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Nonlinear Ergodic Theorems For Reversible Semigroups of Lipschitzian Mappings in Uniformly Convex Banach Spaces

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I. Introduction

Let G be a semitopological semigroup i.e., G is a semigroup with a Hausdorff topology such that for each s $\in$ G the mappings s $\rightarrow$ as and s $\rightarrow$ sa from G to G are continuous. G is called right reversible if any two closed left ideals of G have nonvoid intersection. In this case, (G, $\geqslant$) is a directed system when the binary relation "$\geqslant$" on G is defined by t $\geqslant$ s if and only if

$$\{t\} \cup \overline{Gt} \subset \{s\} \cup \overline{Gs},$$

for all t, s $\in$ G. Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups [18]. Left reversibility of G is defined similarly. G is called reversible if it is both left and right reversible.

Let C be a nonempty closed convex subset of a Banach space X with norm $\|\cdot\|$ and let T be a mapping from C into itself. T is said to be a Lipschitzian mapping if for each n $\geqq$ 1 there exists

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a real number $k_n>0$ such that

$$\| T^n x - T^n y \| \leqq k_n \| x - y \|$$

for all x, y $\in$ C. A Lipschitzian mapping is said to be nonexpansive if $K_n=1$ for all $n \geqq 1$ and asymptotically nonexpansive if $\lim_{n\to\infty} K_n = 1$, respectively [10].

A family $\zeta=\{S(t) : t \in G\}$ of mappings from C into itself is said to be a continuous representation of G on C if ζ satisfies the following ;

(1) $S(ts)x=S(t)S(s)x$ for all t, s $\in$ G and x $\in$ C,

(2) For every x $\in$ C, the mapping $(s, x) \to S(s)x$ from $G \times C$ into C is continuous when $G \times C$ has the product topology.

A continuous representation ζ of G on C is said to be a Lipschitzian semigroup on C if each t $\in$ G, there exists $K_t>0$ such that

$$\| S(t)x - S(t)y \| < K_t \| x - y \|$$

for all x, y $\in$ C.

The first nonlinear ergodic theorem for nonexpansive mappings was established by Baillon [1] : Let C be a closed convex subset of a real Hilbert space H and T a nonexpansive mapping from C into itself. If the set F(T) of fixed points of T is nonempty, then for each x $\in$ C, the Cesaro mean

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to some y $\in$ F(T). In this case, putting y=Px for each x$\in$C, P is a nonexpansive retraction of C onto F(T) such that PT = TP = P and Px $\in$ $\overline{\text{conv}}$ $\{T^n x : n \geqq 1\}$ for each x $\in$ C, where convA is the closure of the convex hull of A. And later extened to Banach spaces Bruck [6], Hirano [14], Reich [25], and others.

A corresponding result for nonexpansive semigroups on C was given by Baillon [2], Baillon–Brezis [3] and Reich [24]. Nonlinear ergodic theorems for general commutative semigroups of nonexpansive mappings were given by Brézis–Browder [4], Hirano–Takahashi [16] and Hirano–Kido–Takahashi [17].

In [26], Takahashi proved the following nonlinear ergodic theorem for a noncommutative semigroup of nonexpansive mappings: Let C be a nonempty closed convex subset of a real Hilbert space H, and let S be an amenable semigroup of nonexpansive mappings t from C into itself. Suppose the set F(S) of all common fixed points of $t \in S$ is nonempty. Then there exists a nonexpansive retraction P of C onto F(S) such that $Pt = tP = P$ for all $t \in S$ and $Px \in \overline{\text{conv}}\,\{tx : t \in S\}$. Furthermore, Hirano–Takahashi [15] extended this result to a Banach space. And, Lau – Takahashi [20] also proved the same result for a reversible semigruop of nonexpansive mappings in Banach spaces. Recently, Ishihara - Takahashi [19] proved the existence of the ergodic retraction for a reversible semigroup of Lipschitziam mappings in Hilbert spaces.

