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PROPERTIES OF A FAMILY OF WEAKLY PSEUDOCONVEX MANIFOLDS

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0. Introduction

The research on a domain of holomorphy is very exciting because this concept is one of the great differences between the theory of functions of one complex variable and that of several complex variables. There are many equivalent conditions to the domain of holomorphy [2,3,4,5]. Levi's conjecture, saying that a pseudoconvex domain in C^n is a domain of holomorphy, was an important unsolved problem for a long time in the theory of functions of several variables but it is solved by K. Oka[9], H J. Bremermann[1] and F. Norguet[8], etc. Thus pseudoconvexity is a very important concept in the theory of functions of several complex variables. Now the reserch on pseudo-convexity over complex manifolds is actively in process. It has developed that a strongly pseudoconvex manifold is a Stein maifold[2]. But the research on a weakly pseudoconvex manifold is advanced recently. Since every strongly pseudoconvex manifold is weakly pseudoconvex, there is the same phenomenon on a weakly pseudoconvex manifold as on a strongly pseudoconvex manifold. But every weakly pseudoconvex manifold is not always strongly pseudoconvex so that there are quietly different phenomena. The aim of this research is clarifying these phenomena.

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In Section 1, we show the difference between the one-variable and several-variable theory of holomorphic functions.

In Section 2, we introduce pseudoconvexity of domains and example which is not a strongly pseudoconvex domain but a pseudoconvex domain.

In Section 3, we delve into a complex n-torus and a complex holomorphic line bundle over a complex n-torus which are important tools for the research on a weakly pseudoconvex manifold.

1. Preliminaries

An open connected set in the space C^n of n complex variables is called a domain. A domain Ω in C^n is said to have a C^j boundary $(j\geq 1)$ if there is a C^j function $\Phi: C^n\rightarrow R$ such that $\Omega=\{z\in C^n : \Phi(z)<0\}$ and

$$\operatorname{grad} \Phi(z)=\left(\frac{\partial\Phi}{\partial z_1},\cdots,\frac{\partial\Phi}{\partial z_n}\right)\neq 0$$

on the boundary $b\Omega$ of Ω. Such a function Φ is called a defining function for Ω. Let $H(\Omega)$ be the set of all holomorphic functions on Ω.

Definition 1.1. An open set Ω is called a domain of holomorphy if there are no open sets Ω_1 and Ω_2 in C^n with the following properties:

1. $\phi\neq\Omega_1\subset\Omega_2\cap\Omega$.
2. Ω_2 is connected and not contained in Ω.
3. For every $f\in H(\Omega)$ there is a function $f_2\in H(\Omega_2)$ (necessarily

uniquely determined) such that $f=f_2$ on Ω_1.

Let $a \in C^n$. Consider the set of pairs (U,f), where U is an open set in C^n, $a \in U$ and f is holomorphic on U. Two such pairs (U,f) and (V,g) are said to be equivalent if there exists a neighborhood W of a, $W \subset V \cap U$ such that $f \mid W = g \mid W$. Let f_a be an equivalence class with respect to this relation and will be called a germ of holomorphic functions at a. We denote by O_a the set of all germs of holomorphic functions at a. $O = \bigcup_{a \in C^n} O_a$ is called the sheaf of germs of holomorphic functions on C^n[7]. We define a topology on O as follows. Let $f_a \in O_a$, and let (U,f) be a pair defining f_a. Let $N(U,f) = \{f_b ; b \in U\}$ where f_b is the germ at b defined by the pair (U,f). By definition, the sets $N(U,f)$, where (U,f) runs over all pairs defining f_a, form a fundamental system of neighborhoods of f_a.

2. Pseudoconvexity of domains

Proposition 2.1. If a domain $\Omega \subset C^n$ is domain of holomorphy, then there is no part of the boundary across which every element in $H(\Omega)$ can be continued analytically.

Proof. Let $\gamma : [0,1] \to \Omega$ be a curve with $\gamma(t) \in \Omega$ for $0 \leq t < 1$ and $a = \gamma(1) \in b\Omega$. We suppose that the germ of f at $\gamma(0)$ can be continued analytically along γ and denoted by F_a the germ at $\gamma(1)$ so obtained. Let (D, F) be a representative of F_a where D is a polydisc and $F \in H(D)$. By definition of analytic continution, there is $\varepsilon > 0$ such that, for $1 - \varepsilon \leq t < 1$, $\gamma(t) \in D$ and the germ of F at $\gamma(t)$ is the same as the germ of f at $\gamma(t)$. Let U be the connected component of $D\cap\Omega$ containing $\{\gamma(t) : 1-\varepsilon\leq t<1\}$. We then have $F|U=f|U$.

