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NOTES ON THE RUSCHEWEYH DERIVATIVES

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Abstract

We introduce the classes $S^*(n, \alpha)$ and $K(n, \alpha)$ of analytic functions by using the symbol $D^\alpha f(z)$ defined by $\frac{z}{(1-z)^{1+\alpha}} * f(z)$ $(\alpha \geq -1)$. The object of the present paper is to derive some properties of these classes.

1. Introduction

Let A(n) be the class of the functions of the form

$$(1.1) \qquad f(z)=z+\sum_{k=n+1}^{\infty} a_n z^k \quad (n \in N=\{1,2,3,\ldots\})$$

which are analytic in the unit disk $U=\{z : |z|<1\}$.

A function $f(z)$ belonging to the class $A(n)$ is said to be in the class $S^*(n)$ if and only if

$$(1.2) \qquad Re\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (z \in U).$$

Further, a function $f(z)$ belonging to the class $A(n)$ is said to be in the class $K(n)$ if and only if

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$$(1.3) \qquad Re\{1+\frac{zf''(z)}{f'(z)}\}>0 \qquad (z \in U).$$

Note that $f(z) \in K(n)$ if and only if $zf'(z) \in S^*(n)$, and that $K(n) \subset S^*(n)$.

For the functions $f_j(z)(j=1,2)$ defined by

$$(1.4) \qquad f_j(z)=z+\sum_{k=n+1}^{\infty} a_{k,j}z^k,$$

we define the Hadamard product (or convolution) $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ by

$$(1.5) \qquad f_1 * f_2(z)=z+\sum_{k=n+1}^{\infty} a_{k,1}\, a_{k,2}\, z^k.$$

Making use of the convolution, Ruscheweyh [4] introduced the symbol $D^{\alpha}f(z)$ by

$$(1.6) \qquad D^{\alpha}f(z)=\frac{z}{(1-z)^{1+\alpha}} * f(z) \qquad (\alpha \geq -1)$$

for $f(z)$ in $A(n)$, which is called the Ruscheweyh derivative of $f(z)$.

A function $f(z)$ belonging to the class $A(n)$ is said to be in the class $S^*(n,\alpha)$ if it satisfies $D^{\alpha}f(z) \in S^*(n)$ for $\alpha \geq -1$. A function $f(z)$ belonging to the class $A(n)$ is said to be in the class $K(n,\alpha)$ if it satisfies $D^{\alpha}f(z) \in K(n)$ for $\alpha \geq -1$.

The classes $S^*(1,\alpha)$ and $K(1,\alpha)$ for $n=1$ were studied by Owa, Fukui, Sakaguchi and Ogawa [3].

2. Some Properties of The Classes $S^*(n, \alpha)$ and $K(n, \alpha)$

In order to derive some properties of our classes, we have to recall here the following results.

Lemma 1 ([1], [3]). for $\alpha \geq -1$, we have

$$z(D^{\alpha}f(z))' = (\alpha+1)D^{\alpha+1}f(z) - \alpha D^{\alpha}f(z). \tag{2.1}$$

Lemma 2([2]). Let $\phi(u,v)$ be a complex valued function,

$\phi : D \to C$, $D \subset C \times C$(C is the complex plane),

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies the following conditions:

(1) $\phi(u,v)$ is continuous in D;

(2) $(1,0) \in D$ and $Re\{\phi(1,0)\} > 0$;

(3) for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -n(1+u_2^2)/2$, $Re\{\phi(iu_2, v_1)\} \leq 0$.

Let $P(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$ be regular in the unit disk U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$Re\{\phi(p(z), zp'(z))\} > 0 \qquad (z \in U),$$

then $Re\{P(z)\} > 0$ $(z \in U)$.

Applying the above lemmas, we derive

Theorem 1. For $\alpha \geq 0$, we have

$$S^*(n, \alpha+1) \subset S^*(n, \alpha).$$

Proof. Define the function $p(z)$ by

$$\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}=p(z) \tag{2.2}$$

for $f(z)$ belonging to the class $S^*(n,\alpha+1)$. Then $p(z)=1+p_nz^n+p_{n+1}z^{n+1}+\cdots$ is regular in U, so using Lemma 1, we have

$$\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}=\frac{1}{\alpha+1}(\alpha+p(z)). \tag{2.3}$$

Taking the logarithmic differentiations of both sides in (2,3), we obtain

$$\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)}=p(z)+\frac{zp'(z)}{\alpha+p(z)}, \tag{2.4}$$

or

$$Re\{\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)}\}=Re\{p(z)+\frac{zp'(z)}{\alpha+p(z)}\}>0. \tag{2.5}$$

Let $p(z)=u=u_1+iu_2$, $zp'(z)=v=v_1+iv_2$, and

$$\phi(u,v)=u+\frac{v}{\alpha+u}. \tag{2.6}$$

Then

(1) $\phi(u,v)$ is continuous in $D=(C-\{\alpha\})\times C$;

(2) $(1,0)\ \varepsilon\ D$ and $Re\{\phi(1,0)\}=1>0$.

(3) for all $(iu_2,v_1)\ \varepsilon\ D$ and such that $v_1\leq -n(1+u_2^2)/2$,

$$Re\{\phi(iu_2,v_1)\}=\frac{\alpha v_1}{\alpha^2+u_2^2}$$
$$\leq -\frac{\alpha n(1+u_2^2)}{2(\alpha^2+u_2^2)}$$
$$\leq 0.$$

Therefore, the function $\phi(u,v)$ satisfies the conditions in Lemma 2. This implies that

$$\text{(2.7)} \qquad Re\{\frac{z(D^\alpha f(z))'}{D^\alpha f(z)}\}=Re\{p(z)\}>0 \qquad (z \in U),$$

this is, that $f(z) \in S^*(n,\alpha)$. Thus we complete the assertion of Theorem 1.