In this paper, we would like to extend the results of Ishihara - Takahashi to uniformly convex Banach spaces with a Fréchet differentiable norm. Our proofs employ the methods of Hirano - Takahashi [15] Ishihara - Takahashi [19], Miyadera - Kobayashi [22], Takahashi - Zhang [27] and Lau - Takahashi [20].

II. Preliminaries and Notations

Let X be a Banach space with the norm $\|\cdot\|$ and X^* its dual. Then, the value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x,$

$x^* \rangle$. With each $x \in X$, we associate the set

$$J(x)=\{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) = \phi$ for each $x \in X$. The multivalued mapping $J : X \longrightarrow X^*$ is called the duality mapping of X. Let $B=\{x \in X : \|x\| = 1\}$ stand for the unit sphere of X. Then the norm of X is said to be Gâteaux differentiable (and X is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x - ty\| - \|x\|}{t}$$

exists for each x and y in B. It is said to be Fréchet differentiable if for each x in B, this limit is attained uniformly for y in B. It is well known that if X is smooth, then the duality mapping J is single-valued. And we also know that if the norm of X is Fréchet differentiable, then J is norm to norm continuous. (see [5], [9] for more details.)

For x and y in X, Sgm[x, y] denotes the set

$$\{\lambda x+(1-\lambda)y : 0 \leq \lambda \leq 1\}$$

In this paper, unless other specified, X will denote a uniformly convex Banach space with modulus of convexity δ. The modulus of convexity of X is the function $\delta : [0, 2] \longrightarrow [0. 1]$ defined by

$$\delta(\varepsilon)=\inf\{1-\frac{x+y}{2} : \|x\| \leq 1, \ \|y\| \leq 1, \ \|x-y\| \geq \varepsilon\}$$

for $0 \leq \varepsilon \leq 2$. X is uniformly convex if and only if $\delta(\varepsilon) > 0$ for $\varepsilon > 0$ ([7], [9] and [23]). It is known that δ is nondecreasing ([13], [21] and [28]). Hence if X is uniformly convex and $\delta(\varepsilon_n) \to 0$, then $\varepsilon_n \to 0$.

III. Lemmas and Propositions

In this section, we prove serveral lemmas and propositions which are crucial for our purpose in next section.

Lemma 1. Let C be a closed convex subset of a uniformly convex Banach space X. Let G be a right reversible semitopological semigroup and $\zeta = \{S(t) : t \in G\}$ a Lipschitzian semigroup on C with $\lim_t \sup k_t \leqq 1$. If $f \in F(\zeta)$, then there exists the limit of

$$\| S(t)x - f \| \quad \text{for all } x \in C.$$

Proof. Since, for all $t \in G$,

$$\begin{aligned} \| S(t)x - f \| &\leqq \| S(t)x - S(t)S(s)x \| + \| S(t)S(s)x - f \| \\ &\leqq \| S(t)x - S(ts)x \| + k_t \| S(s)x - f \| \end{aligned}$$

for each $x \in C$ and $s \in G$. Taking the lim sup as t and fixed s, we obtain

$$\begin{aligned} \limsup_t \| S(t)x - f \| &\leqq (\limsup_t k_t) \| S(s)x - f \| \\ &\leqq \| S(s)x - f \| \end{aligned}$$

for all $s \in G$. Taking the liminf as s, we have

$$\limsup_t \| S(t)x - f \| \leqq \liminf_t \| S(s)x - f \|$$

This completes the proof

Proposition 2. Let X, C, G and ζ be as in Lemma 1. Then

$f(\zeta)$ is nonempty if and only if $\{S(t)x : t \in G\}$ is bounded for any $x \in C$. Furthermore, $F(\zeta)$ is a closed and convex subset of C.

