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PETTIS MEAN CONVERGENCE OF WEAKLY MEASURABLE PETTIS INTEGRABLE MARTINGALES

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1. Introduction

Convergence of martingales of Bochner integrable functions with values in a Banach space has been studied by many authors. A martingale of functions with values in a Banach space with Radon-Nikodym property obeys the basic theory of convergence of scalar-valued functions.

In 1972, J.J. Uhl, Jr.[7] characterized Pettis mean convergent martingales of strongly measurable Pettis integrable functions.

In 1980, K. Musial[3] dealt with the properties of Banach space valued martingales of weakly measurable Pettis integrable functions. And he gave a martingale characterization of Banach spaces in which all measures of σ-finite variation have relatively norm compact ranges.

In this paper, we characterize the Banach space with Weak Radon-Nikodym property(WRNP) in terms of Pettis mean convergence of weakly measurable Pettis integrable martingales.

In the second section preliminary results are gathered. The last section is devoted to the characterization of Banach space with Weak Radon-Nikodym property in terms of Pettis mean convergence of weakly measurable Pettis integrable martingales.

2. Preliminaries

Throughout this paper (Ω, Σ, μ) be a finite measure space. X is called a Banach space with dual X^*. The symbol B(X) is the unit ball of X.

If Σ_0 is a sub-σ-algebra of Σ, then a function $f : \Omega \to X$ is weakly Σ_0-measurable iff the function x^*f is Σ_0-measurable for every $x^* \varepsilon X^*$.

A weakly Σ-measurable function is called weakly measurable.

A weakly Σ-measurable function $f : \Omega \to X$ is called Pettis integrable on Σ_0 iff there exists a set function $\nu : \Sigma_0 \to X$ such that

$$x^*\nu(E) = \int_E x^*f \, d\mu$$

for all $x^* \varepsilon X^*$ and $E \varepsilon \Sigma_0$. In that case we write

$$\nu(E) = (P) - \int_E f d\mu$$

ν is called the indefinite integral of f on Σ_0. f is called Pettis integrable iff it is Pettis integrable on Σ.

$|\nu|$ is called the variation of ν.

The space of all X-valued Pettis integrable functions on (Ω, Σ, μ) is denoted by $P(\Sigma,X)$ will denote the space obtained by identifying functions which are weakly equivalent, endowed the Pettis norm :

$$|f|_p = \sup\{\int_\Omega |x^*f| \, d\mu : x^* \varepsilon B(X^*)\}.$$

It is well known that

$$|f| = \sup\{ \|\int_E f d\mu\| : E \varepsilon \Sigma\}$$

defines an equivalent norm in $P(\Sigma,X)$.

The following extension property of Pettis integrable functions is found in[3].

Proposition 2.1 : Let Σ_0 be a sub-σ-algebra of Σ and $f : \Omega \to X$ be a weakly Σ_0-measurable function which is also Pettis integrable on Σ_0. Then f is Pettis integrable.

Definition 2.2 : If Σ_0 is a sub-σ-algebra of Σ, f is Pettis integrable on Σ_0 then g is called the Pettis conditional expectation of f with respect to Σ_0 if

(a) g is weakly Σ_0-measurable and

(b) (P)-$\int_E fd\mu = (P)$-$\int_E gd\mu$ for every $E\varepsilon\Sigma_0$. In that case we write $g = (P) - E(f|\Sigma_0)$.

Definition 2.3 : Given a directed set $(\Pi,<)$ and a family of σ-algebras $\Sigma_\pi \subset \Sigma$, $\pi\varepsilon\Pi$, the family $(f_\pi, \Sigma_\pi, \pi\varepsilon\Pi) \subset P(\Sigma,X)$ ~~forms and X-valued~~ Pettis martingale on (Ω, Σ, μ) if the following conditions are satisfied :

(a) If $\pi<\rho$, then $\Sigma_\pi \subsetneq \Sigma_\rho$;

(b) f_π is Pettis integrable on Σ_π ;

(c) If $\pi<\rho$, then $(P) - E(f_\rho \mid \Sigma_\pi) = f_\pi$.

Definition 2.4 : A Banach space X has the Weak Radon-Nikodym Property(WRNP) if for every finite complete measure(Ω, Σ, μ) and every μ-continuous measure $\nu : \Sigma\rightarrow X$ of σ-finite variation there exists $f\varepsilon P(\Sigma, X)$ such that

$$\nu(E) = (P) - \int_E fd\mu$$

for every $E\varepsilon\Sigma$.

The following theorem obtained Rybakov[5], will used in the proof of our main theorem.

