On Singular Compactifications

Keun Park and Je Yoon Lee

1. Introduction

In [8], J.P.Guglielmi proposed the open question under which condition on $X$, $\alpha X$ is equal to $\sup\{X \cup S(f) \mid \alpha X \supseteq \cup_{f \in S} S(f)\}$ for any compactification $\alpha X$ of $X$. Since $\alpha X \supseteq \cup_{f \in S} S(f)$ if and only if $f \in S^*$, J.P.Guglielmi's problem can be restated as follows: Under which condition on $X$, is $\alpha X$ equal to $\sup\{X \cup S(f) \mid f \in S^*\}$ for any compactification $\alpha X$ of $X$? R.E.Chandler[4] showed that if $X$ is non-pseudocompact, then the Stone-Cech compactification of $X$ is equal to $\sup\{X \cup S(f) \mid f \in S^*\}$ and also he showed that if $X$ is a retractive space, then the Stone-Cech compactification of $X$ is equal to $\sup\{X \cup S(f) \mid f \in S^*\}$.

In this note, we give some conditions under which $\alpha X$ is equal to $\sup\{X \cup S(f) \mid f \in S^*\}$ for any compactification $\alpha X$ of $X$, and give some examples.

2. Singular compactifications

Throughout this note, the topological space $X$ is assumed to be locally compact and all topological spaces are assumed to be Hausdorff.

In [2], the singular set $S(f)$ is defined as $\bigcap \{\text{Cl}(f(X-F)) \mid F \text{ is compact in } X\}$ for a continuous map of $X$ to $Y$, where "Cl" denotes the closure operator. And $f$ is called singular if $S(f) = Y$. For a singular

The work supported by the Korean Ministry of Education grants 1989.
map \( f \) of \( X \) to a compacce space \( Y \), a Hausdorff compactification, which is called a singular compactification, is defined as follows:

Let the open sets in \( X \) be as they were in the original topology on \( X \). For \( p \in Y \), a basic neighborhood of \( p \) is defined to be any set of the form \( U \cup f^{-1}(U) \) where \( U \) is an open neighborhood of \( p \) in \( Y \). This topological space on \( X \cup S(f) = X \cup Y \) is denoted by \( X \cup S(f) \) (\([4],[5]\)).

We will denote \( C^*(X) \) the set of all continuous and bounded map of \( X \) to \( \mathbb{R} \) (the real line with the usual topology) and \( C^* \) the set of all continuous and bounded maps of \( X \) to \( \mathbb{R} \) with the extension to a Hausdorff compactification \( \alpha X \) of \( X \). Let \( S^* = \{ f \in C^*(X) \mid f \) is singular \( \} \) \( S^n = \{ f \in C^* \mid f \) is singular \( \} \) for any compactification \( \alpha X \) of \( X \). Then, it is true that \( S^* \neq \emptyset \) and \( S^n \neq \emptyset \) since all constant map are singular.

In 1982, B.J. Ball and S. Yokura([1]) introduced the concept of determining set. For each subset \( F \) of \( C^*(X) \), let \( K(F) \) be the subfamily of the family of all Hausdorff compactifications of \( X \) such that \( K(F) \) consists of all compactifications of \( X \) to which each element of \( F \) can be extended. If \( K(F) \) has the smallest element \( \alpha X \), then \( \alpha X \) is said to be determined by \( F \), and denoted by \( \alpha_F X \). It is easily shown that every subset of \( C^*(X) \) determines a compactification of \( X \) if \( X \) is locally compact. And then, he showed the following theorem.

**Theorem 2.1** [1]. For any subset \( F \) of \( C^*(X) \) and any compactification \( \alpha X \) of \( X \), the followings are equivalent.

1. \( F \) determines a compactification and \( \alpha_F X = \alpha X \).
2. \( F \subset C^* \) and \( F^* \) separates points of \( \alpha X - X \) where \( F^* \) is the subset \( \{ f^* \in C^*(\alpha X) : f^* \) is an extension of \( f \) and \( f \in F \} \) of \( C^*(\alpha X) \).
Theorem 2.2. Let $f \in C^*$. Then, $X \cup S(f)$ is the smallest compactification of $X$ to which $f$ can be extended.

