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On the converse of Schur's lemma

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It is well known by Schur's lemma that the endomorphism ring of an irreducible module is a division ring. But there exist a ring and a non-irreducible R-module M such that the endomorphism ring $End_R(M)$ of M is a division ring, thereby the converse of Schur's lemma does not always hold.

So it might be quite interesting to observe the situation for which the converse of Schur's lemma is valid, that is, a given R-module M is irreducible whenever the endomorphism ring $End_R(M)$ is a division. For the convenience we denote this property by (CS), so that a ring R has the property (CS) if and only if a given R-module M is irreducible whenever the endomorphism ring $End_R(M)$ is a division ring.

Initially Hirano and Park [6] observed a certain class of rings with the property (CS), for example they proved that semiprime Goldie rings with (CS) are semi-simple Artinian. Also they showed that von Neumann regular PI rings enjoy the property (CS). By this fact a large class of rings may have the property (CS).

Motivated by their result in [6], we consider in this paper semiprime PI rings with the property (CS). Indeed, we show that for a semiprime PI ring with (CS) R is abelian regular if and only if $Ra=aR$ for each a in R. So by this result every commutative semiprime ring with (CS) is automatically von Neumann reuglar.

We begin with observing the definitions of some kind of regular rings.

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Let R be a ring and x an element of R. Then an element x is said to be *regular* if there exists an element y of R such that $xyx=x$, while x is said to be *right(or left) regular* if there exists y of R such that $x^2y=x$ (or $yx^2=x$). Furthermore, we call x a *strongly regular* if it is both right regular and left regular.

Definition 1. A ring R is said to be *regular, right regular, left regular,* or *strongly regular* if for each element x in R, x is regular, right regular, left regular, strongly regular element.

For example, any direct product of division rings is regular, right or left regular, strongly regular. Also the endomorphism ring of a vector space over a division ring is regular.

Lemma 2. Let x be a strongly regular element of a ring R. Then there exists one and only one element y in R such that $xy=yx$, $x^2y(=yx^2)=x$ and $xy^2(=y^2x)=y$, and in particular, x is regular. For any element r of R with $x^2r=x$, y coincides with xr^2. Moreover, y commutes with every element which is commutative with x.

Proof. Let a and b be two elements of R such that $x^2a=x$, $bx^2=x$. Then

(1) $xa=bx^2a=bx$ so that

(2) $xa^2=bxa=b^2x$. From (1) we have also

(3) $xax=bx^2=x=x^2a=xbx$. Now put $y=xa^2$. It follows then from (1), (2), (3) that $xy=xbxa=xa=bx=bxax=xa^2x=yx$, $x^2y=xyx=xax=x$, $xy^2=bxy=bxa=y$, as desired. Suppose next that y' be any element which satisfies the same equalities as y ; $xy'=y'x$, $x^2y'=x$, $xy'^2=y'$. Then by replacing a, b in (2) by y,y' respectively, we get $y=xy^2=y'^2x=y'$, showing the uniqueness of y.

For the proof of the last assertion, let r be any element of R such that

$xr=rx$. Then we have first $yxr=yrx=yrx^2y=yx^2ry=xry=rxy$, i.e r commutes with $yx=xy$. It follows from this now $yr=y^2xr=yryx=yxry=ryxy=rx$ and this completes our proof.

Let us call an element x of a ring R *π-regular, right π-regular,* or *left π-regular* if a suitable power of x is regular, right regular, or left regular, respectively. Futhermore, we call an element x of R *strongly π-regular* if it is both right π-regular and left π-regular. Now it can readily be seen that a power x^n of x is right(or left) regular if and only if there exists an element y of R such that $x^{n+1}y=x$(or $yx^{n+1}=x^n$).

Definition 3. A ring R is said to be *π-regular, right π-regular, left π-regular* or *strongly π-regular* if every element of R is π-regular, right π-regular, left π-regular or strongly π-regular

Every regular, right regular, left regular, strongly regular ring is π-regular, right π-regular, left π-regular, strongly π-regular.