Proof. Suppose that $\{S(t)x : t \in G\}$ is bounded for any $x \in C$. Since X is uniformly convex, there exists a unique asymptotic center a with respect to C [11] such that

$$\limsup_t \| S(t)x - a \| \leqq \limsup_t \| S(t)x - z \|$$

for all $z \in C - \{a\}$. On the other hand, since for all $s \in G$

$$\| S(st)x - S(s)a \| \leqq K_s \| S(t)x - a \|,$$

taking limsup as s, we have

$$\begin{aligned} \limsup_s \| S(s)x - S(s)a \| &\leqq (\limsup_s k_s) \| S(t)x - a \| \\ &\leqq \| S(t)x - a \| \end{aligned}$$

Taking limsup as t, then we obtain

$$\limsup_t \| S(t)x - S(t)a \| < \limsup_t \| S(t)x - a \|$$

This implies that $a \in F(\zeta)$. The converse follows from Lemma 1. The closedness of $F(\zeta)$ is obvious from the continuity of the elements of ζ. To show convesity of $F(\zeta)$, it is sufficient to show that $z = (x+y)/2 \in F(\zeta)$ for all $x, y \in F(\zeta)$. Let $x, y \in F(\zeta)$, $z = (x+y)/2$ and $x \neq y$. Then we have

$$\lim_t S(t)z = z$$

If not, there exists $\varepsilon > 0$ such that for any $t \in G$, there is $t' \in G$ with $t' \geqslant t$ and

$$\| s(t')z - z \| \geqq \varepsilon.$$

Choose $d > 0$ so small that

$$(R+d)\left[1+\delta\left(\frac{4\varepsilon}{R+d}\right)\right] < R,$$

where $R = \| x - y \| > 0$ and δ is the modulus of convexity of X. Since $\limsup_t k_t \leqq 1$, it follows that there is $t_0 \in G$ such that

$$k_t \| x-y \| \leqq \| x-y \| + d$$

for all $t \geqslant t_0$. Put $u=[S(t_0')z-x]/2$, $v=[y-S(t_0')z-x]/2$ for $t_0' \geqslant t$. Then we have,

$$\| u \| = \frac{1}{2}\| S(t_0')z-x \| \leqq \frac{1}{4}k_0 \| x-y \| \leqq \frac{1}{4}(R+d),$$

$$\| v \| = \frac{1}{2}\| y-S(t_0')z \| \leqq \frac{1}{4}k_0' \| x-y \| \leqq \frac{1}{4}(R+d),$$

and

$$\| u-v \| = S(t_0')z-z \| \geqq \varepsilon.$$

Since X is uniformly convex,

$$\left\| \frac{u+v}{2} \right\| \leqq \frac{1}{4}(R+d)\left[1-\delta\left(\frac{\varepsilon}{R+d}\right)\right] \leqq \frac{1}{4}R$$

and hence

$$\frac{1}{4}R = \frac{1}{4}\| x-y \| = \frac{1}{2}\| u+v \| < \frac{1}{4}R.$$

This is a contradiction. Hence we have

$$\lim_t S(t)z = z.$$

Therefore, we have

$$\begin{aligned} S(s)z &= \lim_t S(s)S(t)z \\ &= \lim_t S(t)z \\ &= \lim_t S(t)z \\ &= z. \end{aligned}$$

This completes the proof.

The next lemma is known [12]. It is a simple consequence of the definition of the modulus of convexity.

Lemma 3. Let X be a uniformly convex Banach space with modulus of convexity δ. If $\|x\| \leq r$, $\|y\| \leq r$, $r < R$ and $\|x-y\| \geq \varepsilon\,(>0)$, then

$$\|\lambda x+(1-\lambda)y\| \leq r[1-2\lambda(1-\lambda)\delta(\tfrac{\varepsilon}{R})]$$

for all $0 \leq \lambda \leq 1$.

the proofs of our following lemmas are based on methods used in [14] and [20].

