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On Singular Compactifications

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1. Introduction

In [8], J.P.Guglielmi proposed the open question under which condition on X, αX is equal to $\sup\{X \cup_f S(f) \mid \alpha X \geq \cup_f S(f)\}$ for any compactification αX of X. Since $\alpha X \geq \cup_f S(f)$ if and only if $f \in S^\alpha$, J.P.Guglielmi's problem can be restated as follows ; Under which condition on X, is αX equal to $\sup\{X \cup_f S(f) \mid f \in S^\alpha\}$ for any conpactification αX of X? R.E.Chandler[4] showed that if X is non-pseudocompact, then the Stone-Cech compactification of X is equal to $\sup\{X \cup_f S(f) \mid f \in S^*\}$ and also he showed that if X is a retractive space, then the Stone-Cech compactification of X is equal to $\sup\{X \cup_f S(f) \mid f \in S^*\}$.

In this note, we give some conditions under which αX is equal to $\sup\{X \cup_f S(f) \mid f \in S^\alpha\}$ for any compactification αX of X, and give some examples.

2 Singular compactifications

Throughout this note, the topological space X is assumed to be locally compact and all topological spaces are assumed to be Hausdorff.

In [2], the singular set S(f) is defined $\cap\{Cl(f(X-F)) \mid F$ is compact in $X\}$ for a continuous map of X to Y, where "Cl" denotes the closure operater. And f is called singular if S(f)=Y. For a singular

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map f of X to a compace space Y, a Hausdorff compactification, which is called a singular compactification, is defined as follows; Let the open sets in X be as they were in the original topology on X. For $p \in Y$, a basic neighborhood of p is defined to be any set of the form $U \cup f^{-1}(U) - F$ where U is an open neighborhood of p in Y. This topological space on $X \cup S(f) = X \cup Y$ is denoted by $X \cup S(f) = X \cup Y$ is denoted by $X \cup_f S(f)$ ([4], [5]).

We will denote $C^*(X)$ the set of all continuous and bounded map of X to R(the real line with the usual topology) and C^α the set of all continuous and bounded maps of X to R with the extension to a Hausdorff compactification αX of X. Let $S^* = \{f \in C^*(x) \mid f$ is singular $S^\alpha = \{f \in C^\alpha \mid f$ is singular$\}$ for any compactification αX of X. Then, it is true that $S^* \neq \phi$ and $S^\alpha \neq \phi$ since all constant map are singular.

In 1982. B.J.Ball and S.Yokura[1] introduced the concept of determining set. For each subset F of $C^*(X)$, let K(F) be the subfamily of the family of all Hausdorff compactifications of X such that K(F) consists of all compactifications of X to which each element of F can be extended. If K(F) has the smallest element αX, then αX is said to be determined by F, and denoted by $\alpha_F X$. It is easily shown that every subset of $C^*(X)$ determines a compactification of X if X is locally compact. And then, he showed the following theorem.

Theorem 2.1 [1]. For any subset F of $C^*(X)$ and any compactification αX of X, the followings are equivalent.

(1) F determines a compactification and $\alpha_F X = \alpha X$.

(2) $F \subset C^\alpha$ and F^α separates points of $\alpha X - X$ where F^α is the subset $\{f^\alpha \in C^*(\alpha X) : f^\alpha$ is an extension of f and $f \in F\}$ **of** $C^*(\alpha X)$.

Theorem 2.2. Let $f\in C^*$. Then, $X\cup_f S(f)$ is the smallest compatification of X to which f can be extended.

Proof. Let $\alpha X=X\cup_f S(f)$ and define $f^\alpha : \alpha X \longrightarrow S(f)$ given by $f^\alpha(x)=f(x)$ if $x\in X$ and $f^\alpha(x)=x$ otherwise. Then, it is easily shown that f^α is a continuous extension of f and it is trivial that f^α separates points of $\alpha X-X$. Since X is locally compact, αX is the smallest compactification of X to which f can be extended, by theorem 2. 1.

Corollary 2.3. For any subset F of S^*, $\sup\{X\cup_f S(f) \mid f\in F\}$ is the smallest compactification of X to which F can be extended.