Theorem 2.5 : Let(Ω, Σ, μ) be a finite complete measure space, let $\nu : \Sigma\rightarrow X^*$ be a μ-continuous vector measure of σ-finite variation. Then there exists a weak* measurable function $f: \Sigma \to X^*$ such that

$$\nu(E) = (w^*) - \int_E f d\mu,\ E \varepsilon \Sigma.$$

Definition 2.6 : A Banach space is said to X have the Compact Range Property if for every finite complete measure space(Ω, Σ, μ) and for every μ-continuous X-valued measure of σ-finite variation has norm compact range.

The following Proposition is found in [3].

Proposition 2.7 : If X has the Weak Radon-Nikodym Property then it has also the Compact Range Property.

3. Pettis mean convergence of Pettis martingales

For a martingale$(f_\pi, \Sigma_\pi, \pi \varepsilon \Pi) \subset P(\Sigma,X)$, define a vector measure $\nu_\pi(E) = (P) - \int_E f_\pi d\mu$ for each $E \varepsilon \Sigma_\pi$. From Proposition 2.1 we can obtain an extension to all of Σ, still denoted by ν_π. If f_π is Bochner integrable, then $|\nu_\pi|(\Omega) = (B) - \int \| f_\pi \| d\mu = \| f_\pi \|_1 < \infty$. Thus the finiteness of the set function $\sup_\pi |\nu_\pi|$ is the same as L_1-uniform boundedness of (f_π).

J.J. Uhl, Jr.[6] proved the following theorem.

Theorem 3.1: The following assertions are equivalent :

(a) X has the Radon-Nikodym property,

(b) For every finite measure space (Ω, Σ, μ) and every martingale$(f_\pi, \Sigma_\pi, \pi \varepsilon \Pi) \subset L_1(\Sigma,X)$ on (Ω, Σ, μ) which is uniformly integrable and the set function $\sup_\pi |\nu_\pi|$ is finite there exists and $f \varepsilon L_1(\Sigma,X)$ such that $\lim_{\pi \to \infty} \| f_\pi - f \| = 0$.

Definition 3.2 : The martingale $(f_\pi, \Sigma_\pi, \pi\varepsilon\Pi) \subset P(\Sigma,X)$ is said to be Pettis uniformly integrable if

$$\lim_{\mu(A)\to 0} \| (P) - \int_A f_\pi d\mu \| = 0 \text{ uniformly in } \pi\varepsilon\Pi.$$

The following Proposition is found in [3].

Proposition 3.3 : Let $(f_\pi, \Sigma_\pi, \pi\varepsilon\Pi)$ be a Pettis martingale defined on (Ω, Σ, μ). Define a set function $\nu : \bigcup_{\pi\varepsilon\Pi} \Sigma_\pi \to X$ by

$$\nu(E) = \lim_\pi \nu_\pi(E).$$

Then the following are equivalent :

(a) ν is μ-continuous and has relatively norm compact range.

(b) The martingale is Pettis-Cauchy in $P(\Sigma,X)$ and all measures ν_π, $\pi\varepsilon\Pi$ has relatively norm compact ranges.

The following theorem is the extension of theorem 2.2[8].

Theorem 3.4 : The following are equivalent :

(a) X has (WRNP).

(b) For every finite complete measure space (Ω, Σ, μ), every Pettis uniformly integrable martingale $(f_\pi, \Sigma_\pi, \pi\varepsilon\Pi) \subset P(\Sigma,X)$ converges in Pettis norm if set function $\sup_{\pi\varepsilon\Pi} |\nu_\pi|$ is σ-finite on (Ω, Σ_{π_0}) for some $\pi_0\varepsilon\Pi$.

Proof : In virtue of Proposition 2.1 we may assume, without loss of generality, that Σ is the completion of $\sigma(\bigcup_{\pi\varepsilon\Pi}\Sigma_\pi)$ with respect to the restriction of μ to $\sigma(\bigcup_{\pi\varepsilon\Pi}\Sigma_\pi)$. Define $\nu : \bigcup_\pi\Sigma_\pi \to X$ by $\nu(E) = \lim_\pi (P) - \int_E f_\pi d\mu$ for each $E\varepsilon\bigcup_\pi\Sigma_\pi$. Since $(f_\pi, \Sigma_\pi)_{\pi\varepsilon\Pi}$ is Pettis uniformly integrable,

$$\lim_{\mu(E)\to 0} \| \nu(E) \| = \lim_{\mu(E)\to 0} \sup_{\pi\varepsilon\Pi} \| (P) - \int_E f_\pi d\mu \| = 0$$

