A Remark on the Existence of Periodic Solutions to the First Order Ordinary Differential Equations

Wan Se Kim and Byung Soo Lee

I. Introduction

It is known that the system of the form

\[ \dot{x} = f(t, x) \]  \hspace{1cm} (E)

where \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous and \( T \)-periodic with respect to \( t \) for some positive constant \( T \), has at least one \( T \)-periodic solution if we assume

(H1) The solution to IVP for (E) are unique

(or \( \langle f(t, x) - f(t, x_0), x - x_0 \rangle < 0, t \in [0, T], x_0, x \in \mathbb{R} \) with \( \| x \| = r \) for some \( r > 0 \)),

(H2) \( \langle f(t, x), x \rangle = 0 \) for all \( (t, x) \in \mathbb{R} \times \mathbb{R}^n \) with \( \| x \| = r \) for some \( r > 0 \).

It is natural to ask whether we can find an appropriate sign condition which is independent of (H1), (H2) and still guarantees the existence of \( T \)-periodic solution to the system (E). The answer is affirmative when we replace (H1) by a generalized sign condition far away from the origin without assuming (H1).

In section 1, we give the answer to our question in BVP view. More precisely, we investigate the existence of \( T \)-periodic solutions
to the BVP

\[ \dot{x} = f(t, x) \]
\[ x(0) = x(T) \]

where \( x = x(t), f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is \( T \)-periodic with respect to \( t \) and continuous (or \( f \) is a Caratheodory function having sublinear growth in \( x \)). The proof is based on Leray-Schauder's continuation theorem.

In section 2, we extend our result to the delay functional differential equations. More precisely, we devote ourselves to prove the existence of \( T \)-periodic solutions to the BVP

\[ \dot{x} = f(t, x) \]
\[ x_0 = x_T \]

where \( x = x(t), x : [-r, 0] \to \mathbb{R}^n, x(s) = x(t+s) \) and \( f : \mathbb{R} \times \mathbb{C}_r \to \mathbb{R}^n \) is a continuous function and takes bounded sets into bounded sets. Here \( r \) is a non-negative constant and \( \mathbb{C}_r \) is the Banach space of continuous mappings \( h : [-r, 0] \to \mathbb{R}^n \) with the norm

\[ \| h \| = \sup_{s \in [-r,0]} | h(s) |. \]

The proof is based on Mawhin's continuation theorem.

Our results are related to these results in [2], [3] which are derived from the method of guiding functions.

II. First Order Ordinary Differential Equations.

Let \( \mathbb{C}_T \) be the Banach space of mappings \( x : \mathbb{R} \to \mathbb{R}^n \) which are continuous and \( T \)-periodic with the norm
A Remark on the Existence of Periodic Solutions to the First Order Ordinary Differential Equations

\[ \| x \|_{C_T} = \sup_{t \in \mathbb{R}} \| x(t) \| \]

\( \| x(t) \| \) is the Euclidean norm of \( x(t) \). Let \( \phi_i \) and \( \mu_i \), \( i = 1, 2, \ldots, s \) be linear independent solutions to the \( T \)-periodic, homogeneous differential equations \( \dot{x} = A(t)x \) and its adjoint \( \dot{y} = -A^*(t)y \) with \( A : \mathbb{R} \to \mathbb{R}^n \) continuous and \( T \)-periodic, respectively. By Gram Schmidt procedure, we may assume

\[ \langle \phi_i, \phi_j \rangle = \langle \mu_i, \mu_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq s. \]

Define

\[
\begin{align*}
P : C_T &\to C_T \text{ by } Px = \sum_{i \leq s} \langle x, \phi_i \rangle \phi_i \\
Q : C_T &\to C_T \text{ by } Px = \sum_{i \leq s} \langle x, \mu_i \rangle \mu_i.
\end{align*}
\]

Then they are projections.

