THE COMPLETE RELATIONS OF TYPES FOR A HIGHER ORDER TYPE-THEORETIC LANGUAGE

Hyang Il Yi

1. Introduction

The Language $L_\tau$ under consideration is called "type-theoretic" because its syntax is based on Russell's simple theory of types, probably most closely resembling the version of type theory in Church (1940). $L_\tau$ will contain both constants and variables in syntax, and it will allow quantification over variables of any category.

We recall the concept of $L_\tau$ recursively.

(1) $e$ is a type.

(2) $t$ is a type.

(3) If $a$ and $b$ are any types, then $\langle a, b \rangle$ is a type.

(4) Nothing else is a type.

In other words,

$e$ is a term, $t$ is a formulas, $\langle e, t \rangle$ is a one-place predicates and $\langle e, \langle e, t \rangle \rangle$ is a two-place predicates.

2. Type model $D$ of a higher-order type language

Let's construct a type model $D$ of a higher-order type theoretic language $L_\tau$. Let $E$ be a singleton of type $e$. Starting from $D_e = \{t\}$ a chain
of approximations of a type model $D$ is built by defining

$$D_{n+1} = E + \langle D_n, D_n \rangle$$

where $+$ represents disjoint sum and $\langle D_n, D_n \rangle$ is the space of all continuous mappings from $D_n$ to $D_n$, and embedding each $D_n$ in $D_{n+1}$ by a suitable projection pair $\langle i_n, P_n \rangle$ of $D_n$ on $D_{n+1}$ where $i_n : D_n \rightarrow D_{n+1}$, $P_n : D_{n+1} \rightarrow D_n$ with the properties $P_n \circ i_n = id_{D_n}$, $i_n \circ P_n \subseteq id_{D_{n+1}}$.

If $d \in D_n$, we identify $d$ with $i_n(d) \in D$.

There we can assume

$$D \subseteq D_1 \subseteq \cdots \subseteq D_n \subseteq \cdots \subseteq D.$$ Let $d_n$ stand for $i_n \circ P_n(d)$. It holds

$$d_n = \lim_{\rightarrow} P_n(d) \subseteq d.$$ Also if $d \in D_n$ then $d_n = d$. Now we may take the type model $D$ of $L_r$ into account of the equational form

$$D = E + \langle D, D \rangle.$$ Defining a partial ordering $\leq$ on $D_n$ by $d \leq f$ if and only if $d(a) \leq f(a)$ for all $a \in D_n$, the set of all continuous functions from $D_n$ to $D_n$ is a complete partial ordered set and the disjoint sum of $E + \langle D_n, D_n \rangle$ is a complete one, too.

3. Complete relations on the model $D$

Definition 2.1

(1) A binary relation $R \subseteq D \times D$ is $\omega$-complete if and only if

$$\langle U(d^{(i)}) \rangle, \langle f^{(i)} \rangle \in R$$

whenever for all $i \in \omega$, $\langle d^{(i)}, f^{(i)} \rangle \in R$ where

$$\langle d^{(i)} \rangle, \langle f^{(i)} \rangle$$

are increasing chains in $D$.

(2) $R \subseteq D \times D$ is complete if and only if $R$ is $\omega$-complete and $(t, t) \in R$.

Proposition 2.1 The following properties of $D$ hold:

(1) $d_0 = t$. 

(2) If $a \in E$, then for all $n \geq 1$ $a = a_n$.
(3) If $d \in D$, $\langle d_n \rangle_{n\omega}$ is an increasing chain in $D$ and $d = \bigcup \langle d_n \rangle_{n\omega}$.
(4) If $f \in D(D)$, then $f_{n+1}(d) = f_{n+1}(d_n) \in D_n$.
(5) If $f \in D(D)$, then $(f(d_n))_n = f_{n+1}(d_n)$.

Let $K = \{k_1, k_2, \ldots, k_m\}$ be a set of basic predetermined types of $D$ and $\Phi$ a set of type variables. The set $T$ of types is defined by:

1. $K, \Phi \subseteq T$.
2. If $a, \beta \in T$ then $\beta(\alpha) \in T$.
3. If $a, \beta \in T$ then $\neg a \in T, \alpha \wedge \beta \in T, \alpha \vee \beta \in T$.
4. $\alpha \rightarrow \beta \in T$.

Now we define a relation $R(\alpha) \subseteq D \times D$ for each term $\alpha$ of the set $T^*$ of closed types. We set the $\alpha$ in the form

$$\alpha = Q_0 \zeta_1 \cdots Q_m \zeta_m \alpha'$$

where $\alpha'$ is either a basic type $K_i$ or a type of the logical forms. Let us define the scope size $l(\alpha) = n$.

1. $l(\forall \zeta \alpha) > l(\forall \zeta \alpha/\zeta)$
2. $l(\exists \zeta \alpha) > l(\exists \zeta \alpha/\zeta)$ for all types $\gamma$.

$R(\alpha)$ is built by successive approximations in the following way.