Lemma 4. Let X, C, G and ζ be as in Lemma 1. Let X be in C, $f \in F(\zeta)$ and $0 < \alpha \leq \beta < 1$. Then for any $\varepsilon > 0$, there exists $t_0 \in G$ such that

$$\| S(t)[\lambda S(s)x+(1-\lambda)f]-[\lambda S(t)S(s)x+(1-\lambda)f] \| < \varepsilon$$

for all $s,\ t \geqslant t_0$ and $\alpha \leq \lambda \leq \beta$.

Proof. Let $\varepsilon > 0$, $c = \min\{2\lambda(1-\lambda) : \alpha \leq \lambda \leq \beta\}$

and $c=\max\{2\lambda(1-\lambda) : \alpha \leqq \lambda \leqq \beta\}$. By Lemma 1, $\lim_t \| S(t)x-f \|$ exists. Put $r=\lim_t \| S(t)x-f \|$ for any $f \in F(\zeta)$. Since G is right reversible and $\limsup_t k_t \leqq 1$, $r=\inf_t \| S(t)x-f \|$. If $r=0$, then there exists $t_0 \in G$ such that

$$\| S(t)x - f \| < \frac{\varepsilon}{Mc'}$$

for all $t \geqslant t_0$, where $M=\sup_{t \geqslant t_0} k_t$. Hence, for $s, t \geqslant t_0$ and $0<\lambda<1$,

$$\begin{aligned}
&\| S(t)[\lambda S(s)x+(1-\lambda)f]-[\lambda S(t)S(s)x+(1-\lambda)f] \| \\
&\leqq \lambda \| S(t)[\lambda S(s)x+(1-\lambda)f]-S(t)S(s)x] \| \\
&\quad +(1-\lambda) \| S(t)[\lambda S(s)x+(1-\lambda)f]-f \| \\
&< \lambda k_t \| \lambda S(s)x+(1-\lambda)f-S(s)x \| \\
&\quad +(1-\lambda)k_t \| \lambda S(s)x+(1-\lambda)f-f \| \\
&= 2\lambda(1-\lambda)k_t \| S(s)x-f \| \\
&\leqq Mc' \| S(s)x-f \| \\
&< \varepsilon.
\end{aligned}$$

Now, let $r>0$. Since δ is nondecreasing, for given $\varepsilon>0$, we can choose $d>0$ so small that

$$(r+d)\left[1-c\delta\left(\frac{4\varepsilon}{r+d}\right)\right]<r,$$

where δ is the modulus of convexity of the norm. And also, since $r=\lim_t \| S(s)x-f \|$ and $\limsup_t k_t \leqq 1$, there exists $t_0 \in G$ such that

$$k_t \| S(s)x-f \| < r+d$$

for all $s, t \geqslant t_0$. For some λ with $\alpha \leqq \lambda \leqq \beta$, we put $u=(1-\lambda)[S(t)z-f]$ $v=\lambda[S(t)S(s)x-S(t)z]$ where $z=\lambda S(s)x+(1-\lambda)f$. Then, we have

$$\begin{aligned}\| u \| &\leqq (1-\lambda)k_t \| z-f \| \\ &= \lambda(1-\lambda)k_t \| S(s)x-f \| \\ &< \lambda(1-\lambda)(r+d) \\ &\leqq \frac{1}{4}(r+d), \\ \| v \| &\leqq \lambda k_t \| S(s)x-z \| \\ &= \lambda(1-\lambda)k_t \| S(s)x-f \| \\ &< \frac{1}{4}(r+d),\end{aligned}$$

and

$$\| u-v \| = \| S(t)[\lambda S(s)x+(1-\lambda)f]-[\lambda S(t)S(s)x+(1-\lambda)f] \| .$$

Suppose that $\| u-v \| \geqq \varepsilon$ for some $\varepsilon > 0$, then by Lemma 3,

$$\begin{aligned}\lambda(1-\lambda) \| S(t)S(s)x-f \| &= \| \lambda u+(1-\lambda)v \| \\ &\leqq \lambda(1-\lambda)(r+d)[1-2\lambda(1-\lambda)\delta(\frac{4\varepsilon}{r+d})] \\ &\leqq \lambda(1-\lambda)(r+d)[1-c\delta(\frac{4\varepsilon}{r+d})] \\ &\leqq \lambda(1-\lambda)r.\end{aligned}$$

And hence,

$$\| S(t)S(s)x-f \| < r$$

for s, t $\succcurlyeq t_0$. This is a contradiction to the fact that $r = \inf_t \| S(t)x-f \|$. This completes the proof.

