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ON GENERALIZED NEAR-FIELDS

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Throughout this paper N stands for a right near-ring. For the basic terminologies and notations, we refer to Pilz [7]. In Murty [6] a near-ring N is called a gerneralized near-field (GNF) if for each a in N there exists a unique b in N such that $a=aba$ and $b=bab$, that is, (N, $\cdot$) is an inverse semigroup. See Howie [1], for properties of inverse semigroups. Recall that a near-ring N is called subcommutative if $aN=Na$ for all a in N and a nearring N is called regular (strongly regular) if for each a in N there exists b in N such that $a=aba$ ($a=ba^2$). In Lee [3], Jat and Choudhary [2], a nearring N is called left bipotent if $Na=Na^2$ for all a in N, and N is called an S-near-ring if a in Na for all a in N. In Ligh and Utumi [4], N is said to have the condition C_1 (C_2) if $Na=aNa(aN=aNa)$ for all a in N. N has IFP if $ab=0$ implies $bxa=0$ for all x in N and a, b in N [7].

The aim of this paper is to show a characterization of a GNF, that is, N is an (left bipotent) S-near-ring with the condition C_1 if and only if it is a generalized near-field.

We need the following lemmas due to Mason [5] and Murty [6].

Lemma 1. If a zero-symmetric near-ring N has no non-zero nilpotent elements, then N has IFP.

Lemma 2. If a near-ring N is a GNF, then it is zero-symmetric and has no non-zero nilpotent elements.

Theorems 3. Suppose N is a strongly regular near-ring. Then N has the condition C_2.

Proof. Let a be in N. Then $a=axa$ for some $x \in N$ by Theorem 3 of Reddy and Murty [8]. Hence for a and $b \in N$, $ab=axab=axa$-$bxa \in aNa$ by Corollary 11 of Reddy and Murty [8]. Therefore $aN=aNa$.

Lemma 4. If a near-ring N has the IFP, then for any a, $n \in N$ and any idempotent $e \in N$, $ane=aene$.

The proof of this lemma is easy and hence omitted.

Corollary 5. If a regular near-ring N has the IFP, then it has the condition C_2.

Proof. Let a be in N. Since N is regular, there exists $x \in N$ such that $a=axa$. Since xa is an indempotent, $xa=xaxa=x(xa)a(xa)=x^2a^2$ by Lemma 4. So $a=axa=ax^2a^2 \in Na^2$. Thus N is strongly regular and hence, by Theorem 3, N has the condition C_2.

Theorem 6. Let a zero symmetric near-ring N have no non-zero nilpotent elements. Then N is regular if and only if it has the condition C_2.

Proof. If N is regular, by Lemma 1 and Corollary 5, N has the condition C_2.

For the converse, assume that N has the condition C_2. Then, for any $a \in N$, there exists $x \in N$ such that $a^2=axa$. Thus we have $(a-ax)a=0$, By Lemma 1, $a(a-ax)=0$ and $ax(a\text{-}ax)=0$. Hence we have $(a-ax)^2$

$=0$. Since N has no non-zero nilpotent elements, $a=ax$. Since N has the condition C_2, we have $a=ax=aya$ for some $y \in N$. Hence N is regular.

Theorem 7. (Murty [6]). The following are equivalent;
(1) N is a GNF.
(2) N is regular and each idempotent is central.
(3) N is regular and subcommutative.

Theorem 8 (Lee [3]). N is a left bipotent S-near-ring if and only if it is strongly regular.

Now we prove our main theorem.

Theorem 9. The following are equivalent.
(1) N is a GNF.
(2) N is regular and subcommutative.
(3) N is an S-near-ring with the condition C_1.
(4) N is a left bipotent S-near-ring with the condition C_1.

Proof. (1)→(2) Follows by theorem 7.

(2)→(3). Let $xa \in Na$ and $a=aya$ for some $y \in N$. Since N is subcommutative, $xa=az$ for some $z \in N$. Therefore by Theorem 7, $xa=az=ayaz=azya \in aNa$. Hence N has the condition C_1. So That N is an S-near-ring.

(3)→(4). Let a be in N with $a^2=0$. Since N is an S-near-ring with C_1, there exists $x \in N$ such that $a=axa$. since $xa \in Na=aNa$, $xa=aya$ for some $y \in N$. So $a=axa=a(aya)=a^2ya=0$. Thus N has no non-zero nilpotent element. Hence by Proposition 9.43 of Pilz

[7], for each idempotent $e \in N$ and $n \in N$, $en = ene$. Therefore N is strongly regular by Theorem 12 of Reddy and Murty [8]. Thus by Theorem 8, N is a left bipotent S-near-ring.

(4)→(1) From Theorem 8, N is strongly regular. From the hypothesis and Theorem 3, N is subcommutative. N is regular by Theorem 3 of Reddy and Murty [8]. Hence N is GNF by Theorem 7.

Remarks. The condition that N is a S-near-ring is essential in Theorem 9. As an example consider the following ;

Example 10 (See Pilz [7], p. 340 (E), (0,7,0,7). Let $(N, +)$ (where $N = \{0,a,b,c\}$) be the klein four group. Define multiplication as follows ;

·	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

Then $(N, +, \cdot)$ is a near-ring which satisfies the conditions C_1 and C_2. But N is not an S-near-ring and hence N is not a GNF.

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