Nonlinear semigroups on locally convex spaces

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Abstract

Let $E$ be a locally convex Hausdorff space and let $\Gamma$ be a calibration for $E$. In this note we proved that if $E$ is sequentially complete and a multi-valued operator $A$ in $E$ is $\Gamma$-accretive such that $D(A) \subset R (I + \lambda A)$ for all sufficiently small positive $\lambda$, then $A$ generates a nonlinear $\Gamma$-contraction semigroup $\{T(t): t > 0\}$. We also proved that if $E$ is complete, $\Gamma$ is a dually uniformly convex calibration, and an operator $A$ is $m$-$\Gamma$-accretive, then the initial value problem

$$\begin{cases}
\frac{d}{dt} u(t) + Au(t) & \geq 0, \ t > 0, \\
u(0) = x
\end{cases}$$

has a solution $u : [0, \infty) \rightarrow E$ given by $u(t) = T(t)x = \lim_{n \rightarrow \infty} (I + \frac{n}{t} A)^{n} x$ for each $x \in D(A)$.

1. $\Gamma$-completions

Let $E$ be a locally convex space and let $\Gamma$ be a calibration for $E$, i.e., $\Gamma$ is a direct set of semi-norms on $E$ which induces the topology of $E$. For $p \in \Gamma$, a sequence $\{x_n\}$ in $E$ is called a $p$-Cauchy...
sequence if \( p(x_i - x_j) \to 0 \) as \( i, j \to \infty \). Two \( p \)-Cauchy sequences \( \{x_i\} \) and \( \{y_i\} \) are said to be equivalent if \( p(x_i - y_i) \to 0 \) as \( i \to \infty \). Let \( \{x_i\} \) be a \( p \)-Cauchy sequence and \( \mathcal{X} \) be the set of all \( p \)-Cauchy sequences in \( E \) which are equivalent to \( \{x_i\} \). Such a set \( \mathcal{X} \) is called a \( p \)-class on \( E \). The set of all \( p \)-classes on \( E \) will be denoted by \( E[p] \) and it will be called the \( p \)-completion of \( E \). For \( x, y \in E[p] \) and real numbers \( \alpha, \beta \), \( \alpha x + \beta y \) is defined to be the \( p \)-class which contains a \( p \)-Cauchy sequence \( \{\alpha x_i + \beta y_i\} \) for some \( \{x_i\} \subseteq \mathcal{X} \) and \( \{y_i\} \subseteq \mathcal{Y} \). Then \( E[p] \) is a real vector space.

For \( x \in E[p] \), we define
\[
p(x) = \lim_{i \to \infty} p(x_i) \quad \text{for} \quad \{x_i\} \subseteq \mathcal{X}.
\]

Then the value \( p(x) \) does not depend on the choice of \( \{x_i\} \) from \( \mathcal{X} \).

In is obvious that \( p \) is a norm on \( E[p] \) and, with this norm, \( E[p] \) is a Banach space. The family of Banach spaces \( \{E[p] : p \in \Gamma \} \) defined in this way will be called the \( \Gamma \)-completion of \( E \). We denote by \( S_p(x) \) the \( p \)-class which contains the \( p \)-Cauchy sequence whose terms are all identical to \( x \). Then the zero element of the Banach space \( E[p] \) is \( S_p(0) \) and we have

\[
p(S_p(x)) = p(x) \quad \text{for every} \quad x \in E.
\]

Let \( \{E[p] : p \in \Gamma \} \) be the \( \Gamma \)-completion of \( E \). First we have a linear and continuous map

\[
S_p : E \to E[p] : x \mapsto S_p(x),
\]
which satisfies the equality \( p(S_t(x)) = p(x) \) for every \( x \in E \). Next, when \( p \geq q \) in \( \Gamma \), that is, \( q(x) \geq p(x) \) for every \( x \in E \), we have the natural embedding

\[
T_{q,p} : E[q] \to E[p],
\]

which maps every \( x \in E[q] \) to be the \( p \)-class which contains elements of \( x \). Obviously, this map is linear,

\[
p(T_{q,p}(x)) \leq q(x) \quad \text{for every } x \in E[q]
\]

and

\[
T_{q,p} \cdot S_q = S_p.
\]

2. \( \Gamma \)-contractions and \( \Gamma \)-accretive operators

Let \( E \) and \( F \) be locally convex spaces and let \( \Gamma \) be a calibration for \((E,F)\). In other words, each \( p \in \Gamma \) has the \( E \)-component \( p_E \) and the \( F \)-component \( p_F \) and \( \Gamma_E = \{ p_E : p \in \Gamma \} \) and \( \Gamma_F = \{ p_F : p \in \Gamma \} \) are calibrations for \( E \) and \( F \), respectively. We shall denote the embeddings \( S_{p_E} \) and \( S_{p_F} \) by the same \( S_p \).