For any domain Ω of C, by the theorem of Weierstass[6], there is a holomorphic function which cannot be continued analytically at everyt point of $b\Omega$. Hence, by Proposition 2.1, Ω is a domain of holomorphy. This is not true for arbituary domains in C^n, $n>1$, as was firstly pointed out by F. Hartogs[6].

E. E. Levi showed that the boundary of a domain of holomorphy is not arbitrary and has a kind of convexity called pseudoconvexity.

Definition 2.2. A domain Ω in C^n with C^2 boundary is said to be (Levi)-pseudoconvex if there exists a defining function Φ for Ω such that $L(\Phi)$ is positive semi-definite on holomorphic tangent vectors to $b\Omega$ (i.e.,

$$L(\Phi)(z)(w)=\sum_{j,k=1}^{n}\frac{\partial^2\Phi}{\partial z_j\,\partial\bar{z}_k}(z)w_j\bar{w}_k\geq 0$$

for all $z\in b\Omega$, for $w\in C^n$ satisfying

$$\sum_{j=1}^{n}\frac{\partial\Phi}{\partial z_j}(z)\,w_j=0).$$

A domain of holomorphy is pseudoconvex [2,4,5] but the assertion that a (Levi)-pseudoconvex domain is a domain of holomorphy is known as the Levi problem.

Definition 2.3. Let Ω be a domain in C^n such that $\Omega\subset\subset C^n$ and it has a C^2 boundary. Ω is said to be strongly (or strictly) pseudoconvex if there exists a defining function Φ for Ω such that the Levi form $L(\Phi)$ of Φ is positive definite on holomorphic tangent vectors to $b\Omega$.

Example 2.4. Let Ω be the "solid torus" in C^2 whose major circle of rotation lies in the x_1-x_2 plane and whose minor circle of rotation is in the y_1, y_2 directions. This region is described by

$$\Omega=\{(z_1, z_2)\in C^2 : |z_1|^2+|z_2|^2+\frac{9}{4}-3\sqrt{x_1^2+x_2^2}<r^2\}$$

where $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$.

Let $\Phi(z_1, z_2)=|z_1|^2+|z_2|^2+\frac{9}{4}-3\sqrt{x_1^2+x_2^2}-r^2$ on Ω.

Then Φ is a defining function for Ω. The Levi form is

$$|w_4|^2-\frac{3|w_1|^2}{4\sqrt{x_4^2+x_2^2}}+|w_2|^2=\frac{3|w_2|^2}{4\sqrt{x_4^2+x_2^2}}+\frac{3|w_1x_1+w_2x_2|^2}{4\sqrt{(x_1^2+x_2^2)}}$$

Thus the Levi form of Φ is positive definite on $b\Omega$ if $0<r<\frac{3}{4}$ and positive semidefinite if $r\leqq\frac{3}{4}$. Hence Ω is strongly pseudoconvex only if $0<r<\frac{3}{4}$ and pseudoconvex if $r\leqq\frac{3}{4}$.

3. The line bundle and pseudoconvexity on manifolds

Now we consider the pseudoconvexity on complex manifolds.

Definition 3.1. Let X be an n-dimensional complex manifold. We say that X is a strongly [weakly] pseudoconvex manifold of dimension

n if there is a function $\Phi \in C^\infty(X,R)$ such that

(1) The Levi form $L(\Phi)$ of Φ is everywhere positive definite [semi-definite] (*i.e.*, $L(\Phi)(z)>0$ [$\geqq 0$] on C^n where $z \in X$).

(2) $X_c=\{z \in X : \Phi(z)<c\}$ is relatively compact in X for any $c \in R$.

Definition 3.2. A complex manifold Ω of n dimension (which is countable at infinity) is said to be a Stein manifold if

(1) Ω is holomorphically convex.

(2) Ω is holomorphically separable.

(3) For every $z \in \Omega$, one can find n functions $f_1, \dots, f_n \in H(\Omega)$ which form a coordinate system at z.

Example 3.3. It was known that Ω is a Stein manifold if, and only if Ω is a strongly pseudoconvex manifold [2,4]. Each domain of holomorphy in C^n is a Stein manifold [4]. Hence every domain of holomorphy is a strongly pseudoconvex manifold. We know that a pseudoconvex domain is a domain of holomorphy whence every pseudoconvex domain is a strongly pseudoconvex manifold.

