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Generalized Special Linear Group

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1. Introduction

Let K be an algebraically closed field of characteristic $p > 0$ and F be a subfield of K. The group SL(n, F) of all $n \times n$ matrices with determinant 1, over the field F, is a member of a large, important family of groups which arise naturally as covering groups of certain subgroups of automorphism groups of simple Lie algebras. The structure and representations of these groups largely depend on those of the corresponding Lie algebras. The standard reference for the study of these groups are Borel [1], Steinberg [7], Carter [2] and Humphreys [3]. Representations of these groups have been discussed by Humphreys [4], Jeya Kumar [5], Srinivasan [6] and by this author in [8], [9], [10] and [11]. In this note we try to see what happens to these groups if we take the get of all $n \times n$ matrices over the field F with determinant ± 1, when $n = 2$. We restrict ourselves to the structure of the group. Representation of these groups will be discussed elsewhere.

2. Generalized Special Linear Group

Consider the set of all $n \times n$ matrices over a field F, with determinant ± 1. Denote this set by GSL(n, F). Clearly GSL(n, F) forms a group under matrix multiplication. SL(n, F), the special linear group, of all $n \times n$ matrices over the field F, with determinant 1 is a subgroup of GSL(n, F). We call GSL(n, F) as the Generalized Special Linear Group of matrices of order n over F.

3. Generators for the Generalized Special Linear Group GSL(2,F)

Let $x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $x_{-\alpha}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $x_\beta(t) = \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix}$.
$x_{-\beta}(t) = \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}$, $t \in F$. Clearly these elements belong to GSL(2, F).
Further $x_\alpha(t)$, $x_{-\alpha}(t)$, $t \in F$ generate SL(2, F).

Now consider any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ GSL(2, F). Then $ad-bc=\pm 1$.
If $ad-bc = +1$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ SL(2, F) and therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is generated by $x_\alpha(t)$ and $x_{-\alpha}(t)$, $t \in F$. But

$$x_{-\alpha}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t & -1 \end{pmatrix} = x_{-\beta}(0) \quad x_{-\beta}(-t)$$

Therefore $x_{-\alpha}(t)$ is generated by $x_{-\beta}(s)$, $s \in F$. Therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad-bc = 1$ is generated by $x_\alpha(t)$ and $x_{-\beta}(t)$, $t \in F$.

Let $ad-bc = -1$.

Case (i) Let $c \neq 0$. Now,

$$\begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_2 & -1 \end{pmatrix} = \begin{pmatrix} 1+t_1t_2 & -t_1 \\ t_2 & -1 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_2 & -1 \end{pmatrix} \begin{pmatrix} 1 & t_3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+t_1t_2 & -t_1 \\ t_2 & -1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+t_1t_2 & t_3+t_1t_2-t_1 \\ t_2 & t_2t_3-1 \end{pmatrix}$$

Let $t_2=c$ and choose t_1 and t_3 such that $1+t_1t_2=a$

and $t_2t_3-1=d$. Then since $\det\begin{pmatrix}1+t_1t_2 & t_3+t_1t_2-t_1\\ t_2 & t_2t_3-1\end{pmatrix}$
$=\det\begin{pmatrix}1 & t_1\\ 0 & 1\end{pmatrix}\cdot\det\begin{pmatrix}1 & 0\\ t_2 & -1\end{pmatrix}\cdot\det\begin{pmatrix}1 & t_3\\ 0 & 1\end{pmatrix}=1\cdot(-1)\cdot 1=-1.$
$(1+t_1t_2)\ (t_2t_3-1)-t_2(t_3+t_1t_2-t_1)=-1$ gives $a\cdot d-c(t_3+t_1t_2-t_1)=-1$.

Hence $b=(t_3+t_1t_2-t_1)$ ($\because ad-bc=-1$). Therefore
$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ t_2 & -1\end{pmatrix}\begin{pmatrix}1 & t_3\\ 0 & 1\end{pmatrix}$ where
t_1, t_2, t_3 are given by

$$\left.\begin{aligned} t_2&=c\\ 1+t_1t_2&=a\\ t_2t_3-1&=d\end{aligned}\right\}\qquad (1)$$

Thus $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=x_\alpha(t_1)\ x_{-\beta}(t_2)\ x_\alpha(t_3)$ where t_1, t_2, t_3 are given by (1). Thus $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is generated by $x_\alpha(t)$, $x_{-\beta}(t)$, $t\in F$.

Case (ii) Let $b\neq 0$ Now,
$\begin{pmatrix}1 & 0\\ t_1 & 1\end{pmatrix}\begin{pmatrix}1 & t_2\\ 0 & -1\end{pmatrix}=\begin{pmatrix}1 & t_2\\ t_1 & t_1t_2-1\end{pmatrix}$ and hence
$\begin{pmatrix}1 & 0\\ t_1 & 1\end{pmatrix}\begin{pmatrix}1 & t_2\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ t_3 & 1\end{pmatrix}=\begin{pmatrix}1 & t_2\\ t_1 & t_1t_2-1\end{pmatrix}\begin{pmatrix}1 & 0\\ t_3 & 1\end{pmatrix}$
$=\begin{pmatrix}1+t_2t_3 & t_2\\ t_1+(t_1t_2-1)t_3 & t_1t_2-1\end{pmatrix}.$

Take $t_2=b$ and choose t_1 and t_3 such that $1+t_2t_3=a$, $t_1t_2-1=d$,
Then, since $\det\begin{pmatrix}1+t_2t_3 & t_2\\ t_1+(t_1t_2-1)t_3 & t_1t_2-1\end{pmatrix}=-1$
$(1+t_2t_3)\ (t_1t_2-1)-t_2\ t_1+(t_1t_2-1)t_3=-1$

i.e.