Proof. Since $\sup\{X\cup_f S(f) \mid f\in F\} \geq X\cup_f S(f)\}$ for any f in F, it follows that F can be extended to $\sup\{X\cup_f S(f) \mid f\in F\}$. If αX is a Hausdorff compactification of X to which F can be extended, then for any f in F, $\alpha X \geq X\cup_f S(f)$ by theorem 2.2. Hence, we have that $\alpha X \geq \{X\cup_f S(f) \mid f\in F\}$, and so $\sup\{X\cup_f S(f) \mid f\in F\}$ is the smallest compactification of X to which F can be extended.

Using above Theorem 2.1 and Corollary 2.3, we obtain the following corollary which is proved by R. E. Chandler and G. D. Faulkner[4].

Corollary 2.4. $\alpha X=\sup\{X\cup_f S(f) \mid f\in S^\alpha\}$ if and only if $\{f^\alpha \in C^*(\alpha X) \mid f^\alpha$ is an extension of f and $f\in S^\alpha\}$ separates points of $\alpha X-X$.

3. Main results

For a completely regular space X, X is called a retractive space if and only if there is a retration of βX onto $\beta X-X$.

Theorem 3.1[6]. If X is a retractive space, then X is locally compact and pseudocompact.

Lemma 3.2[8]. For a compactification αX of X, αX is a singular compactification if and only if there exists a retraction of αX onto $\alpha X - X$.

Theorem 3.3[4]. If X is non-pseudocomapact, then

$$\beta X = \sup\{X \cup_f S(f) \mid f \in S^*\}.$$

We generalize the theorem of R.E.Chandler : "If X is a retractive space, then $\beta X = \sup\{X \cup_f S(f) \mid f \in S^*\}$." and give some answers on the problem given by J. P. Guglielmi.

Theorem 3.4. If αX is a singular compactification, then αX is equal to $\sup\{X \cup_f S(f) \mid f \in S^*\}$.

Prooof. Suppose that h is a singular map of X to Y where $S(h) = Y$ is compact and $\alpha X = X \cup_h S(h)$. Sincd the compact space Y is embeddable to the cube, there exists and embedding π of Y to $\prod_{\lambda \in \Lambda} I_\lambda$ where and I_λ is a closed interval in R. For any $\lambda \in \Lambda$, let $p_\lambda : \prod_{\lambda \in \Lambda} I_\lambda \longrightarrow I_\lambda$ be a projection and let $F = \{p_\lambda \circ \pi \circ h \mid \lambda \in \Lambda\}$. Then, it is obvious that F is a subset of $C^*(X)$. Since $\alpha X = X \cup_h S(h)$, there exists unique extension h^α of h to αX with $h^\alpha(\alpha X) = Y$. Hence, we have that for any $\lambda \in \Lambda$, $p_\lambda \circ \pi \circ h^\alpha$ is an extension of $p_\lambda \circ \pi \circ h$ to αX. And so, $p_\lambda \circ \pi \circ h \in C^\alpha$. Next, we will show that $p_\lambda \circ \pi \circ h$ is singular. Let $Z = Cl(p_\lambda \circ \pi(Y))$, then Z is compact since it is closed in the compact space I_λ. For any compact subset A of X,

$$\begin{aligned} Z &\supseteq Cl(p_\lambda \circ \pi \circ h(X - A) \\ &= Cl(Cl(p_\lambda \circ \pi \circ h(X - A))) \end{aligned}$$

$\supseteq Cl(p_\lambda \circ \pi(Cl(h(X-A))))$ since p_λ is continuous
$= Cl(p_\lambda \circ \pi(Y))$ since h is singular
$= Z.$