**Proposition.** Suppose \( A(t) \) and \( b(t) \) are continuous and \( T \)-periodic on \( \mathbb{R} \). The equation

\[ \dot{x} = A(t)x + B(t) \]

has a \( T \)-periodic solution if and only if

\[ Qb = 0. \]

If (II.2) is satisfied, then (II.1) has unique \( T \)-periodic solution such that \( Px = 0 \).

Now let

\[ C_{T-0} = \{ x \in C_T \mid Px = 0 \} \]

\[ C_{T-0} = \{ x \in C_T \mid Qx = 0 \} . \]

Define \( K : C_{T-0} \to C_{T-0} \), \( b \to x \), where \( x \) is a solution to (II.1). Then \( K \) is well-defined, linear and \( K(0) = 0 \). And since \( I - Q : C_T \to C_{T-0} \)
is linear, $K(I-Q) : C_T \to C_T$ is well-defined, linear and bounded. Moreover, $K(I-Q) : C_T \to C_T$ is a compact operator.

You may find the above mentioned results in [3].

**Lemma II.1** Let $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $F(t,x)$ be continuous function, then $H : C_T \to C_T$, $x \to Hx = F(t,x(\cdot))$ is a continuous and maps bounded sets into bounded sets.

**Lemma II.2** If $A$ is a positive definite operator, then there is $c > 0$ such that $\langle Ax, x \rangle > c \| x \|^2$ for all $x \in \mathbb{R}^n$.

**Theorem II.1** Let $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $T$-periodic function with respect to $t$. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric and positive definite linear operator and $\langle f(t,x), Ax \rangle > 0$ for $\| x \| > r$ for some $r > 0$. Then BVP

(E) $\dot{x} = f(t, x)$

(B) $x(0) = x(T)$

has at least one solution.

**Proof** Let $D(L) = C_T \cap C^1[0,T]$. Define an operator $L : D(L) \subset C_T \to C_T$ by $Lx = \dot{x} - x$ for $x \in D(L)$, then $\dot{x} - x = 0$ has only trivial $T$-periodic solution which implies $P = Q = 0$. Hence for each $f \in C_T$ the $T$-periodic solution to $\dot{x} = x + f$ exists uniquely.

Therefore $L^{-1} : C_T \to C_T$, $f \to x$ exists and is a compact operator. Now consider a substitution operator

$$N : C_T \to C_T, \quad x \to -x(\cdot) + f(\cdot, x(\cdot)).$$

Then $N$ is continuous and maps bounded sets into bounded sets. Therefore, $x \in C_T$ is a solution to the BVP (B) (E) if and only if
x ∈ D(L) and x satisfies

\[ (I.3) \quad Lx = Nx, \text{ or equivalently} \]

\[ (I.4) \quad x = L^{-1}Nx \]

Since \( L^{-1} \) is a completely continuous and \( N \) is continuous and maps bounded sets into bounded sets, the composition \( L^{-1}N : C_T \to C_T \) is continuous and compact.

By using Leray-Schauder's degree argument, if all possible solution \( x \) to the family of equations

\[ (I.5) \quad x = L^{-1}Nx, \quad 0 \leq \lambda \leq 1, \]

are bounded in \( C_T \) independently of \( \lambda \), then (I.4) has a solution. If \((x, \lambda)\) solves (I.5), then \((x, \lambda)\) solves

\[ (I.6) \quad Lx = \lambda Nx, \quad 0 \leq \lambda \leq 1, \]

and \( x \) is a solution to the \( T \)-periodic BVP of the equation

\[ (I.7) \quad \dot{x} = (I - \lambda)x + \lambda f(t,x), \quad 0 \leq \lambda \leq 1 \]

When \( \lambda = 0 \) by our assumption, we have only trivial \( T \)-periodic solution. Thus the proof will be completed if we show that the solution to (I.6), for \( 0 \leq \lambda \leq 1 \), are bounded in \( C_T \) independently of \( \lambda \). To this end, define \( \phi : R^r \to R \) by \( \phi(x) = \langle Ax, x \rangle \). Let \( M = \sup_{\|x\| \leq \sigma} \phi(x) \). Then since \( \lim_{\|x\| \to \infty} \phi(x) = \infty \), for \( M > M \), there \( r_0 > 0 \) such that \( \phi(x) > M \) whenever \( \|x\| > r_0 \).

