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THE COMPLETE RELATIONS OF TYPES FOR A HIGHER ORDER TYPE-THEORETIC LANGUAGE

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1. Introduction

The Language L_T under consideration is called "type-theoretic" because its syntax is based on Russell's simple theory of types, probably most closely resembling the version of type theory in Church(1940). L_T will contain both constants and variables in syntax, and it will allow quantification over variabes of any category.

We recall the concept of L_T recursively.

(1) e is a type.

(2) t is a type

(3) If a and b are any types, then $\langle a,b\rangle$ is a type.

(4) Nothing else is a type.

In other words,

e is a term, t is a formulas, $\langle e,t\rangle$ is a one-place predicates and $\langle e,\langle e,t\rangle\rangle$ is a two-place predicates.

2. Type model D of a higher-order type language

Let's construct a type model D of a higher-order type theoretic language L_T. Let E be a singleton of type e. Starting from $D_0 = \{t\}$ a chain of approximations of a type model D is built by defining

$$D_{n+1} = E + \langle D_n, D_n \rangle$$

where + represents disjoint sum and $\langle D_n, D_n \rangle$ is the space of all continuous mappings from D_n to D_n, and embedding each D_n in D_{n+1} by a suitable projection pair (i_n, P_n) of D_n on D_{n+1} where $i_n : D_n \to D_{n+1}$, $P_n : D_{n+1} \to D_n$ with the properties $P_n \circ i_n = id_{D_n}$, $i_n \circ P_n \subseteq id_{D_{n+1}}$.
If $d \varepsilon D_n$, we identify d with $i_{n\infty}(d) \varepsilon D$.
There we can assume

$$D_o \subseteq D_1 \subseteq \cdots \subseteq D_n \subseteq \cdots \subseteq D.$$

Let d_n stand for $i_{n\infty} \circ P_{\infty n}(d)$. It holds

$$d_n = i_{n\infty} \circ P_{\infty n}(d) \subseteq d.$$

Also if $d \varepsilon D_n$, then $d_n = d$. Now we may take the type model D of L_T into account of the equational form

$$D = E + \langle D, D \rangle.$$

Defining a partial ordering $\leq$ on D_n by $d \leq f$ if and only if $d(a) \leq f(a)$ for all $a \varepsilon D_n$, the set of all continuous functions from D_n to D_n is a complete partial ordered set and the disjoint sum of $E + \langle D_n, D_n \rangle$ is a complete one, too.

3. Complete relations on the model D

Definition 2.1

(1) A binary relation $R \subseteq D \times D$ is ω-complete if and only if
$(\cup \langle d^{(i)} \rangle_{i\varepsilon\omega}, \cup \langle f^{(i)} \rangle_{i\varepsilon\omega}) \varepsilon R$
whenever for all $i \varepsilon \omega$, $(d^{(i)}, f^{(i)}) \varepsilon R$ where
$\langle d^{(i)} \rangle_{i\varepsilon\omega}$, $\langle f^{(i)} \rangle_{i\varepsilon\omega}$ are increasing chains in D.

(2) $R \subseteq D \times D$ is complete if and only if R is ω-complete and $(t,t) \varepsilon R$.

Proposition 2.1 The following properties of D hold :

(1) $d_o = t$.

(2) If $a \varepsilon E$, then for all $n \geq 1$ $a = a_n$.

(3) If $d \varepsilon D$, $\langle d_n \rangle_{n\varepsilon\omega}$ is an increasing chain in D and $d = U \langle d_n \rangle_{n\varepsilon\omega}$.

(4) If $f \varepsilon \langle D,D \rangle$, then $f_{n+1}(d) = f_{n+1}(d_n) \varepsilon D_n$.

(5) If $f \varepsilon \langle D,D \rangle$, then $(f(d_n))_n = f_{n+1}(d_n)$.

Let $K = \{k_1, k_2, \cdots, k_m\}$ be a set of basic predetermined types of D and Φ a set of type variables. The set T of types is defined by :

(1) K, $\Phi \subseteq T$.

(2) If α, $\beta \varepsilon T$ then $\beta(\alpha) \varepsilon T$

(3) If α, $\beta \varepsilon T$ then $\neg\alpha \varepsilon T$, $\alpha \wedge \beta \varepsilon T$, $\alpha \vee \beta \varepsilon T$,
$\alpha \rightarrow \beta \varepsilon T$ and $\alpha \leftrightarrow \beta \varepsilon T$.

(4) If $\alpha \varepsilon T$, $\alpha \not\varepsilon \Phi$, then $\forall \zeta \alpha \varepsilon T$ and $\exists \zeta \alpha \varepsilon T$ for all $\zeta \varepsilon \Phi$.

Now we define a relation $R(\alpha) \subseteq D \times D$ for each term α of the set T^o of closed types. We set the α in the form

$$\alpha = Q_1\zeta_1 \cdots\cdots Q_n\zeta_n\alpha'$$

where α' is either a basic type K_i or a type of the logical forms. Let us define the scope size $l(\alpha) = n$.

(i) $l(\exists\zeta\alpha) > l(\alpha[\exists\zeta\alpha/\zeta])$

(ii) $l(\forall\zeta\alpha) > l(\alpha[\gamma/\zeta])$ for all types γ.

R(α) is built by successive approximations in the following way.

Definition 2.2 Let $K_1, \cdots, K_n$ be complete relations over D.

(1) $R(\alpha)_n \subseteq D_n \times D_n(\alpha \varepsilon T^o, n > 0 \mid$ is defined by :

(a) $R(\alpha)_o = \{(t,t)\}$ for all $\alpha \varepsilon T^o$.

(b) $R(K_i)_{n+1} = K_i$.