Proof. Let $\alpha X = X \cup S(f)$ and define $f^* : \alpha X \to S(f)$ given by $f^*(x) = f(x)$ if $x \in X$ and $f^*(x) = x$ otherwise. Then, it is easily shown that $f^*$ is a continuous extension of $f$ and it is trivial that $f^*$ separates points of $\alpha X - X$. Since $X$ is locally compact, $\alpha X$ is the smallest compactification of $X$ to which $f$ can be extended, by theorem 2.1.

Corollary 2.3. For any subset $F$ of $S^*$, $\sup\{X \cup S(f) \mid f \in F\}$ is the smallest compactification of $X$ to which $F$ can be extended.

Proof. Since $\sup\{X \cup S(f) \mid f \in F\} \geq X \cup S(f)$ for any $f$ in $F$, it follows that $F$ can be extended to $\sup\{X \cup S(f) \mid f \in F\}$. If $\alpha X$ is a Hausdorff compactification of $X$ to which $F$ can be extended, then for any $f$ in $F$, $\alpha X \geq X \cup S(f)$ by theorem 2.2. Hence, we have that $\alpha X \geq \{X \cup S(f) \mid f \in F\}$, and so $\sup\{X \cup S(f) \mid f \in F\}$ is the smallest compactification of $X$ to which $F$ can be extended.

Using above Theorem 2.1 and Corollary 2.3, we obtain the following corollary which is proved by R.E. Chandler and G.D. Faulkner[4].

Corollary 2.4. $\alpha X = \sup\{X \cup S(f) \mid f \in S^*\}$ if and only if $f^* \in C^*(\alpha X)$ $| f^*$ is an extension of $f$ and $f \in S^* |$ separates points of $\alpha X - X$.

3. Main results

For a completely regular space $X$, $X$ is called a retractive space if and only if there is a retraction of $\beta X$ onto $\beta X - X$. 
Theorem 3.1[4]. If \( X \) is a retractive space, then \( X \) is locally compact and pseudocompact.

Lemma 3.2[8]. For a compactification \( \alpha X \) of \( X \), \( \alpha X \) is a singular compactification if and only if there exists a retraction of \( \alpha X \) onto \( \alpha X - X \).

Theorem 3.3[4]. If \( X \) is non-pseudocompact, then

\[
\beta X = \sup \{ X \cup S(f) \mid f \in S^* \}.
\]

We generalize the theorem of R.E.Chandler: "If \( X \) is a retractive space, then \( \beta X = \sup \{ X \cup S(f) \mid f \in S^* \} \)." and give some answers on the problem given by J.P.Guglielmi.

Theorem 3.4. If \( \alpha X \) is a singular compactification, then \( \alpha X \) is equal to \( \sup \{ X \cup S(f) \mid f \in S^* \} \).

Proof. Suppose that \( h \) is a singular map of \( X \) to \( Y \) where \( S(h) = Y \) is compact and \( \alpha X = X \cup hS(h) \). Since the compact space \( Y \) is embeddable to the cube, there exists and embedding \( \pi \) of \( Y \) to \( \prod_{\lambda \in \Lambda} I_\lambda \) where and \( I_\lambda \) is a closed interval in \( R \). For any \( \lambda \in \Lambda \), let \( p_\lambda : \prod_{\lambda \in \Lambda} I_\lambda \to I_\lambda \) be a projection and let \( F = \{ p_\lambda \circ \pi \circ h \mid \lambda \in \Lambda \} \). Then, it is obvious that \( F \) is a subset of \( C^*(X) \). Since \( \alpha X = X \cup hS(h) \), there exists unique extension \( h^* \) of \( h \) to \( \alpha X \) with \( h^*(\alpha X) = Y \). Hence, we have that for any \( \lambda \in \Lambda \), \( p_\lambda \circ \pi \circ h^* \) is an extension of \( p_\lambda \circ \pi \circ h \) to \( \alpha X \). And so, \( p_\lambda \circ \pi \circ h \in C_\lambda \). Next, we will show that \( p_\lambda \circ \pi \circ h \) is singular.

