{p}R_1 - \frac{1}{n-p}R_2 = 0.$$

on the other hand, (2.3) and $B_{0\,ij^*kl^*} = 0$ yields

(2.6)
$$R_{ij^*kl^*} = -\frac{1}{2n}\left\{\frac{n+2}{n(n+1)}R - \frac{2}{n-p}R_2\right\} g_{ij^*}g_{kl^*} - \frac{1}{2n(n+1)}R\, g_{il^*}g_{kj^*},$$

and (2.4) and $B_{0\,xy^*uv^*} = 0$ yields

(2.7)
$$R_{xy^*uv^*} = -\frac{1}{2n}\left\{\frac{n+2}{n(n+1)}R - \frac{2}{p}R_1\right\} g_{xy^*}g_{uv^*} - \frac{1}{2n(n+1)}R\, g_{xv^*}g_{uy^*}.$$

From (2.6), we obtain

$$R_{ij^*} = \frac{1}{2n}\left\{\frac{n+2+np}{n(n+1)}R - \frac{2}{n-p}R_2\right\} g_{ij^*}.$$

Thus we get

$$\frac{n+2+np}{n^2(n+1)}R - \frac{1}{p}R_1 - \frac{2}{n(n-p)}R_2 = 0. \tag{2.8}$$

From (2.5), (2.8) and $R - R_1 - R_2 = 0$, we see that

$$R = R_1 = R_2 = 0 \text{ when } p>1.$$

Thus, by (2.6) and (2.7), we obtain

$$R_{ij^*kl^*} = 0 \text{ and } R_{xy^*uv^*} = 0.$$

Hence we have

Theorem 2.1. There is no locally product Kaehlerian metrics with vanishing conformal curvature tensor field except for flat.

Combining Theorem A and Theorem 2.1, we obtain.

Corollary 2.2. There is no locally product Kaehlerian metrics with constant holomorphic sectional curvature except for flat.

References

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