(c) $(d_1,d_2) \varepsilon R(\beta(\alpha))_{n+1} \leftrightarrow d_i \varepsilon \langle D,D \rangle_{n+1}$ or $d_i = t (i = 1,2)$ and for all $(f_1,f_2) \varepsilon R(\alpha)_n$, $(d_1(f_1),d_2(f_2)) \varepsilon R(\beta(\alpha))_{n-1}$(if $d_i = t$, then $d_i(f_i) = t$).

(d) $(d_1,d_2) \varepsilon R(\neg\alpha)_{n+1} \leftrightarrow d_i \varepsilon \langle D,D \rangle_{n+1}$ or $d_i = t (i = 1,2)$ and for all ($f_1,f_2) \varepsilon R(\neg\alpha)_n, (d_1(f_1),d_2(f_2)) \notin R(\alpha)_n$.

(e) $(d_1,d_2)\varepsilon R(\alpha\wedge\beta)_{n+1}\leftrightarrow(d_1,d_2)\varepsilon R(\alpha)_{n+1}$ and $(d_1,d_2)\varepsilon R(\beta)_{n+1}$.

(f) $(d_1,d_2)\varepsilon R(\alpha\vee\beta)_{n+1}\leftrightarrow(d_1,d_2)\varepsilon R(\alpha)_{n+1}$ or $(d_1,d_2)\varepsilon R(\beta)_{n+1}$.

(g) $(d_1,d_2)\varepsilon R(\alpha\rightarrow\beta)_{n+1}\leftrightarrow d_i\varepsilon\langle D,D\rangle_{n+1}$ or $d_i = t(i = 1,2)$ and for all $(f_1,f_2)\varepsilon R(\alpha)_n$, $(d_1(f_1), d_2(f_2))\varepsilon R(\beta)_n$.

(h) $(d_1,d_2)\varepsilon R(\alpha\leftrightarrow\beta)_{n+1}\leftrightarrow(d_1,d_2)\varepsilon R(\alpha\rightarrow\beta)_{n+1}$ and $(d_1,d_2)\varepsilon R(\beta\rightarrow\alpha)_{n+1}$.

(i) $(d_1,d_2)\varepsilon R(\forall\zeta\alpha)_{n+1}\leftrightarrow$for all $\gamma\varepsilon T^{\circ}$, $(d_1,d_2)\varepsilon R(\alpha[\gamma/\zeta])_{n+1}$.

(j) $(d_1,d_2)\varepsilon R(\exists\zeta\alpha)_{n+1}\leftrightarrow(d_1,d_2)\varepsilon R(\alpha[\exists\zeta\alpha/\zeta])_{n+1}$.

(2) $(d,f)\varepsilon R(\alpha)\leftrightarrow$for all n, $(d_n,f_n)\varepsilon R(\alpha)_n$.

Now the following results can be obtained from the definition 2.2.

Theorem 2.1. It holds the following :

(1) $R(\alpha)_n\subsetneq R(\alpha)_{n+1}$.

(2) If$(d,f)\subsetneq R(\alpha)_{n+1}$, then $(d_n,f_n)\varepsilon R(\alpha)_n$.

(3) $R(\alpha)_n\subsetneq R(\alpha)$.

Proof. The third assertion is an immediate consequence of (1) and (2). Let's use the simultaneous induction on $l(\alpha)$. For n = 0 the proof is trivial. If α is the one of basic types, i.e., $l(\alpha) = 0$ then (1) follows by definition 2.2 (1)~(b) and (2) follows by proposition 2.1,(2).

Let us consider the case of formulations. Let $\gamma\equiv\beta(\alpha)$ ($l(\gamma) = 0$).

(1) Let $(d,f)\varepsilon R(\beta(\alpha))_n$ and take $(a,b)\varepsilon R(\alpha)_n$.

We have $d(a) = d(a_{n-1})$, $f(b) = f(b_{n-1})$ by proposition 2.1,(4). By(2) We have $(a_{n-1}, b_{n-1})\varepsilon R(\alpha)_{n-1}$ and therefore by (1)

$$(d(a),f(b))\varepsilon R(\beta(\alpha))_{n-2}\subsetneq R(\beta(\alpha))_{n-1}.$$

Hence $(d, f)\varepsilon R(\beta(\alpha))_{n+1}$.

(2) Let $(d,f)\varepsilon R(\beta(\alpha))_{n+1}$. Take $(a, b)\varepsilon R(\alpha)_{n-1}\subsetneq R(\alpha)_n$ by (1). Hence

we have $(d(a),f(b))\varepsilon R(\beta(\alpha))_{n-2}$.
And by (2), $((d(a))_{n-1}, (f(b))_{n-1})\varepsilon R(\beta(\alpha))_{n-2} \subsetneq R(\beta(\alpha))_{n-1}$.
Now by proposition 2.1 (5), it holds

$$d_n(a) = (d(a))_{n-1},\ f_n(b) = (f(b))_{n-1}.$$

$$\text{Hence}(d_n,f_n)\varepsilon R(\beta(\alpha))_n.$$

In the case of disjunction and conjunction, (1) and (2) follow by the definition 2.2 (1)~(e), (f). The proof of cases $\neg\alpha$, $\alpha\rightarrow\beta$ and $\alpha\leftrightarrow\beta$ is similarly to the case of formulation $\beta(\alpha)$. Finally the cases of quantification is proved by induction on $l(\alpha)$ and the properties of scope size since

$(d,f)\varepsilon R(\forall\zeta\alpha)_n$ if and only if $\forall\gamma\varepsilon T^{\circ}$, $(d,f)\varepsilon R(\alpha[\gamma/\zeta])_n$.
and
$(d,f)\varepsilon R(\exists\zeta\alpha))_n$ if and only if $(d,f)\varepsilon R(\alpha[\exists\zeta\alpha/\zeta])_n$.

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