Let \( Z = \text{Cl}(p_\lambda \circ \pi(Y)) \), then \( Z \) is compact since it is closed in the compact space \( I_\lambda \). For any compact subset \( A \) of \( X \),

\[
Z \supset \text{Cl}(p_\lambda \circ \pi \circ h(X - A)) = \text{Cl}(\text{Cl}(p_\lambda \circ \pi \circ h(X - A)))
\]
\[ \subseteq \text{Cl}(\alpha \circ \pi(\text{Cl}(h(X-A)))) \] since \( p_A \) is continuous

\[ = \text{Cl}(\alpha \circ \pi(Y)) \] since \( h \) is singular

\[ = Z. \]

Hence, we have that \( Z = \cap \{ \text{Cl}(\alpha \circ \pi \circ h(X-A)) \mid A \text{ is compact in } X \} \). Therefore, for any \( \lambda \in \Lambda, p_A \circ \pi \circ h \) is singular. Finally, we will show that \( F^* = \{ f^* \mid f^* \text{ is an extension of } f \text{ to } \alpha X, f \in F \} \) separates points of \( \alpha X - X \), that is, for any \( y_1, y_2(\neq) \in \alpha X - X \), there exists a \( \lambda \in \Lambda \) such that \( (p_A \circ \pi \circ h)^*(y_1) \neq (p_A \circ \pi \circ h)^*(y_2) \) where \( (p_A \circ \pi \circ h) \) is an extension of \( p_A \circ \pi \circ h \) to \( \alpha X \). In the above progress, we know that \( h^*(y_1) = y_1 \) and \( h^*(y_2) = y_2 \) since \( y_1, y_2 \in \alpha X - X \) and \( \alpha X = X \cup \text{S}(h) \).

Since \( \pi \) is embedding, we have that \( \pi(y_1) \neq \pi(y_2) \). Since \( p_A \circ \pi \circ h \) is an extension of \( p_A \circ \pi \circ h \) and the extension is unique, \( (p_A \circ \pi \circ h)^* \) is equal to \( p_A \circ \pi \circ h \). The fact that \( \pi(y_1) \neq \pi(y_2) \) implies that there is a \( \lambda \in \Lambda \) such that \( p_A \circ \pi(y_1) \neq p_A \circ \pi(y_2) \), and so \( (p_A \circ \pi \circ h)^*(y_1)p_A \circ \pi \circ h^*(y_2) = (p_A \circ \pi \circ h)^*(y_2) \). Hence, \( F^* \) separates points of \( \alpha X - X \). Therefore, by Corollary 2.4, \( \alpha X = \sup \{ X \cup S(f) \mid f \in F \} \). But since \( F \subseteq S^* \), \( \alpha X = \sup \{ X \cup S(f) \mid f \in F \} \leq \sup \{ X \cup S(f) \mid f \in S^* \} \) and so, \( \alpha X = \sup \{ X \cup S(f) \mid f \in S^* \} \) because of \( \sup \{ X \cup S(f) \mid f \in S^* \} \leq \alpha X \) by Corollary 2.3. This completes the proof.

**Corollary 3.5.** If \( X \) is a retractive space, then \( \alpha X = \sup \{ X \cup S(f) \mid f \in S^* \} \) for any compactification \( \alpha X \) of \( X \).

**Proof.** Let \( r \) be a retraction of \( \beta X \) to \( \beta X - X \) and let \( \phi \) be the natural projection of \( \beta X \) to \( \alpha X \). Define \( r' : \alpha X \to \alpha X - X \) given by \( r' = \phi \mid_{\alpha X - X} \circ r \circ \phi \). Then, it is obvious that \( r' \) is well-defined, continuous map with \( r' |_{\alpha X - X} = 1_{\alpha X - X} \). Hence, \( \alpha X = \sup \{ X \cup S(f) \mid f \in S^* \} \) by Lemma 3.2 and Theorem 3.4.
We obtain R. E. Chandler's Theorem as a Corollary.

**Corollary 3.6.** If $X$ is a retractive space, then the Stone-Cech Compactification $\beta X$ of $X$ is equal to $\sup\{X \cup S(f) \mid f \in S^*\}$.

In above, we showed that if $X$ is a retractive space, then $\alpha X=\sup\{X \cup S(f) \mid f \in S^*\}$ for any compactification $\alpha X$, and this implies that $\beta X=\sup\{X \cup S(f) \mid f \in S^*\}$. But, the converses don't hold as you see in examples below. First, we give an example in which $\beta X=\sup\{X \cup S(f) \mid f \in S^*\}$, but $\alpha X\neq\sup\{X \cup S(f) \mid f \in S^*\}$ for some compactification $\alpha X$ of $X$.