Remark. From the definition of the Ruscheweyh derivative, it is easy to see that

$$z(D^\alpha f(z))'=\frac{z}{(1-z)^{1+\alpha}} * zf'(z)=D^\alpha(zf'(z)).$$

With the help of Theorem 1 and Remark, we prove

Theorem 2. For $\alpha \geq 0$, we have

$$K(n,\alpha+1) \subset K(n,\alpha).$$

Proof. Note that

$$f(z) \in K(n,\alpha+1) \Leftrightarrow D^{\alpha+1}f(z) \in K(n)$$
$$\Leftrightarrow z(D^{\alpha+1}f(z))' \in S^*(n)$$

$$\Leftrightarrow D^{\alpha+1}(zf'(z)) \in S^*(n)$$
$$\Leftrightarrow zf'(z) \in S^*(n,\alpha+1)$$
$$\Leftrightarrow zf'(z) \in S^*(n,\alpha)$$
$$\Leftrightarrow D^{\alpha}(zf'(z)) \in S^*(n)$$
$$\Leftrightarrow z(D^{\alpha}f(z))' \in S^*(n)$$
$$\Leftrightarrow D^{\alpha}f(z) \in K(n)$$
$$\Leftrightarrow f(z) \in K(n,\alpha).$$

This completes the proof of Theorem 2.

Finally, we drive

Theorem 3. If the function $f(z)$ defined by (1.1) is in the class $S^*(n,\alpha)$, then

(2.8) $$Re\{(\frac{D^{\alpha}f(z)}{z})^{\beta-1}\} > \frac{n}{2(\beta-1)+n} \quad (z \in U),$$

where $1<\beta\leq(n-2)/2$.

Proof. Defining the function $p(z)$ by

(2.9) $$(\frac{D^{\alpha}f(z)}{z})^{\beta-1} = \gamma+(1-\gamma)p(z)$$

for $f(z) \in S^*(n,\alpha)$, where $\gamma=\frac{n}{2(\beta-1)+n}$, we see that $p(z)=1+p_n z^n+p_{n+1} z^{n+1}+\ldots$ is regular in U. Making the logarithmic differentiations of both sides in (2.9), we have

(2.10) $$\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}=1+\frac{(1-\gamma)zp'(z)}{(\beta-1)\{\gamma+(1-\gamma)p(z)\}}.$$

Therefore

$$(2.11)\qquad Re\{1+\frac{(1-\gamma)zp'(z)}{(\beta-1)\ \{\gamma+(1-\gamma)p(z)\}}>0\quad (z\ \varepsilon\ U).$$

Letting $p(z)=u=u_1+iu_2$, $zp'(z)=v=v_1+iv_2$, and

$$(2.12)\qquad \phi(u,v)=1+\frac{(1-\gamma)v}{(\beta-1)\ \{\gamma+(1-\gamma)u\}},$$

we obtain that

(1) $\phi(u,v)$ is continuous in $D=(C-\{\frac{\gamma}{\gamma-1}\})\times C$;
(2) $(1,0)\ \varepsilon\ D$ and $Re\{\phi(1,0)\}=1>0$;
(3) for all $(iu_2,v_1)\ \varepsilon\ D$ and such that $v_1\leq -n(1+u_2^2)/2$,

$$\begin{aligned} Re\{\ \phi(iu_2,v_1)\}&=1+\frac{\gamma(1-\gamma)v_1}{(\beta-1)\ \{\gamma^2+(1-\gamma)^2u_2^2\}}\\ &\leq 1-\frac{n\gamma(1-\gamma)(1+u_2^2)}{2(\beta-1)\ \{\gamma^2+(1-\gamma)^2u_2^2\}}\\ &\leq 0 \end{aligned}$$

for $1<\beta\leq(n+2)/2$. Therefore, with the aid of Lemma 2, we conclude that $Re\{p(z)\}>0$ $(z\ \varepsilon\ U)$, that is, that

$$(2,13)\qquad Re\{(\frac{D^\alpha f(z)}{z})^{\beta-1}\}>\gamma=\frac{n}{2(\beta-1)+n}\quad (z\ \varepsilon\ U).$$

Corollary. If the function $f(z)$ defined by (1.1) is in the class $K(n,\alpha)$, then

$$(2.14)\qquad Re\{(D^{\alpha}f(z))'\}^{\beta-1} > \frac{n}{2(\beta-1)+n} \qquad (z \in U),$$

where $1<\beta\leq(n+2)/2$

Proof. Noting that

$$\begin{aligned} f(z) \in K(n,\alpha) &\Leftrightarrow D^{\alpha}f(z) \in K(n) \\ &\Leftrightarrow z(D^{\alpha}f(z))' \in S^*(n) \\ &\Leftrightarrow D^{\alpha}(zf'(z)) \in S^*(n) \\ &\Leftrightarrow zf'(z) \in S^*(n,\alpha), \end{aligned}$$

we completes the proof of Corollary.

References

1. S. Fukui and K. Sakaguchi, An extension of a theorem of S. Ruscheweyh, Bull. Fac. Edu. Wakayama Univ. Nat. Sci. 29(1980), 1-3.
2. S. S. Miller and P. T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65(1978), 289-305.
3. S. Owa, S. Fukui, K. Sakaguchi and S. Ogawa, An application of the Ruscheweyh derivatives, Internat. J. Math. Sci. 4(1986), 721-730.
4. S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49(1975), 109-115.

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