We prove that for any possible \( T \)-periodic solution \( x \) to (I.7), we have
To do this, define \( v : \mathbb{R} \rightarrow \mathbb{R}, \ t \rightarrow \phi(x(t)) \), then \( v \) is of class \( C^1 \) and \( T \)-periodic and such that

\[
(II.9) \quad \dot{v}(t) = 2\langle Ax(t), \dot{x}(t) \rangle = 2(1-\lambda)\langle Ax(t), x(t) \rangle + 2\lambda \langle Ax(t), f(t, x(t)) \rangle,
\]

for all \( t \in \mathbb{R} \). For every value \( t_0 \) of \( t \) such that

\[
v(t_0) = \sup_{t \in [0,T]} v(t) = \sup_{t \in [0,T]} v(t),
\]

we have \( v(t_0) = 0 \), since \( v \) can be extended on the whole of \( \mathbb{R} \). If \( \|x(t_0)\| > r \), then \( \langle f(t_0), Ax(t_0) \rangle > 0 \). Thus

\[
v(t_0) = 2(1-\lambda)\langle Ax(t_0), x(t_0) \rangle + 2\lambda \langle Ax(t_0), f(t_0, x(t_0)) \rangle > 0
\]

which is impossible. Hence \( \|x(t_0)\| \leq r \).

If there exists \( t_i \in [0,T] \) such that \( \|x(t_i)\| > r \), then \( v(t_i) = \langle Ax(t_i), x(t_i) \rangle > M_0 \) and so \( M_0 < v(t_i) \leq \sup_{t \in [0,T]} \langle Ax(t), x(t) \rangle = \sup_{t \in [0,T]} \langle Ax(t), x(t) \rangle = M \)

which is a contradiction. Hence we have \( \|x(t)\| \leq r \) for all \( t \in [0,T] \) for every possible \( T \)-periodic solution to (II.7).

So we have that every solution \((x, \lambda)\) to (II.5) has an \( a' \) priori hound in \( C_T \) independently of \( \lambda \). Therefore, by Leray-schauder’s continuation theorem, \( \dot{x} = L^{-1}N x \) has a solution, or \( \dot{x} = f(t, x) \) has a solution in \( C_T \).

**Corollary II.1** Let \( f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be continuous and \( T \)-periodic function with respect to \( t \). Let \( \langle f(t, x), x \rangle \geq 0 \) for \( \|x\| \geq r \) for some \( r > 0 \). Then BVP (E) (B) has at least one solution.

**Theorem II.2**. Let \( f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be continuous and \( T \)-periodic function with respect to \( t \) with \( \|f(t, x)\| \leq \alpha \|x\| + \beta \) for some \( \alpha, \beta \).
A Remark on the Existence of Periodic Solutions to the
First Order Ordinary Differential Equations

0<\alpha<1/T, \beta>0 for all (t, x) \in \mathbb{R} \times \mathbb{R}^n. Let \( A : \mathbb{R}^n \to \mathbb{R}^n \) be symmetric linear operator and has no eigenvalue with zero real part, and \( \langle f(t, x), Ax \rangle > 0 \) for \( \| x \| > r \) for some \( r > 0 \). Then BVP (B) (E) has at least one solution in \( C_T \).

