ON GENERALIZED NEAR-FIELDS

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Throughout this paper \( N \) stands for a right near-ring. For the basic terminologies and notations, we refer to Pilz [7]. In Murty [6] a near-ring \( N \) is called a generalized near-field (GNF) if for each \( a \) in \( N \) there exists a unique \( b \) in \( N \) such that \( a=aba \) and \( b=bab \), that is, \( (N, \cdot) \) is an inverse semigroup. See Howie [1], for properties of inverse semigroups. Recall that a near-ring \( N \) is called subcommutative if \( aN=Na \) for all \( a \) in \( N \) and a near-ring \( N \) is called regular (strongly regular) if for each \( a \) in \( N \) there exists \( b \) in \( N \) such that \( a=aba \) \( (a=baa) \). In Lee [3], Jat and Choudhary [2], a near-ring \( N \) is called left bipotent if \( Na=Na^2 \) for all \( a \) in \( N \), and \( N \) is called an S-near-ring if \( a \) in \( Na \) for all \( a \) in \( N \). In Ligh and Utumi [4], \( N \) is said to have the condition \( C_1 \) \( (C_2) \) if \( Na=aNa(aN=Na) \) for all \( a \) in \( N \). \( N \) has IFP if \( ab=0 \) implies \( bxa=0 \) for all \( x \) in \( N \) and \( a, b \) in \( N \) [7].

The aim of this paper is to show a characterization of a GNF, that is, \( N \) is an (left bipotent) S-near-ring with the condition \( C_1 \) if and only if it is a generalized near-field.

We need the following lemmas due to Mason [5] and Murty [6].

**Lemma 1.** If a zero-symmetric near-ring \( N \) has no non-zero nilpotent elements, then \( N \) has IFP.
Lemma 2. If a near-ring $N$ is a GNF, then it is zero-symmetric and has no non-zero nilpotent elements.

Theorems 3. Suppose $N$ is a strongly regular near-ring. Then $N$ has the condition $C_2$.

Proof. Let $a$ be in $N$. Then $a=axa$ for some $x \in N$ by Theorem 3 of Reddy and Murty [8]. Hence for $a$ and $b \in N$, $ab=axab=axab-bxa \subseteq aNa$ by Corollary 11 of Reddy and Murty [8]. Therefore $aN = aNa$.

Lemma 4. If a near-ring $N$ has the IFP, then for any $a, n \in N$ and any idempotent $e \in N$, $ane = aeae$.

The proof of this lemma is easy and hence omitted.

Corollary 5. If a regular near-ring $N$ has the IFP, then it has the condition $C_2$.

Proof. Let $a$ be in $N$. Since $N$ is regular, there exists $x \in N$ such that $a=axa$. Since $xa$ is an idempotent, $xa=xax=axa=axa=axa$ by Lemma 4. So $a=axa-axa \subseteq N^2$. Thus $N$ is strongly regular and hence, by Theorem 3, $N$ has the condition $C_2$.

Theorem 6. Let a zero symmetric near-ring $N$ have no non-zero nilpotent elements. Then $N$ is regular if and only if it has the condition $C_2$.

Proof. If $N$ is regular, by Lemma 1 and Corollary 5, $N$ has the condition $C_2$.

For the converse, assume that $N$ has the condition $C_2$. Then, for any $a \in N$, there exists $x \in N$ such that $a=axa$. Thus we have $(a-ax)a=0$. By Lemma 1, $a(a-ax)=0$ and $ax(a-ax)=0$. Hence we have $(a-ax)^2$.
= 0. Since \( N \) has no non-zero nilpotent elements, \( a = ax \). Since \( N \) has the condition \( C_0 \), we have \( a = ax = aya \) for some \( y \in N \). Hence \( N \) is regular.

**Theorem 7.** (Murty [6]). The following are equivalent:

1. \( N \) is a GNF.
2. \( N \) is regular and each idempotent is central.
3. \( N \) is regular and subcommutative.

**Theorem 8** (Lee [3]). \( N \) is a left bipotent S-near-ring if and only if it is strongly regular.

Now we prove our main theorem.

**Theorem 9.** The following are equivalent.

1. \( N \) is a GNF.
2. \( N \) is regular and subcommutative.
3. \( N \) is an S-near-ring with the condition \( C_i \).
4. \( N \) is a left bipotent S-near-ring with the condition \( C_i \).

**Proof.** (1) \( \rightarrow \) (2) Follows by theorem 7.

(2) \( \rightarrow \) (3). Let \( xa \in Na \) and \( a = aya \) for some \( y \in N \). Since \( N \) is subcommutative, \( xa = ax \) for some \( x \in N \). Therefore by Theorem 7, \( xa = ax = aya = aya \in aNa \). Hence \( N \) has the condition \( C_i \). So that \( N \) is an S-near-ring.

(3) \( \rightarrow \) (4). Let \( a \) be in \( N \) with \( a^2 = 0 \). Since \( N \) is an S-near-ring with \( C_i \), there exists \( x \in N \) such that \( a = axa \). Since \( xa \in Na = aNa \), \( xa = aya \) for some \( y \in N \). So \( a = axa = a(aya) = aya = 0 \). Thus \( N \) has no non-zero nilpotent element. Hence by Proposition 9.43 of Pilz
for each idempotent \( e \in N \) and \( n \in N \), \( en = ene \). Therefore \( N \) is strongly regular by Theorem 12 of Reddy and Murty [8]. Thus by Theorem 8, \( N \) is a left bipotent \( S \)-near-ring.

(4)\( \rightarrow \)(1) From Theorem 8, \( N \) is strongly regular. From the hypothesis and Theorem 3, \( N \) is subcommutative. \( N \) is regular by Theorem 3 of Reddy and Murty [8]. Hence \( N \) is GNF by Theorem 7.

**Remarks.** The condition that \( N \) is a \( S \)-near-ring is essential in Theorem 9. As an example consider the following:

**Example 10** (See Pilz [7], p. 340 (E), (0,7,0,7). Let \( (N, +) \) (where \( N = \{0, a, b, c\} \)) be the klein four group. Define multiplication as follows:

\[
\begin{array}{cccc}
\cdot & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & a \\
b & 0 & 0 & 0 & 0 \\
c & 0 & a & 0 & a \\
\end{array}
\]

Then \( (N, +, \cdot) \) is a near-ring which satisfies the conditions \( C_1 \) and \( C_2 \). But \( N \) is not an \( S \)-near-ring and hence \( N \) is not a GNF.

**References**

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