The following lemma is a direct consequence of Lau-Takahashi [20].

Lemma 5. Let C be a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm and $\{S(t)x : t \in G\}$ a bounded net in C. Let $z \in \bigcap_{t_0 \in G} \overline{\text{conv}} \{S(t)x : t \succcurlyeq t_0\}$,

$y \in C$ and $\{P_t\}$ a net of element in C with $P_t \in Sgm[y, S(t)x]$ and $\| P_t - z \| = \min\{ \| u - z \| : u \in Sgm[y, S(t)x]\}$ If $\{P_t\}$ converges strongly to y as t, then $y = z$.

By using Lemma 4 and Lemma 5, we can prove the following lemma.

Lemma 6. Let X be a uniformly convex Banach space with a Fréchet differentiable norm. Let C, G and ζ be as in Lemma 1 and $\{S(t)x : t \in G\}$ a bounded net for some $x \in C$. Then for any

$$z \in \bigcap_{s \in G} \overline{\text{conv}} \{S(t)x : t \geqslant s\} \cap F(\zeta) \text{ and } y \in F(\zeta),$$

there exists $t_0 \in G$ such that

$$\langle S(t)x - y, J(y - z) \rangle \leqq 0$$

for all $t \geqslant t_0$.

Proof. If $y = z$ or $x = y$, then the result is obvious. So, let $y \neq z$ and $x = y$. For any $t \in G$, taking a unique element $P_t \in Sgm[y, S(t)x]$ such that

$$\| P_t - z \| = \min\{ \| u - z \| : u \subset Sgm[y, S(t)x]\}.$$

Then, since $y \neq w$, $\{P_t\}$ doesn't converge to y from Lemma 5. Hence, we obtain $c > 0$ such that for any $t' \in G$, there is $t \in G$ with $t' \geqslant t$ and

$$\| P_{t'} - y \| \geqq c.$$

Setting $P_{t'} = \alpha_{t'} S(t')x + (1 - \alpha_{t'})y$, $0 \leqq \alpha_{t'} \leqq 1$, then there exists $c_0 > 0$ so small that $\alpha_{t'} \geqq c_0$ (in fact, since $x \neq y$ and $y \in F(\mathfrak{S})$,

$$C \leqq \| p_{t'} - y \| = \| \alpha_{t'} S(t')x + (1-\alpha_{t'})y - y \|$$
$$\leqq \alpha_{t'} k_{t'} \| x - y \| .$$

Hence, put $c_o = \frac{c}{k \| x-y \|}$, where $K = \sup_t k_t$. Letting $K = \lim_t \| S(t)x - y \|$, we have $K > 0$. If not, then we have $\lim_t S(t)x = y$, and so $\lim_t P_t = y$ which contradicts. Now, we can choose $r > 0$ with $K > r$ such that

$$\frac{R}{R-\varepsilon} > 1 - \delta\left(\frac{c_o r}{R+\varepsilon}\right),$$

where δ is the modulus of convexity of the norm and $R = \| z - y \|$ (>0). Fix $\varepsilon_o < \varepsilon$. Then by Lemma 4, there exists t_1 such that

$$\| S(s)[c_o S(t)x + (1-c_o)y] - [c_o S(s)S(t)x - (1-c_o)y] \| < \varepsilon_o (<\varepsilon)$$

for all $s, t \geqslant t_1$. Fix $t' \in G$ with $t' \geqslant t_1$ and $\| P_{t'} - y \| > c$ Then, since $\alpha_{t'} \geqq c_o$ (>0), we have

$$\| c_o S(t')x + (1-c_o)y \|$$
$$(1 - \frac{C_o}{\alpha t'})y + \frac{C_o}{\alpha t'} [\alpha t' S(t')x + (1-\alpha t')y]$$
$$\in \mathrm{sgm}[y, \ Pt']$$