**Proof** Let \( D(L) = C_T \cap C^1[0, T] \). Define an operator \( L : D(L) \subseteq C_T \to C_T \) by \( Lx = x - \varepsilon Ax \), where \( \varepsilon \) such that \( \varepsilon T \| A \| + \alpha T \langle 1 \rangle \), for \( x \in D(L) \), then for each \( f \in C_T \), the \( T \)-periodic solution \( x \) to \( \dot{x} = \varepsilon Ax + f \) exists uniquely. Therefore \( L^{-1} : C_T \to C_T, f \mapsto x \) exists and is a compact operator. Now we consider a substitution operator

\[
N : C_T \to C_T, \quad x \mapsto -\varepsilon Ax(\cdot) + f(\cdot, x(\cdot))
\]

Then \( N \) is continuous and maps bounded sets into bounded sets. Therefore, \( x \in C_T \) is a solution to the BVP (B) (E) if and only if \( x \in D(L) \) and \( x \) satisfies

\[
(I.10) \quad Lx = Nx, \quad \text{or}
\]

\[
(I.11) \quad x = L^{-1}Nx.
\]

Since \( L^{-1} \) is a completely continuous and \( N \) is continuous and maps bounded sets into bounded sets, the composition \( L^{-1}N : C_T \to C_T \) is continuous and compact. By using Leray-Schauder's degree argument, if all solution \( x \) to the family of equations.

\[
(I.12) \quad x = \lambda L^{-1}Nx, \quad 0 \leq \lambda \leq 1,
\]

are bounded in \( C_T \) independent of \( \lambda \), then (II.10) has a solution. If \( (x, \lambda) \) solves (II.12), then \( (x, \lambda) \) solves

\[
(I.13) \quad Lx = \lambda Nx, \quad 0 \leq \lambda \leq 1,
\]

and \( x \) is solution to the \( T \)-periodic BVP of the equation.
If $\lambda = 0$ we have only trivial $T$-periodic solution. Thus, the proof will be completed if we show that the solution to (II.12), for $0 < \lambda \leq 1$, are bounded in $C_T$ independently of $\lambda$. To this end, let $(x, \lambda)$ be any solution to (II.13) with $0 < \lambda \leq 1$ then
\[
\|x\| = (1 - \lambda) \|Ax\| + \lambda \|f(t, x)\| \quad (0 < \lambda \leq 1)
\]
\[
\leq \|Rx\| + \|f(t, x)\|
\leq \epsilon \|A\| \|x\| + \alpha \|x\| + \beta
\]
\[
= (\epsilon \|A\| + \alpha) \|x\| + \beta.
\]
If $\|x(t)\| \geq r$ for all $t \in [0, T]$, then
\[
0 = \int_0^T \langle x(t), Ax(t) \rangle dt
\]
\[
= (1 - \lambda) \epsilon \int_0^T \langle Ax(t), A(t) \rangle dt + \lambda \int_0^T \langle f(t, x(t)), Ax(t) \rangle dt > 0
\]
which is impossible. Hence there is a $t_0 \in [0, T]$ such that $\|x(t_0)\| < r$.

Since $x(t) = x(t_0) + \int_{t_0}^t x(t) dt$, $\|x\| \leq r + \int_{t_0}^t \|x\| dt = r + T \|x\|$. Therefore,
\[
\|x\| \leq r + [\epsilon T \|A\| + \alpha T] \|x\| + \beta T, \quad \text{or}
\]
\[
[I - \epsilon T \|A\| - \alpha T] \|x\| \leq r + \beta T.
\]
Since $\epsilon T \|x\| + \alpha T < 1$, we have
\[
\|x\| \leq (r + \alpha T) / (1 - \epsilon T \|A\| - \alpha T).
\]

Hence, we have that every solution $(x, \lambda)$ to (II.12) has an a priori bound in $C_T$ independently of $\lambda$. Therefore, by the Leray-Schauder's continuation Theorem, $x = L^{-1}N_x$ has a solution, or $\dot{x} = f(t, x)$ has a solution in $C_T$.