Lemma 4. Let a and b in a ring R satisfy $x^{n+1}a=x^n$, $bx^{m+1}=x^m$ for some m, n>0. Then they satify $x^{m+1}a=x^m$, $bx^{n+1}=x^n$, too.

Proof. When $m \geqq n$, $x^{m+1}a=x^m$ follows immediately from $x^{n+1}a=x^n$. Suppose now $m<n$. Then $x^m=bx^{m+1}$ implies $x^m(=b^2x^{m+2}=\cdots)=b^{n-m}x^n$, and so we obtain $x^{m+1}a=b^{n-m}x^{n+1}x=b^{n-m}x^n=x^m$. Similarly, we can verify the validity of $bx^{n+1}=x^n$.

Now we prove

Lemma 5. Let x be a strongly π-regular element of a ring R. Suppose that x^n is right regular element. Then x^n is strongly regular, and moreover there exists an element y of R such that $xy=yx$ and $x^{n+1}y=x^n$.

Proof. That x^n is strongly regular is an immediate consequence of Lemma 4. Now from Lemma 2, it follows that there exists an element y of R such that $x^{2n}y=x^n$ and y commutes with every element which is commutative with x^n, however the latter condition implies that $xy=yx$ because x commutes x^n. Denoting $x^{n-1}y$ again by y, y is evidently the desired element.

Now we define the *index* of a strongly π-regular element x as the least integer n such that x^n is right regular. By Lemma 4, the index n is characterized also as the least integer such that x^n is left regular.

Definition 6. The element x of a ring R is nilpotent if $x^n=0$ for some positive integer n.

Lemma 7. Let x be strongly π-regular element of index n, and y an element such that $xy=yx$ and $x^{n+1}y=x^n$(as in Lemma 5). Then $x\text{-}x^2y$ is a nilpotent element of index n.

Proof See [2, Lemma 4]

Theorem 8. Let R be a PI ring. Then the following are equivalent ;

a) R is π-regular.
b) Each prime ideal of R is primitive.
c) Each prime ideal of R is maximal.
d) R is left(right) π-regular.
e) Each prime factor ring of R is von Neumann regular.

Proof. See [4, Theoren 2.3]

Now we return to the main discussion.

Theorem 9. Semiprime right Goldie ring with (CS) is semisimple Artinian.

Proof. See [6, Theorem 2]

Corollary 10. PI ring with (CS) is π-regular

Proof. Assume that R is a PI ring with (CS). Then for a prime ideal P of R, R/P is Goldie by Posner's Theorem. Now it is easily checked that every factor ring of R also satisfies the condition (CS). Particularly, R/P has (CS). So by Theorem 9, R/P is simple Artinian. Hence by Theorem 8, R is π-regular.

Theorem 11. Let R be a commutative ring. Then the following conditions are equivalent :

a) R satisfies the property(CS).

b) R is π-regular.

Proof. See [6, Theorem 4]

By Corollary 10 and Theorem 11, it may be suspected that von Neumann regular rings satisfy the property (CS). Unfortunately, following example erases the possibility for von Neumann regular rings having the property (CS).

Example 12. [Hirano and Park]. For a division ring D, let V be a left vector space over D. Then the left self injective von Neumann regular ring $R=End_D(V)$ satifies the property (CS) if and only if V is finite dimensional vector space over D. In fact, if V is finite dimensional, then obviously R has the property (CS). Observe that V is an irreducible as a right R-module V. Also we have that $V=eR$ for some $e=e$ in R. Now let E(V) be the injective envelope of the right R-module V. Then obviously $E(V)=eQ$, where Q is the right maximal quotient ring of R. Now $End_R(E(V))=eQe$, and eQe is the right maximal quotient ring of eRe. But since $eRe=D$, we have that $eRe=eQe$. So by the property (CS), $E(V)$ is irreducible R-module and hence $E(V)=V$, i.e., $V=eR$ is an injective R-module. Therefore by [3, Proposition 19.46], V is finite dimensional over D

By this suggestive example, the following question was naturally raised by Hirano and Park[6] ; "Is a semiprime PI ring with the property (CS) von Neumann regular ? " But there is a counterexample for this question by Huh. So we prove that, for a semiprime PI ring R with (CS), R is abelian regular if and only if $Ra=aR$ for each a in R.