**Example 3.7.** Let $X=\mathbb{R}$ with the usual topology. Since $\mathbb{R}$ is non-pseudocompact, $\beta X=\sup\{X \cup S(f) \mid f \in S^*\}$ by Theorem 3.3. Let $\alpha X$ be the compactification of $X$ with two points remainder. If $\alpha X$ is a singular compactification, then there exists a singular map $f: X \rightarrow \{a, b\}$. Then, $f(X)=\{a\}$ or $f(X)=\{b\}$ since $\mathbb{R}$ is connected and $f$ is continuous. But this contradicts to $\text{Cl}(f(X))=\{a, b\}$. Hence, $\alpha X$ is not a singular compactification. If $\alpha X=\sup\{X \cup S(f) \mid f \in S^*\}$, then $X \cup S(f) \leq \alpha X$ for any $f$ in $S^*$, and so, $X \cup S(f)$ is the one-point compactification for any $f \in S^*$. Hence, $\alpha X=\sup\{X \cup S(f) \mid f \in S^*\}$ is the one-point compactification, which is a contradiction.

Next, we give an example in which $X$ is not a retractive space, but $\alpha X=\sup\{X \cup S(f) \mid f \in S^*\}$ for any compactification $\alpha X$ of $X$.

**Lemma 3.8[4].** If $f \in C^*$, then $f^*(\alpha X-X)=S(f)$.

Recall that a directed set $\Lambda$ is a partially ordered set with the following property: for any $\alpha, \beta \in \Lambda$, there is a $\lambda$ in $\Lambda$ such that $\alpha \leq \lambda$ and $\beta \leq \lambda$. 
Lemma 3.9[9] Let $X$ be a compact space, $\Lambda$ a directed set and $A_\alpha$ a closed, connected and non-empty subset of $X$ for any $\alpha \in \Lambda$. If $A_\beta \subseteq A_\alpha$ for $\alpha, \beta \in \Lambda$ with $\beta \leq \alpha$, then $\bigcap_{\alpha \in \Lambda} A^\alpha$ is connected.

Example 3.10. Let $X = [0, \infty)$ with the usual topology. Then, $X$ is not a retractive space by Theorem 3.1, since $X$ is not pseudocompact. First, we will show that $aX - X = \bigcap_{\alpha=1}^{\infty} \text{Cl}_a(X_a)$ for any compactification $aX$ of $X$ where $X_\alpha = [n, \infty)$. Since $aX = \text{Cl}_a(X) = \text{Cl}_a(X_a) \cup [0, n]$ and $[0, n] \subseteq X$, we have that $aX = \text{Cl}_a(X_a) \cup [0, n] - X \subseteq \text{Cl}_a(X_a)$. Hence, $aX - X \subseteq \bigcap_{\alpha=1}^{\infty} \text{Cl}_a(X_a)$. For converse, if $x \in X$, then there exists an open neighborhood $U$ of $x$ in $X$ such that $\text{Cl}_a(U)$ is compact since $X$ is locally compact. And there exists an $n$ in $\mathbb{N}$ such that $\text{Cl}_a(U) \subseteq [0, n]$. And so, $x \not\in \text{Cl}_a(X - \text{Cl}_a(U))$ and $\text{Cl}_a(X - \text{Cl}_a(U)) \supseteq \text{Cl}_a(X - [0, n]) = \text{Cl}_a(X_a)$. Hence, $aX - X \supseteq \text{Cl}_a(X_a) \supseteq \bigcap_{\alpha=1}^{\infty} \text{Cl}_a(X_a)$. Since $X_\alpha$ is connected for any $\alpha$, we have that $aX - X$ is connected for any compactification $aX$ of $X$ by Lemma 3.9. Given compactification $aX$ of $X$, let $p, q(\neq) \in aX - X$, then there exists a continuous map $f$ from $aX$ to $[0, 1]$ with $f(p) = 0$ and $f(q) = 1$. Since $f$ is continuous and $aX - X$ is connected, $f(aX - X)$ is connected. Because of $f(a - X) \in [0, 1]$, $f(p) = 0$ and $f(q) = 1$, we have that $f(aX - X) = [0, 1]$. Let $g = f|_{aX}$. Since $g$ has an extension $f$ to $aX$, by Lemma 3.8, $S(g) = f(aX - X) = [0, 1]$. Hence, we have that $g$ is a singular map. And so, $g$ is an element of $S^*$ and $f(p) \neq f(q)$. Therefore, $S^*$ separates points of $aX - X$. By Corollary 2.4, we know that $aX = \sup \{X \cup S(f) \mid f \in S^*\}$. 
References


Department of Mathematics
University of Ulsan