**Corollary II.12** Let $f : Rx^u \to R^u$ be continuous function and
A Remark on the Existence of Periodic Solutions to the
First Order Ordinary Differential Equations

T-periodic function with respect to $t$ with $\|f(t, x)\| \leq \alpha \|x\| + \beta$ for some $\alpha$, $\beta \geq 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. And $\langle f(t, x), x \rangle \geq 0$ for $\|x\| > \tau$, for some $\tau > 0$.

Then BVP (B) (E) has at least one solution in $C_r$.

Example

$$\dot{x} = ax + bx^3 + e(t)$$
$$x(0) = x(T)$$

where $e : \mathbb{R} \to \mathbb{R}$ is continuous, $T$-periodic and $b > 0$, has at least one $T$-periodic solution.

III. First Order Ordinary Delay Functional Differential Equations

Let us denote by $C_T$ the Banach space of continuous and $T$-periodic mappings $x : \mathbb{R} \to \mathbb{R}^n$ with the norm $\|x\|_{C_T} = \sup_{t \in \mathbb{R}} \|x(t)\|$ where $\cdot$ is the Euclidean norm in $\mathbb{R}^n$. For some $r > 0$ let $C_r$ be the Banach space of continuous mapping $\phi : [-r, 0] \to \mathbb{R}^n$ with the norm $\|\phi\|_{C_r} = \sup_{\theta \in [-r, 0]} \|\phi(\theta)\|$. When $r = 0$, $C_r$ is naturally identified to $\mathbb{R}^n$.

If $x \in C_T$ and $t \in T$, we shall denote by $x_t$ the element of $C_r$ defined by

$$x_t : [-r, 0] \to \mathbb{R}^n, \ \theta \to x(t + \theta).$$

We note that,

$$\|x_t\|_{C_r} = \sup_{\theta \in [-r, 0]} \|x(t + \theta)\| \leq \sup_{t \in \mathbb{R}} \|x(t)\| = \|x\|_{C_T}.$$

When $r = 0$ the mapping $x_t$ will be naturally identified with the element $x(t)$ of $\mathbb{R}^n$. Moreover we shall sometimes identify, without further comment, a constant mapping in $C_T$ or $C_r$ with the element of $\mathbb{R}^n$.
given by its constant value.

Let \( f : R x C \rightarrow R, (t, \phi) \rightarrow f(t, \phi) \)

be \( T \)-periodic with respect to \( t \), continuous and take bounded sets into bounded sets. Let us consider the functional differential equation.

(III.1) \[ x = f(t, x_0). \]

If we define the Banach space by \( X = \{ x \subseteq C, x_0 = x_T \} \) and

\[
\begin{align*}
\text{Dom} L &= X \cap C^\prime[0, T] \cap C_T \\
L : \text{Dom} L &\rightarrow C_T, x \rightarrow x, \\
N : C_T &\rightarrow C_T, x \rightarrow f(x, x).
\end{align*}
\]

then \( \text{Ker} L = R^n \), \( \text{Im} L = \{ y \subseteq C_T : \int_0^T y(s)ds = 0 \} \).

Let us introduce the continuous projectors

\[
\begin{align*}
P : C_T &\rightarrow C_T, x \rightarrow x(0) \\
Q : C_T &\rightarrow C_T, x \rightarrow 1/T \int_0^T x(s)ds.
\end{align*}
\]

Then for each \( x \subseteq C_T \)

\[ \| Qx \|_{C_T} < \| x \|_{C_T} \]

and \( \text{Im} Q \) is the subspace of \( C_T \) of constant mappings, and the following sequence is exact:

\[
C_T \overset{P}{\rightarrow} \text{Dom} L \subset C_T \overset{L}{\rightarrow} C_T \overset{Q}{\rightarrow} C_T
\]