**Definition 2.2** Let $K, \cdots, K_n$ be complete relations over $D$.

1. $R(\alpha)_n \subseteq D_n \times D_n (\alpha \in T^*, n > 0)$ is defined by:
   1. $R(\alpha)_n = \{(t, t)\}$ for all $\alpha \in T^*$.
   2. $R(K)_n = K_n$.
   3. $(d_i, d_j) \in R(\beta(\alpha))_{n+1}$ if $d_i \in D(D)_{n+1}$ or $d_i = t(i = 1, 2)$ and for all $(f_i, f_j) \in R(\alpha)_n$, $(d_i(f_i), d_j(f_j)) \in R(\beta(\alpha))_{n-1}$ (if $d_i = t$, then $d_j(f_i) = t$).
   4. $(d_i, d_j) \in \forall R(\neg \alpha)_{n+1}$ if $d_i \in D(D)_{n+1}$ or $d_i = t(i = 1, 2)$ and for all $(f_i, f_j) \in \forall R(\neg \alpha)_n, (d_i(f_i), d_j(f_j)) \in R(\alpha)_n$. 

(e) \((d_i, d_j) \in R(\alpha \land \beta)_{n+1} \rightarrow (d_i, d_j) \in R(\alpha)_{n+1}\) and
\((d_i, d_j) \in R(\beta)_{n+1}.

(f) \((d_i, d_j) \in R(\alpha \lor \beta)_{n+1} \rightarrow (d_i, d_j) \in R(\alpha)_{n+1}\) or
\((d_i, d_j) \in R(\beta)_{n+1}.

(g) \((d_i, d_j) \in R(\alpha \rightarrow \beta)_{n+1} \rightarrow (d_i, d_j) \in R(\alpha)_{n+1}\) or \((d_i, d_j) \in R(\beta)_{n+1}.

(h) \((d_i, d_j) \in R(\alpha \rightarrow \beta)_{n+1} \rightarrow (d_i, d_j) \in R(\alpha)_{n+1}\) and
\((d_i, d_j) \in R(\beta)_{n+1}.

(i) \((d_i, d_j) \in R(\forall \psi \alpha)_{n+1} \rightarrow \text{for all } \gamma \in T^*,
\((d_i, d_j) \in R(\alpha[\psi/\gamma])_{n+1}.

(j) \((d_i, d_j) \in R(\exists \psi \alpha)_{n+1} \rightarrow (d_i, d_j) \in R(\alpha[\exists \psi \alpha/\alpha])_{n+1}.

(2) \((d, f) \in R(\alpha)_{n+1} \rightarrow \text{for all } n, (d_i, d_j) \in R(\alpha)_{n}.

Now the following results can be obtained from the definition 2.2.

**Theorem 2.1.** It holds the following:

(1) \(R(\alpha)_{n} \subseteq R(\alpha)_{n+1}.

(2) If \((d, f) \subseteq R(\alpha)_{n+1}, \text{ then } (d, f) \in R(\alpha)_{n}.

(3) \(R(\alpha)_{n} \subseteq R(\alpha).

**Proof.** The third assertion is an immediate consequence of (1) and (2). Let’s use the simultaneous induction on \(\iota(\alpha)\). For \(n = 0\) the proof is trivial. If \(\alpha\) is the one of basic types, i.e., \(\iota(\alpha) = 0\) then (1) follows by definition 2.2 (1) \(\sim\) (b) and (2) follows by proposition 2.1, (2).

Let us consider the case of formulations. Let \(\gamma = \beta(\alpha)\) (\(\iota(\gamma) = 0\)).

(1) Let \((d, f) \in R(\beta(\alpha))_{n}\) and take \((a, b) \in R(\alpha)_{n}.

We have \(d(a) = d(a_{n-1}), f(b) = f(b_{n-1})\) by proposition 2.1, (4). By (2)
We have \((a_{n-1}, b_{n-1}) \in R(\alpha)_{n-1}\) and therefore by (1)

\((d(a), f(b)) \in R(\beta(\alpha))_{n-1} \subseteq R(\beta(\alpha))_{n-1}.

Hence \((d, f) \in R(\beta(\alpha))_{n+1}.

(2) Let \((d, f) \in R(\beta(\alpha))_{n+1}.\) Take \((a, b) \in R(\alpha)_{n+1} \subseteq R(\alpha)_{n}\) by (1). Hence
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we have \((d(a), f(b)) \in R(\beta(\alpha))\)\(_n\).

And by (2), \(((d(a))\_n\_1, (f(b))\_n\_1) \in R(\beta(\alpha))\) \(\subseteq R(\beta(\alpha))\)\(_{n-1}\).

Now by proposition 2.1 (5), it holds
\[d_n(a) = (d(a))\_n\_1, f_n(b) = (f(b))\_n\_1.\]
Hence \((d_n, f_n) \in R(\beta(\alpha))\)\(_n\).

In the case of disjunction and conjunction, (1) and (2) follow by the definition 2.2 (1) \((e), (f)\). The proof of cases \((\text{\textasciitilde} \alpha, \alpha \rightarrow \beta)\) and \(\alpha \rightarrow \beta\) is similarly to the case of formulation \(\beta(\alpha)\). Finally the cases of quantification is proved by induction on \(I(\alpha)\) and the properties of scope size since
\[(d, f) \in R(\forall \xi \alpha), \text{ if and only if } \forall \gamma \in T^e, (d, f) \in R(\alpha[\gamma/\xi])\)\(_n\).

and
\[(d, f) \in R(\exists \xi \alpha), \text{ if and only if } (d, f) \in R(\alpha[\exists \xi \alpha/\xi])\)\(_n\).

REFERENCES

5. Lambek J., From Types to Sets, Advances in Math. 36. 113~64, 1980
of ICALP 74, Lecture Notes in Computer Science 14, 141~156, 1974.

Department of Applied Mathematics
National Fisheries University of Pusan
Pusan 608~737
Korea