For our main results, we need some definitions and well known standard facts.

Definition 13. A regular ring is said to be *abelian* if all idempotents in R are central.

Obviously, any commutative regular ring is abelian, and any direct product of division rings is abelian. On the other hand, the endomorphism ring of a vector space V over a division ring is abelian only when the dimension of V is at most 1.

Lemma 14. If e is an idempotent in a semiprime ring R, then the following conditions are equivalent ;

a) e is central in R.
b) e commutes with every idempotent in R.
c) eR is a two-sided ideal of R.
d) Re is a two-sided ideal of R.
e) $(1-e)\ Re=0$.
f) $eR(1-e)=0$.

Proof a)⇒c) is trivial

c⇒e) ; Since eR is a left ideal, Re⊂eR, whence (1-e) Re=0.

e)⇒a) ; Since (1-e) Re=0, we see that Re⊂eR, whence eR is a left ideal of R. Then eR(1-e) is a left ideal of R such that $(eR(1-e))^2=0$, hence eR(1-e)=0. Given any r in R, we have er(1-e)=0 as well as (1-e)re=0, whence er=ere=re. Hence e is central.

a)⇔d)⇔f), by symmetry.

a)⇔b) is obvious.

b)⇔e) ; Given any x in (1-e) Re, we see that e+x is an idempotent, hence e commutes with e+x. Then e commutes with x so that x=xe =ex=0. Therefore (1-e) Re=0.

Theorem 15. For a regular ring R, the following conditions are equivalent ;

a) R is abelian.

b) R/P is a division ring for all prime ideals P of R.

c) R has no nonzero nilpotent elements

d) All right (left) ideals of R are two-sided.

e) every nonzero right (left) ideal of R contains a nonzero central idempotents.

Proof. See [5, Theorem 3.2].

Theorem 16. A ring R is strongly regular if and only if it is abelian regular.

Proof. See [5, Theorem 3.5]

Theorem 17. Let R be a semiprime PI ring with (CS). Then for each element x in R, Rx=xR if and only if R is abelian regular.

Proof. Suppose that R is abelian regular. Then for each x in R, there exists an element y in R such that $xyx=x$ and $xy=yx=e$ is an idempotent element in R. Since R is abelian regular, $Rx=Re=eR=xyR\subset xR$. Similarly, $xR\subset Rx$, and so $Rx=xR$ for each x in R.

Conversely, suppose that $Rx=xR$ for each x in R. Since R is a semiprime PI ring with (CS) for any prime ideal P, R/P is a prime PI ring. According to Posner's Theorem, R/P is a left and right Goldie ring. Then R/P is a prime right Goldie ring with (CS). Hence R/P is a simple Artinian by Theorem 9. Therefore R/P is von Neumann reuglar. Since R is PI and each prime factor ring of R is von Neumann regular, R is left and right π-regular by Theorem 8. In the other words, R is strongly π-regular. We will claim that R is a reduced ring. For each x in R with $x^n=0$ for some positive integer n, Rx is nilpotent ideal because $Rx=xR$ for each x in R. Since R is semiprime, $Ra=0$ and so $a=0$. Hence R is reduced ring. But since R is strongly π-regular, for each element x in R there exists y in R such that $xy=yx$, $x^{n+1}y=x^n$ and $x-x^2y$ is a nilpotent element of index n by Lemma 7. Hence $x-x^2y=0$ and so $x=x^2y$ for each element x in R. Therefore R is strongly regular and so R is abelian regular by Theorem 16.

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