which implies

\[ \text{Ker} L = \text{Im} P, \text{Im} L = \text{Ker} Q. \]
and
\[ C_T = \text{Im} P \oplus \text{Ker} P = \text{Ker} L \oplus \text{Ker} P, \quad C_T = \text{Im} Q \oplus \text{Ker} Q = \text{Im} Q \oplus \text{Im} L \]
as topological sums.
Thus we have \( C_T / \text{Im} L \cong \text{Im} Q \).
\[ \text{Im} P = \{ x(0) : x \in P \} = \mathbb{R}^n, \]
\[ \text{Im} Q = \{ 1 / T \int_0^T x(s) ds : x \in C_T \} = \mathbb{R}^n. \]

\[ \dim \text{Ker} L = n = \dim \text{Im} Q = \dim C_T / \text{Im} L = \dim \text{CoKer} L \otimes \infty, \]
\( L \) is linear and \( \text{Im} L \) is closed in \( C_T \). Hence \( L \) is Fredholm mapping of index zero and there exists an isomorphism \( J : \text{Im} Q \rightarrow \text{Ker} L \).

If we consider the restriction
\[ L_p = L \mid_{\text{Dom} \cap \text{Ker} P} : \text{Dom} L \cap \text{Ker} P \rightarrow \text{Im} L, \]
than \( L_p \) is bijective, so that its algebraic inverse
\[ K_p = L_p^{-1} : \text{Im} L \rightarrow \text{Dom} L \cap \text{Ker} P \]
is defined and \( K_p(y)(t) = x(t) = \int_0^t y(s) ds \)
We will denote \( K_{p0} : C_T \rightarrow \text{Dom} L \cap \text{Ker} P \) the generalized inverse of \( L \) defined by \( K_{p0} = K_p(I - Q) \).
Then \( K_{p0} \) is a compact operator by Arzela-Ascoli theorem. \( K_{p0} N \) takes bounded sets into relatively compact sets since \( N \) takes bounded sets into bounded sets. You may find the following Lemma in Mawhin [1], Mawhin and Gains [2].

**Lemma III.1** With the assumption and notations above, \( N \) is \( L \)-compact on each bounded subset of \( C_T \).
Theorem III.1 Let $f : R^2 \times C \rightarrow R^n$ be $T$-periodic with respect to $t$, continuous and takes bounded sets into bounded sets. Let $A : R^n \rightarrow R^n$ be a symmetric and positive definite linear operator such that $\langle f(t, x), Ax \rangle \geq 0$, $\|x\| \geq r$ for some $r > 0$. Then BVP

(E) \quad \dot{x} = f(t, x)
(B) \quad x_0 = x_T

has at least one solution.

Proof. We will apply Mawhin's continuation theorem to our proof. Now it is easy to see $x \in C_T$ is a solution BVP (E) (B) if and only if $x \in \text{Dom}L$ and

(III.1) \quad Lx = Nx.

Since $L$ is a Fredholm mapping of index zero and $N$ is $L$-compact, by Mawhin's continuation theorem if there exits a bounded open set $G$ in $C_T$ such that

(a) for each $\lambda \in \mathbb{R}$, $\lambda \in \mathbb{R}$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \in \partial G$.

(b) $QNx \neq 0$ for each $x \in \text{Ker} L \cap \partial G$ and

\[ d(QN|_{\text{Ker} L^*}, G \cap \text{Ker} L, 0) \neq 0 \]

where $d$ is the Brouwer topological degree.

Then the equation $Lx = Nx$ has at least one solution in $\text{Dom}L \cap G$.

Now we prove (a). For this purpose, let $(x, \lambda)$ be any solution to

(III.2) \quad Lx = \lambda Nx,
then \((x, \lambda)\) is a solution to BVP

(B) \[ \dot{x} = \lambda f(t, x) \]
\[ x_0 = x_T \]

Let \(M = \sup_{\|x\| \leq r} \langle Ax, x \rangle\), then since \(\lim_{\|x\| \to \infty} \langle Ax, x \rangle = \infty\),
for \(M_0 > M\), there exists \(r_0 > r\) such that \(\langle Ax, x \rangle > M_0\) whenever \(\|x\| > r_0\).