Hence, we have that $Z=\cap\{Cl(p_\lambda \circ \pi \circ h(X-A)) \mid A$ is compact in $X\}$. Therefore, for any $\lambda \in \Lambda$, $p_\lambda \circ \pi \circ h$ is singular. Finally, we will show that $F^a=\{f^a \mid f^a$ is an extension of f to αX, $f \in F\}$ separates points of $\alpha X-X$, that is, for any y_1, $y_2(\neq) \in \alpha X-X$, there exists a $\lambda \in \Lambda$ such that $(p_\lambda \circ p \circ h)^a\ (y_1) \neq (p_\lambda \circ \pi \circ h)^a\ (y_2)$ where $(p_\lambda \circ \pi \circ h_a)$ is an extension of $p_\lambda \circ \pi \circ h$ to αX. In the above progress, we know that $h^a(y_1)=y_1$ and $h^a(y_2)=y_2$ since y_1, $y_2 \subset \alpha X-X$ and $\alpha X=X\cup_h S(h)$. Since π is embedding, we have that $\pi(y_1)\neq\pi(y_2)$. Since $p_\lambda \circ \pi \circ h^a$ is an extension of $p_\lambda \circ \pi \circ h$ and the extension is unique, $(p_\lambda \circ \pi \circ h)^a$ is equal to $p_\lambda \circ \pi \circ h^a$. The fact that $\pi(y_1)\neq\pi(y_2)$ implies that there is a $\lambda \in \Lambda$ such that $p_\lambda \circ \pi(y_1)\neq p_\lambda \circ \pi(y_2)$, and so $(p_\lambda \circ \pi \circ h)^a\ (y_1) p_\lambda \circ \pi \circ h^a\ (y_1)=p_\lambda \circ \pi(y_1)\neq p_\lambda \circ \pi(y_2)=p_\lambda \circ \pi \circ h^a(y_2)=(p_\lambda \circ \pi \circ h)^a(y_2)$. Hence, F^a separates points of $\alpha X-X$. Therefore, by Corollary 2.4, $\alpha X=\sup\{X \cup_f S(f) \mid f \in F\}$. But since $F \subseteq S^a$, $\alpha X=\sup\{X \cup_f S(f) \mid f \subset F\} \leq \sup\{X\cup_f S(f) \mid f \subset S^a$ and so, $\alpha X=\sup\{X\cup_f S(f) \mid f \in S^a$ because of $\sup\{X\cup_f S(f) \mid f \in S^a\} \leq \alpha X$ by Corollary 2.3. This completes the proof.

Corollary 3.5. If X is a retractive space, then $\alpha X=\sup\{X \cup_f S(f) \mid f \in S^a\}$ for any compactification αX of X.

Proof. Let r be a retraction of βX to $\beta X-X$ and let ϕ be the natural projection of βX to αX. Define $r': \alpha X \to \alpha X-X$ given by $r'=\phi\mid_{\alpha X-X} \circ r \circ \phi^{-1}$. Then, it is obvious that r' is well-defined, ontinuous map with $r'\mid_{\alpha X-X}=1_{\alpha X-X}$. Hence, $\alpha X=\sup\{X \cup_f S(f) \mid f \in S^a\}$ by Lemma 3.2 and Theorem 3.4.

We obtain R. E. Chandler's Theorem as a Corollary.

Corollary 3.6. If X is a retractive space, then the Stone-Cech Compactification βX of X is equal to $\sup\{X\cup_f S(f) \mid f\in S^*\}$.

In above, we showed that if X is a retractive space, then $\alpha X=\sup\{X\cup_f S(f) \mid f\in S^\alpha\}$ for any compactification αX, and this implies that $\beta X=\sup\{X\cup_f S(f) \mid f\in S^*\}$. But, the converses don't hold as you see in examples below. First, we give an example in which $\beta X=\sup\{X\cup_f S(f) \mid f\in S^*\}$, but $\alpha X\neq\sup\{X\cup_f S(f) \mid f\in S^\alpha\}$ for some compactification αX of X.

Example 3.7. Let X=R with the usual topology. Since R is non-pseudocompact, $\beta X=\sup\{X\cup_f S(f)\in S^*\}$ by Theorem 3.3. Let αX be the compactification of X with two points remainder. If αX is a singular compactification, then there exists a singular map $f: X\longrightarrow \{a, b\}$. Then, $f(X)=\{a\}$ or $f(X)=\{b\}$ since R is connected and f is continuous. But this contradicts to $Cl(f(X))=\{a, b\}$. Hence, αX is not a singular compactification. If $\alpha X=\sup\{X\cup_f S(f) \mid f\in S_\alpha\}$, then $X\cup_f S(f) \leq \alpha X$ for any f in S^α, and so, $X\cup_f S(f)$ is the one-point compactification for any $f\in S^\alpha$. Hence, $\alpha X=\sup\{X \cup_f S(f) \mid f\in S^\alpha\}$ is the one-point compactification, which is a contradiction.

Next, we give an example in which X is not a retractive space, but $\alpha X=\sup\{X\cup_f S(f) \mid f\in S^\alpha\}$ for any compactification αX of X.

Lemma 3.8[4]. If $f\in C^\alpha$, then $f^\alpha(\alpha X-X)=S(f)$.