Put $\lambda \frac{C_o}{\alpha t'}$. Then we have

$$\| c_o S(t')x + (1-c_o)y \|$$
$$= \| \lambda y + (1-\lambda)P_{t'} - z \|$$
$$\leqq \lambda \| y - z \| + (1-\lambda) \| P_{t'} - z \|$$
$$\leqq \lambda \| y - z \| + (1-\lambda) \| y - z \|$$
$$= R.$$

Hence we obtain

$$\begin{aligned}
&\| c_o S(s)S(t')x+(1-c_o)y-z \| \\
&\quad \leqq \| S(s)[c_o S(t')x+(1-c_o)y]-[c_o S(s)S(t')x+(1-c_o)y] \| \\
&\quad + \| S(s)[c_o S(t')x+(1-c_o)y]-z \| \\
&\quad \leqq k_s \| c_o S(t')x+(1-c_o)y-z \| + \varepsilon_o \\
&\quad \leqq k_s R+\varepsilon_o
\end{aligned}$$

for all s, $t' \geqq t_1$. Since $\limsup_t k_t < 1$, there exists $t_2 \in G$ such that $k_s R+\varepsilon_o < R+\varepsilon$ for all $s \geqq t_2$. Furthermore, Since $\lim_t \| S(t)x-y \| = K > r$, there exists $t_3 \in G$ such that $\| S(t)x-y \| > r$ for all $t \geqq t_3$. Now, let $t_0 \in G$ with $t_0 \geqq t_i$, $i=1, 2, 3$ and fix $t' \geqq t_0$. Then we have

$$\begin{aligned}
\| c_0 S(s)S(t)x+(1-c_0)y-z \| &\leqq k_s R+\varepsilon_0 \\
&< R+\varepsilon
\end{aligned}$$

for all $s \geqq t_0$. On the other hand, since

$$\| y-z \| = R < R+\varepsilon$$

and

$$\begin{aligned}
\| [c_0 S(s)S(t')x+(1-c_0)y-z]-(y-z) \| &= c_0 \| S(s)S(t')x-y \| \\
&= c_0 \| S(st')x-y \| \\
&\geqq c_0 r
\end{aligned}$$

for all $s \geqq t_0$, by uniform convexity, we have

$$\begin{aligned}
&\| \frac{c_o}{2} S(s)S(t')x+(1-\frac{c_o}{2})y-z \| \\
&= \frac{1}{2} \| [c_0 S(s)S(t')x+(1-c_0)y+z]+(y-z) \|
\end{aligned}$$

$$\leqq (R+\varepsilon)\left[1-\delta\left(\frac{c_\circ r}{R+\varepsilon}\right)\right]$$
$$< R$$

for all $s \succcurlyeq t_0$.

Letting $u_s = \frac{c_\circ}{2} S(s)S(t')x + (1-\frac{c_\circ}{2})y$, since $-\| u_s - z \| > -\| y-z \|$,

we have

$$\begin{aligned} \| u_s + \alpha(y-u_s) - z \| &= \| (1-\alpha)u_s) + \alpha y - z \| \\ &= \| (\alpha-1)(z-u_s) + \alpha(y-z) \| \\ &\geqq \alpha \| y-z \| - (\alpha-1) \| z-u_s \| \\ &\geqq \alpha \| y-z \| - (\alpha-1) \| y-z \| \\ &= \| y-z \| \end{aligned}$$