Let us define \(v : \mathbb{R} \to \mathbb{R}\) by
\[ v(t) = \langle Ax(t), x(t) \rangle \quad \text{for all } t \in \mathbb{R} \]

Then, \(v\) is of class \(C^1\) and \(T\)-periodic such that
\[ \dot{v}(t) = 2 \langle Ax(t), \dot{x}(t) \rangle = 2\lambda \langle Ax(t), f(t, x(t)) \rangle \quad \text{for all } t \in \mathbb{R} \]

For every value \(t_0\) of \(t\) such that
\[ v(t_0) = \sup_{t \in [0, T]} v(t) = \sup_{\|x(t)\| \leq r_0} v(t) \]
we have \(\dot{v}(t_0) = 0\) if \(\|x(t_0)\| > r\), then \(\langle f(t_0, x(t_0)), Ax(t_0) \rangle > 0\).

Thus
\[ \dot{v}(t_0) = 2\lambda \langle Ax(t_0), f(t_0, x(t_0)) \rangle > 0, \]
which is impossible. Hence \(\|x(t_0)\| < r\).

If there exists \(t_i\) in \([0, T]\) such that \(\|x(t_i)\| > r_0\), then
\[ v(t_i) = \langle Ax(t_i), x(t_i) \rangle > M_0 \]
and so
\[ M_0 \leq v(t_i) < \sup_{\|x\| \leq r_0} \langle Ax, x \rangle = M \]
which is impossible. Hence, we have \(\|x(t)\| \leq r_0\) for all \(t \in [0, T]\),
i.e.,
\[ \| x \| = \sup_{t \in [0,T]} \| x(t) \| < r_0 \]
for every possible solution to (III.2). Therefore every solution \((x, \lambda)\) of (III.2) is such that \(x \in \partial G\) where \(G\) is an open ball in \(C_\tau\) with radius \(p > r_0\) and centered at origin.

Now we will show that the condition (b) is satisfied. Since \(\langle f(t, x), Ax \rangle > 0\) for \(\| x \| > r\), we obtain
\[ \langle Aa, \int_0^t f(t, a)dt \rangle > 0 \]
for every \(a \in \mathbb{R}^n\) such that \(\| a \| > r\) and hence \(QN_x \neq 0\) for each \(x \in \text{Ker}L \cap \partial G\) and for each \(\lambda \in \mathbb{R}\), \((1-\lambda)A + \lambda QN(\cdot) = 0\) for every \(c \in \partial G \cap \text{Ker}L\). Hence, by the homotopy invariant property of Brouwer degree, we have
\[ d(1-\lambda)A + \lambda QN |_{\text{Ker}L} \cap (\text{Ker}L, 0) = d(QN |_{\text{Ker}L} \cap (\text{Ker}L, 0)) = d(A |_{\text{Ker}L} \cap (\text{Ker}L, 0)) = [\text{sgn}(\text{det}^*)] [\text{sgn}(\text{det}A^*)] \neq 0, \]
Since \(A\) is positive definite linear operator, where \(f^*, A^\prime\) are the matrix representation of \(f\) and \(A\) in some some basis in \(\text{Ker}L\). Thus
\[ d(QN |_{\text{Ker}L} \cap (\text{Ker}L, 0)) \neq 0 \]
Hence the conditions (a), (b) are satisfied and our proof is completed.
A Remark on the Existence of Periodic Solutions to the
First Order Ordinary Differential Equations

Example

\[ \dot{x}(t) = ax(t) + bx(t-r) + cx(t-r) + dx(t) + e(t). \]

where a, b, c, d are constant with \(|c| > |d|\) and \(e: \mathbb{R} \to \mathbb{R}\) is continuous and \(T\)-periodic, has at least one \(T\)-periodic solution.

REFERENCES


Department of Mathematics
Dong-A University
Pusan 604-714, Korea

Department of Mathematics
Kyungsung University
Pusan 608-736, Korea