Take $2n$ vectors $w_1, \dots, w_{2n}$, $w_k=(x_{k1}, \dots, x_{kn}) \in C^n \simeq R^{2n}$ so that the w_k are linearly independent over R. Let $\Gamma(w_1, \dots, w_{2n})$ denote the lattice subgroup of C^n defined by $\{m_1w_1+\cdots+m_{2n}w_{2n} : m_j \in Z, j=1, \dots, 2n\}$. We define the complex n-torus T^n to be the quotient space $T^n=C^n/\Gamma$. The quotient map is a local homeomorphism.

Lemma 3.4. T^n is compact.

Proof. To avoid confusion we only think the lemma in the case of $n=1$. For $z \in C$, $z=m_1w_1+m_2w_2+r_1w_1+r_2w_2$ where m_1, $m_2 \in Z$ and

$0\leq r_1$, $r_1\leq 1$. Hence $[z]=[r_1w_1+r_1w_1]\in T^1$. Let $S=\{r_1w_1+r_2w_2 : r_1,r_2\in R$ with $0\leq r_1$, $r_2\leq 1\}$ and then the function $T^1\to S$ defined by $[z]=[r_1w_1+r_2w_2]\to r_1w_2+r_2w_2$ is a homeomorphism. Since S is compact, T^1 is also compact.

Definition 3.5. Let X be an n-dimensional complex manifold. An n-dimensional complex holomorphic vector bundle E over X consist of an n-dimensional complex manifold E and holomorphic map $\pi : E\to X$ which satisfy :

(1) There is an open covering $\{U_j : j\in I\}$ of X and biholomorphic functions $\theta_j : \pi^{-1}(U_j)\to U_j\times C^n$ such that

$$\begin{tikzcd} E\supset \pi^{-1}(U_j) \arrow[r, "\theta_j"] \arrow[dr, "\pi"'] & U_j\times C^n \arrow[d, "\pi_j"] \\ & X\supset U_j \end{tikzcd}$$

commutes, where $\pi_j(z_j, w)=z_j$.

(2) For $U_j\cap U_k$, there is a holomorphic function $\theta_{jk} : U_j\cap U_k\to \mathrm{GL}(C^n)$ such that $\theta_j \circ \theta_k^{-1}(x,z_k)=(x,z_j)$, $x\in U_j\cap U_k$, z_j, $z_k\in C^n$ if and only if $\theta_{jk}(x)z_k=z_j$.

These functions θ_{jk}, the so-called transition functions, satisfy the (cocycle) conditions.

Definition 3.6. Let E and F be vector bundles and $\{\theta_{jk}\}$ and $\{\eta_{jk}\}$ transition functions of E and F, respectively. We say that E and F are equivalent if there are holomorphic maps $h_j : U_j\to GL(C^n)$

such that $\eta_{jk}h_k = h_j\theta_{jk}$ for all $j,k \in I$.

Remark. 1. If E is a 1-dimensional complex manifold, we shall refer to E as a line bundle.

2. We shall always identify equivalent holomorphic line bundles and $HLB(X)$ denote the group of all equivalence classes of holomorphic line bundles.

Let X be a Hausdorff paracompact space and let O be a sheaf over X. Fix a locally finite covering $\mathscr{U} = \{U_j\}_{j\in I}$ of X.

Let $C^0(\mathscr{U},O) = \{(g_j)_{j\in I} : g_j \in H(U_j)\}$ and let $C^p(\mathscr{U},O) = \{(g_{j_0 j_1 \cdots j_p})_{j_0,\ldots,j_p \in I} : g_{j_0 j_1 \ldots j_p} \in H(U_{j_0 \ldots j_p})$ and $g_{j_0 \ldots j_p}$ is skew-symmetric in the indices $j_0,\ldots,j_p\}$. Set $f = (f_{j_0 \ldots j_p})_{j_0,\ldots,j_p \in I} \in C^p(\mathscr{U},O)$. We define

$\delta^{p+1} : C^p(\mathscr{U},O) \to C^{p+1}(\mathscr{U},O)$ by $\delta^{p+1}(f) = ((\delta^{p+1}f)_{(j_0 \ldots j_{p+1})})$

$= \sum_{k=0}^{p+1} (-1)^k f_{j_0, \ldots \hat{j_k}, \ldots j_{p+1}}$, where $\wedge$ means "omit". We define

$$Z^p(\mathscr{U},O) = \{f \in C^p(\mathscr{U},O) : \delta^{p+1}f = 0\},$$

$$B^p(\mathscr{U},O) = \delta^p(C^{p-1}(\mathscr{U},O)) \subset Z^p(\mathscr{U},O), \text{ and}$$

$$H^p(\mathscr{U},O) = Z^p(\mathscr{U},O)/B^p(\mathscr{U},O).$$

Theorem 3.7. $HLB(T^n)$ is canonically isomorphic to $H^1(T^n, O^*)$ where O^* is the sheaf over T^n of nonvanishing holomorphic functions in which the module operation on each stalk is multiplication.