Recall that a directed set Λ is a partially ordered set with the following property ; for any α, $\beta\in\Lambda$, there is a λ in Λ such that $\alpha\leq\lambda$ and $\beta\leq\lambda$.

Lemma 3.9[9] Let X be a compact space, Λ a directed set and A_α a closed, connected and non-empty subset of X for any $\alpha\in\Lambda$. If $A_\alpha\in A_\beta$ for α, $\beta\in\Lambda$ with $\beta\leq\alpha$, then $\bigcap_{\alpha\in\Lambda} A^\alpha$ is connected.

Example 3.10. Let $X=[0,\infty)$ with the usual topology. Then, X is not a retractive space by Theorem 3.1, since X is not pseudocompact. First, we will show that $\alpha X-X=\bigcap_{n=1}^{\infty} Cl_{\alpha X}(X_n)$ for any compactification αX of X where $X_n=[n,\kappa)$. Since $\alpha X=Cl_{\alpha X}(X)=Cl_{\alpha X}(X_n\cup[0,n])=Cl_{\alpha X}(X_n)\cup[0,n]$, and $[0,n]\subseteq X$, we have that $\alpha X=Cl_{\alpha X}(X_n)\cup[0,n]-X\subseteq Cl_{\alpha X}(X_n)$. Hence, $\alpha X-X\subseteq\bigcap_{n=1}^{\infty} Cl_{\alpha X}(X_n)$. For converse, if $x\in X$, then there exists an open neighborhood U of x in X such that $Cl_X(U)$ is compact since X is locally compact. And there exists an n in N such that $Cl_X(U)\subseteq[O,n)$. And so, $x\notin Cl_{\alpha X}(X-Cl_X(U))$ and $Cl_{\alpha X}(X-Cl_X(U))\supseteq Cl_{\alpha X}(X-[0,n))$, this implies that $x\notin Cl_{\alpha X}(X-[0,n))=Cl_{\alpha X}(X_n)$. Hence, $\alpha X-X\supseteq Cl_{\alpha X}(X_n)\supseteq\bigcap_{n=1}^{\infty} Cl_{\alpha X}(X_n)$. Since X_n is connected for any n, we have that $\alpha X-X$ is connected for any compactifiaction αX of X by Lemma 3.9. Given compactification αX of X, let p, $q(\neq)\in\alpha X-X$, then there exists a continuous map f from αX to [0, 1] with $f(p)=0$ and $f(q)=1$. Since f is continuous and $\alpha X-X$ is connected, $f(\alpha X-X)$ is connected. Because of $f(\alpha-X)\in[0,1]$, $f(p)=0$ and $f(q)=1$, we have that $f(\alpha X-X)=[0,1]$. Let $g=f|_X$. Since g has an extension f to αX, by Lemma 3.8, $S(g)=f(\alpha X-X)=[0,1]$. Hence, we have that g is a singular map. And so, g is an element of S^α and $f(p)\neq f(q)$. Therefore, S^α separates points of $\alpha X-X$. By Corollary 2.4, we know that $\alpha X=\sup\{X\cup_f S(f)\mid f\in S^\alpha\}$.

References

1. B.J.Ball and S.Yokura, Compactifications determined by subsets of C*(X), Topology and its Applications, 15(1983), 1−6.
2. G.L.Cain, R.E.Chandler and G.D.Faulkner, Singular sets and remainders, Trans. Amer. Math. Soc, 268(1981), 161−171.
3. R.E.Chandler, Hausdorff compactifications, Marcel Dekker, Inc., New York, 1976.
4. R.E.Chandler and G.D.Faulkner, Singular compactifications ; The order structure, Proc. of Amer. Math Soc, 100(1987)
5. R.E.Candler, G.D.Faulkner, J.P.Guglielmi and M.C.Memory, Generalizing the Alexandorff-Urysohn double circumference construction, Proc. Amer Math. Soc., 83(1981), 606−608
6. J.B.Conway, Projections and retractions, Proc. of Amer. Math. Soc., 17(1966). 843−847.
7. L.Gillman and M.Jerison, Rings of continuous functions, D.Van Nostrand Co. Princeton, 1960.
8. J.P.Guglielmi, Compactifications with wingular remainders, PH.D.Thesis, North Carolina State University, 1986
9. S.Willard, General topology, Addison-Wesley, 1970.

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