on $E\varepsilon U_\pi \Sigma_\pi$. Since $\sup_\pi | \nu_\pi |$ is σ-finite on (Ω, Σ_{π_0}) for some $\pi_0 \varepsilon \Pi$, there exists a partition $\{E_n\}$ of Ω into disjoint sets in Σ_{π_0} such that $\sup_\pi | \nu_\pi | (E_n) < \infty$ for all $n\varepsilon N$. For a fixed n, if $\alpha \subset U_\pi \Sigma_\pi$ is a finite partition of E_n then there exists $\pi_1 \leq \pi_0$ such that
$\alpha \subset \Sigma_{\pi 1}$. Consequently one has

$$\sum_{A\varepsilon\alpha} \| \nu(A) \| = \sum_{A\varepsilon\alpha} \| (P) - \int_A f_{\pi 1} d\mu \| = \sum_{A\varepsilon\alpha} \| \nu_{\pi 1}(A) \|$$

$$\leq | \nu_{\pi 1} | (E_n) \leq \sup_\pi | \nu_\pi | (E_n) < \infty.$$

Thus ν has σ-finite variation on $U_\pi \Sigma_\pi$. Hence, there exists a measure $\nu : \Sigma \to X$ being the unique extension of ν to the whole of Σ. ν is μ-continuous and of σ-finite variation. Since X has (WRNP), there exists an $f : \Omega \to X$ such that $\nu(E) = (P) - \int_E f d\mu$ whenever $E\varepsilon\Sigma$. Since $\nu(\Sigma)$ is a relatively norm compact set, it follows from Proposition 3.3 that
$(f_\pi, \Sigma_\pi)_{\pi\varepsilon\Pi}$ is Pettis-Cauchy in $P(\Sigma,X)$. Applying the theorem of Doob and Helms to the scalar-valued martingales $(x^* f_\pi, \Sigma_\pi)_{\pi\varepsilon\Pi}$, $x^* \varepsilon X^*$, we get the convergence $\lim_\pi \| f_\pi - f \|_p = 0$.

Conversely let (Ω,Σ,μ) be a fixed finite complete measure space and $F : \Sigma \to X$ be a μ-continuous vector measure of σ-finite variation. There exists a countable partition $\pi_0 = \{E_n\}$ of Ω such that $| F | (E_n) < \infty$. Let Π be the class of all partitions π of Ω into Σ which are refinements of π_0 then Π be a directed set by refinement.
For each $\pi\varepsilon\Pi$, define

$$f_\pi = \sum_{E\varepsilon\pi} \{ F(E)/\mu(E) \} \chi_E.$$

Let Σ_π be the σ-algebra generated by π. Then $(f_\pi,\Sigma_\pi)_{\pi\varepsilon\Pi}$ be a Pettis martingale in $P(\Sigma,X)$. Next we will show that $\sup_\pi | \nu_\pi | (E_n) < \infty$, for all n, so

$\sup_n |\nu_n|$ is σ-finite on (Ω,Σ_{n0}). For a fixed n, let $A_1,A_2,\ldots,A_m$ be a partition of E_n and $A_i\varepsilon\Sigma_n$.

Then we have

$$\sum_{i=1}^{m} \|\nu_n(A_i)\| = \sum_{i=1}^{m} \|(P) - \int_{A_i} f_n d\mu\| = \sum_{i=1}^{m} \|F(A_i)\|$$
$$\leq |F|(E_n)$$

so $\qquad |\nu_n|(E_n) \leq |F|(E_n)$ for all $\pi\varepsilon\Pi$.

Thus we have $\sup_n |\nu_n|(E_n) \leq |F|(E_n) < \infty$.

$(f_n,\Sigma_n)_{n\varepsilon\Pi}$ is Pettis uniformly integrable and Pettis-Cauchy in $P(\Sigma,X)$. It follows from Theorem 2.5 that there exists a weak* measurable function $f: \Omega\to X^{**}$ such that $x^*F(A) = \int_A x^* f d\mu$ for every $A\varepsilon\Sigma$ and $x^*\varepsilon X^*$. From the Cauchy condition for the martingale and the Doob and Helms theorem we can obtain the relation

$$\lim_n \sup \{\int_\Omega |x^*(f_n - f)| d\mu : x^*\varepsilon B(X^*)\} = 0.$$

Since by assumption there exists $g\varepsilon P(\Sigma,X)$ such that $\lim_n \|f_n - g\|_p = 0$, we get

$$x^*F(A) = \int_A x^* g d\mu$$

for every $A\varepsilon\Sigma$ and for all $x^*\varepsilon X^*$ and hence X has (WRNP).

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