for all $\alpha \geqq 1$. Hence, by Theorem 2.5 in [8], we have

$$\langle u_s + \alpha(y-u_s) - y,\ J(y-z)\rangle \geqq 0$$

for all $\alpha \geqq 1$, and hence

$$\langle u_s - y,\ J(y-z)\rangle \leqq 0$$

for all $s \succcurlyeq t_0$. Therefore

$$\begin{aligned} &\frac{c_\circ}{2} \langle S(s)S(t')x - y,\ J(y-z)\rangle \\ &\quad = \langle \frac{c_\circ}{2} S(s)S(t_0)x + (1-\frac{c_\circ}{2})y - y,\ J(y-z)\rangle \\ &\quad \leqq 0 \end{aligned}$$

and hence

$$\langle S(s)S(t')x - y,\ J(y-z)\rangle \leqq 0$$

for all $s \succcurlyeq t_0$. Let $t \succcurlyeq t'$. Then, $t \in \{t'\} \cup Gt'$. Since we may assume that $t \in Gt'$, there exists a net $\{g_\alpha\} \in G$ with $g_\alpha t' \longrightarrow t$. Therefore, we obtain

$$\langle S(t)x-y,\ J(y-z)\rangle \leqq 0$$

for all $t \succcurlyeq t'$. This completes the proof.

New, we prove the following proposition which play a crucial role in the proof of our main theorem in this paper.

Proposition 7. Let C be a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm, G a right reversible semitopological semigroup and $\zeta=\{S(t): t \in G\}$ a Lipschitzian semigroup on C with

$$\limsup_{t} k_t \leqq 1$$

If $\{S(t)x : t \in G\}$ is bounded for any $x \in C$, then the set $\bigcap_{s \in G} \overline{\text{conv}}\{S(t)x : t \succcurlyeq s\} \cap F(\zeta)$. consists of at most one point.

Proof Let $y, z \in \bigcap_{\sigma \in G} \text{conv}\{S(t)x : t \succcurlyeq s\} \cap F(\zeta)$. Then, since $(y+z)/2 \in F(\zeta)$, it follows from Lemma 6 that there is $t_0 \in G$ such that

$$\langle S(tt_0)x-\frac{y-z}{2},\ J(\frac{y-z}{2}-z)\rangle \leqq 0$$

for every $t \in G$. Since $y \varepsilon$ conv $\{S(tt_0)x : t \in G\}$ we have

$$\langle y-\frac{y-z}{2},\ J(\frac{y-z}{2}-z)\rangle \leqq 0$$

and hence

$$\langle y-z,\ J(y-z)\rangle \leqq 0$$

This implies $y=z$.

IV. Nonlinear Erogodic Theorem

Now, we can prove a nonlinear ergodic theorem for reversible semigroups of Lipschitzian mappings in uniformly convex Banach spaces with a Fréchet differentiable norm.

Theorem 8. Let C be a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm and let G be a right reversible semitopological semigroup. Let $\zeta=\{S(t):t \in G\}$ be a Lipschitzian semigroup on C with $\limsup_t k_t \leq 1$. If $\{S(t)x : t \in G\}$ is bounded for any $x \in C$, then the following statements are equivalent :

(1) $\bigcap_{s\in G} \overline{\text{conv}}\ \{S(t)x : t \succcurlyeq s\} \cap F(\zeta)$ is nonempty for each $x \in C$,

(2) There exists a retraction (ergodic retraction) P of C onto $F(\zeta)$ such that $PS(t)=S(t)P=P$ for all $t \in G$ and $Px \in \overline{\text{conv}}\ \{S(t)x : t \in G\}$ for every $x \in C$.