$ad - b\ t_1 + (t_1 t_2 - 1)t_3 = -1$

and since $ad - bc = -1$ it follows that $b = t_1 + (t_1 t_2 - 1)t_3$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_3 & 1 \end{pmatrix}$$

$$= x_{-\alpha}(t_1)\ x_\beta(t_2)\ x_{-\alpha}(t_3) \text{-------}(2)$$

But we have already noted that $x_{-\alpha}(t)$ is generated by $x_{-\beta}(s)$, $s \varepsilon F$. Also,

$$x_\beta(t) = \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = x_\alpha(-t)\ x_{-\beta}(0).$$

Hence $x_\beta(t)$ is generated by $x_\alpha(t)$, $x_{-\beta}(t)$, $t \varepsilon F$. Therefore by (2)

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is generated by $x_\alpha(t)$, $x_{-\beta}(t)$, $t \varepsilon F$.

Case(iii) If both $b=0$ and $c=0$ then $ad - bc = -1$ gives $a \neq 0$ and $d = -a^{-1}$. Therefore

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -a^{-1} \end{pmatrix} = \begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and by what we have seen in case (i) both the elements on the right hand side and therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are generated by $x_\alpha(t)$ and $x_{-\beta}(t)$, $t \varepsilon F$.

Thus to sum up we have ,

Theorem : GSL(2, F) is generated by $x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $x_{-\beta}(t) = \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}$, $t \varepsilon F$.

Recall that SL(2, F) is also generated by $x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t\ \varepsilon\ F$ and $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Similar to this result, we have for GSL(2, F).

Theorem : GSL(2, F) is generated by $x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t \varepsilon F$ and $\omega' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Proof : By the previous theorem, GSL(2, F) is generated by

$x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $x_{-\beta}(t) = \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}$. But,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1-t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1-t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1-t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}$$

Therefore

$$x_{-\beta}(t) = x_\alpha(1)\omega^1 \; x_\alpha(-1) \; \omega^1 \; x_\alpha(1-t) \; \omega^1$$

Therefore GSL(2, F) is generated by $x_\alpha(t)$, $t \varepsilon F$ and ω^1. Hence the theorem.

4. Order of the generalized special linear group

When F is the finite field F_q, $q=p^n$ of finite characteristic $p > 0$, GSL(2, F_q) is a finite group. Now we find its order. Recall that SL(2, F_q) is of order $q(q^2-1)$. This can be proved as follows. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any element in SL(2, F_q), then a can take any of the q values in F_q. Case (i) When a takes $q-1$ nonzero values in F_q, b can take any of the q values and c can take any of the q values. But once a, b, c are fixed, since $ad-bc=1$, d is fixed and therefore can take only one value. Thus we have $(q-1)\ q \cdot q \cdot 1=(q-1)q^2$ choices for a, b, c, d in this case.

Case(ii) When a takes the value 0, b can take only $q-1$ nonzero values since if b is also 0 then $ad-bc=0$ which is not possible. But then, since $ad-bc=1$, $-bc=1$ and therefore $c=-b^{-1}$ i.e., for a given choice of b, c is fixed. But d can take any of the q values. So we have in all $1 \cdot (q-1) \cdot q \cdot 1=(q-1)q$ choices for a, b, c, d in this case.

Thus in all there are

$$
\begin{aligned}
(q-1)q^2+(q-1)q &= (q-1)\ (q^2+q) \\
&= (q-1)\ (q+1)q \\
&= (q^2-1)q
\end{aligned}
$$

Choices for a, b, c, d. Therefore the order of $SL(2, F_q)$ is $q(q^2-1)$.

But in case of $GSL(2, F_q)$ in case(i), once a, b, c are fixed, for each choice of a, b, c the element of has two values since $ad-bc=\pm 1$.

Therefore there are $2(q-1)q^2$ choices in all for a, b, c, d. In case(ii) again, when $a=0$, b can take any of the $(q-1)$ nonzero values. Then c can take only two values namely $\pm b^{-1}$. But d can take any of the q values. So we have again $2(q-1)q$ choices for a, b, c, d. Thus in all there are

$$
\begin{aligned}
2(q-1)q^2+2(q-1)\ q &= 2(q-1)\ (q^2+q) \\
&= 2(q-1)\ (q+1)q \\
&= 2(q^2-1)q
\end{aligned}
$$

elements in $GSL(2, F_q)$. Thus we have proved.

Theorem : $0(GSL(2, F_q)) = 2(q^2-1)q$.

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