Proof. Let E be a line bundle and $\{\theta_{jk}\}$ the transition functions of E. Since $\{\theta_{jk}\}$ satisfies the cocycle conditions, $(\theta_{jk}) \in Z^1(U,O^*)$ where $\mathscr{U} = \{U_j\}_{j\in I}$ be an open covering of T^n. If E and F are equivalent, then there are nonvanishing holomophic function $h_j : U_j \to C^*$ such that

$\eta_{jk}=h_j\theta_{jk}h_k^{-1}$ or $\eta_{jk}=\theta_{jk}h_jh_k^{-1}$, that is, $(\eta_{jk})=(\theta_{jk})(\delta(h_j))$. Thus an equivalence class of bundles defines an element $[(\theta_{jk})]\in H^1(\mathscr{U},O^*)\subset H^1(T^n,O^*)$.

Conversely, let $\xi\in H^1(T^n,O^*)$. Then $\xi=[(\theta_{jk})]$ and we can find an open covering $\mathscr{U}=\{U_j\}$ of T^n such that $(\theta_{jk})\in Z^1(\mathscr{U},O^*)$[4]. Consider a set $Z=\bigcup_j\{U_j\times C\}$. We define relation "$\sim$" on Z by $(j,x,z)\sim(k,y,z)$ iff $x=y$ and $z_j=\theta_{jk}(x)z_k$. Since (θ_{jk}) satisfies the cocycle conditions, "$\sim$" is an equivalence relation. Put $E(\xi)=Z/\sim$. Define a map $\pi: E(\xi)\to T^n$ by $\pi[(j,x,z_j)]=x$ and then π is holomorphic. Define a map θ_j: $\pi^{-1}(U_j)\to U_j\times C$ by $\theta_j[(j,x,z_j)]=(x,z_j)$ and then θ_j is biholomorphic. Let $x\in U_j\cap U_k$. If $\theta_j\theta_k^{-1}(x,z)=(x,z_j)$, then $\theta_j\theta_k^{-1}(x,z)=\theta_j[(k,x,z_k)]=(x,z_j)$. Since $\theta_j[(j,x,z_j)]=(x,z_j)$, $(j,x,z_j)\sim(k,x,z_k)$ so that $\theta_{jk}z_k=z_j$. Hence $E(\xi)$ is a holomorphic line bundle over T^n with transition functions $\{\theta_{jk}\}$. The map $\xi\to E(\xi)$ is an isomorphism of groups of $H(T^n,O^*)$ with $HLB(T^n)$. So, we shall always identify equivalent holomorphic line bundles and we call $H^1(T^n,O^*)$ the group of holomorphic line bundles over T^n.

References

1. H. J. Bremermann, Uber die Áquivalenz der pesudokonvexen Gebiete und der Holomorphieqebiete in Raume von n komplexen Veránderlichen, Math. Ann., 128(1954), 63−91.
2. M. Field, Several complex variables and complex manifold I, II. London Math. Soc. Lec. Note Ser, 75 Cambridge Univ. Press, London, 1982.
3. H. Grauert and K. Fritzsche, Several complex variables, Springer-Verlag Inc., New York, 1976.
4 L. Hormander, An introduction to complex analysis in several variables, Second

ed., North-Holland Publ. Co., Amsterdom, 1979.
5. S. G. Krantz, Functional theory of several complex variables, John Wiley & Sons, Inc., Canada, 1982.
6. R. Narasimhan, Complex analysis in one variable, Birkhäuser Boston, Inc., Switzerland, 1985.
7. ________________, Several complex variables, Univ. of Chicago Press, Chicago, 1971.
8. F. Norguet, Sur les domaines d'holomorphie des functions uniformes de plusieurs variables complexes (passage du local au global), Bull. Soc. Math. France, 82(1954), 137-159.
9. K. Oka, Sur les fonctions analytiques de plusleurs variables, VI. Domaines pseudoconvexes, Tohoku Math. J., 49(1942), 15-52

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