Proof. (2) $\Longrightarrow$(1). Since $S(t)Px=Px$ for all $x \in C$ and $t \in G$, $Px \in F(\zeta)$. And also, since G is right reversible, $t \succcurlyeq s$ implies the existence of a net $\{g_\alpha\} \in G$ such that $g_\alpha s \to t$. Then we have

$$Px=PS(s)x \in \overline{\text{conv}}\ \{S(t)S(s)x : t \in G\} = \overline{\text{conv}}\ \{S(t)x : t \succcurlyeq s\}$$

for all $s \in G$. Hence we have

$$Px \in \bigcap_{s\in G} \text{conv}\ \{S(t)x : t \succcurlyeq s\} \cap F(\zeta).$$

(1) $\Rightarrow$(2). Let $x \in C$. By Proposition 7,

$\bigcap_{s \in G} \overline{\text{conv}} \{S(t)x : t \succcurlyeq s\} \cap F(\mathfrak{S})$ is a singleton. Hence, for each $x \in C$, define a function $P : C \longrightarrow \bigcap_{s \in G} \text{conv} \{S(t)x : t \succcurlyeq s\} \cap F(\mathfrak{S}) = \{z\}$ by $Px = z$. Then P is welldefined on C and it is a retraction of C onto $F(\mathfrak{S})$ and

$$Px \in \text{conv} \{S(t)x : t \in G\}$$

For all $t \in C$, $S(t)P = P$ is obvious. Furthermore, let $s \in G$ and $t_0 \in G$ be fixed. Since if $t \succcurlyeq s$, $t \in \{s\} \cup \overline{Gs}$. Then we have $tt_0 \in \{st_0\} \cup \overline{Gst_0}$ and hence $tt_0 \succcurlyeq st_0$. Therefore, we obtain

$$\{S(t)S(t_0)x : t \succcurlyeq s\} \in \{S(h)x : h \succcurlyeq st_0\}.$$

and also

$$\overline{\{S(t)S(t_0)x : t \succcurlyeq s\}} \in \overline{\{S(h)x : h \succcurlyeq st_0\}}.$$

On the other hand, if $h \succcurlyeq st_0$, then $h \in \{st_0\} \cup Gst_0$.
If hst_0, then

$$S(h)x = S(s)S(t_0)x \in \overline{\{S(t)S(t_0)x : t \succcurlyeq s\}}.$$

If $h \in Gst_0$, then there is a net $\{g_\alpha\} \in G$ such that $g_\alpha st_0 \rightarrow h$. So $S(h) = \lim_{\alpha} S(g_\alpha st_0)$, hence

$$S(h)x \in \{S(t)S(t_0)x : t \succcurlyeq s\}.$$

Therefore, we have

$$\overline{\{S(h)x : h \succcurlyeq st_0\}} \in \overline{\{S(t)S(t_0)x : t \succcurlyeq s\}}.$$

Consequently, we have $\overline{\text{conv}} \{S(h)x : h \succcurlyeq st_0\} = \overline{\text{conv}}\{S(t)S(t_0)x : t$ $s\}$.

Hence, if $z \in \bigcap_{s \in G} \text{conv}\,\{S(t)x : t \geqslant s\}$, then $z \in \bigcap_{s \in G} \overline{\text{conv}}\,\{S(h)x : h \geqslant st_0\} = \bigcap_{s \in G} \overline{\text{conv}}\,\{S(t)S(t_0)x : t \geqslant s\}$.

Therefore, $\bigcap_{s \in G} \overline{\text{conv}}\,\{S(t)x : t \geqslant s\} \in \bigcap_{s \in G} \overline{\text{conv}}\,\{S(t)S(t_0)x : t \geqslant s\}$ for $t_0 \in G$ be fixed. Hence, we have $\bigcap_{s \in G} \overline{\text{conv}}\,\{S(t)S(t_0)x : t \geqslant s\} \cap F(\zeta) = \bigcap_{s \in G} \overline{\text{conv}}\,\{S(t)x : t \geqslant s\} \cap F(\zeta)$.

Therefore, for $t_0 \in G$, we have

$$PS(t_0)x = Px$$

for each $x \in C$. Hence $PS(t) = P$ for all $t \in G$. This completes the proof.

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