We shall deal with multi-valued operators. By a multi-valued operator \( A \) in \( E \) we mean that \( A \) assigns to each \( x \in D(A) \) a subset \( Ax \neq \emptyset \) of \( E \), where \( D(A) = \{ x \in E : Ax \neq \emptyset \} \). And \( D(A) \) is called the domain of \( A \), and the range of \( A \) is defined by \( R(A) = \bigcup_{x \in D(A)} Ax \).

Let \( A \) be a multi-valued operator from \( E \) into \( F \), that is, \( A \) is a subset of \( E \times F \). For \( p \in \Gamma \) and \( [x,y] \in A \), we set
\[ S_p([x,y]) = [S_p(x), S_p(y)]. \]

Then \( S_p(A) \subseteq E[p] \times F[p] \) and we set

\[ A_p = \overline{S_p(A)} \]

where the closure is taken in the product \( E[p] \times F[p] \) of Banach spaces \( E[p] \) and \( F[p] \). Hence \( A_p \) is always closed and \( A_p = (A)_p \).

**Lemma 2.1**[6]. (i) \( \overline{A} = \bigcap_{p \in T} S_p^{-1}(A_p) \),

(ii) \( \overline{D(A)} = \bigcap_{p \in T} S_p^{-1}(\overline{D(A)}) \),

(iii) \( \overline{D(A)} = \overline{S_p(D(A))} \).

**Lemma 2.2**[6]. Assume that \( q \geq 1 \) in \( T \). Then for every \( x = \frac{e}{q} D(A_q) \),

(i) \( T_{q,q'} x \in D(A_p) \),

(ii) \( T_{q,q'} A_q x = A \frac{e}{q} T_{q,q'} x \).

Recall that a multi-valued operator \( A \) in a Banach space \( X \) with its norm \( \| \cdot \| \) is said to be accretive if for each \( x, y \in D(A) \), \( y \in Ax \), \( y \in Ax \), and for every \( \lambda > 0 \), the following inequality holds

\[ \| (x + y) - (x + y) \| \geq \| x - y \| \].

Moreover, if \( R(I + \lambda A) = X \) then \( A \) is said to be \( m \)-accretive.
Let $\Gamma$ be a calibration for a locally convex space $E$.

**Definition 2.3.** An operator $f$ from a subset $D(f)$ of $E$ into $E$ is said to be a $\Gamma$-contraction if

$$p(f(x)) - f(y)) \leq p(x-y)$$

for all $p \in \Gamma$ and $x,y \in D(f)$.

When $f$ is a $\Gamma$-contraction and $p \in \Gamma$, $\{f(x)\}$ is $p$-Cauchy sequence whenever $\{x\}$ is a $p$-Cauchy sequence, hence for every $x \in S_p(D(f))$ we can set

$$f_p(x) = \lim_{n} S_p(f(x)).$$

Then $f_p$ is a contraction of $S_p(D(f))$ into $E[p]$ and

$$f_p \cdot S_p = S_p \cdot f.$$

**Definition 2.4.** An operator $A \in E \times E$ is said to be $\Gamma$-accretive if, for every $\lambda > 0$, $(I + \lambda A)^{-1}$ is a single-valued $\Gamma$-contraction. If, furthermore, $R(I + \lambda A) = E$, then $A$ is said to be $m$-$\Gamma$-accretive. Where $I$ is an identity operator on $E$.

**Lemma 2.5[6].** For any operator $A \in E \times E$ and $\lambda > 0$,

(i) $(I + \lambda A)_p = I + \lambda A_p$ for all $p \in \Gamma$,

(ii) $(I + \lambda A)^{-1}_p = (I + \lambda A_p)^{-1}$.

**Lemma 2.6[6].** (i) If $A$ is $m$-$\Gamma$-accretive, every $A_p$ is $m$-accretive,
(ii) If $E$ is complete, $A$ is closed and every $A_p$ is $m$-accretive, then $A$ is $m$-$\Gamma$-accretive,

(iii) $A$ is $\Gamma$-accretive if and only if every $A_p$ is accretive,

(iv) A $m$-$\Gamma$-accretive operator $A \subseteq E \times E$ is closed in $E \times E$.

(v) If $A$ is $m$-$\Gamma$-accretive and $x \in D(A)$, then $Ax$ is closed.

3. Theorems

**Definition 3.1.** Let $E$ be a locally convex space with a calibration $\Gamma$ and let $\{T(t) : t \geq 0\}$ be a family of nonlinear operators from a closed subset $C$ of $E$ into itself satisfying the following conditions:

(i) $T(0) = I$ (identity), $T(t+s) = T(t)T(s)$ for $t, s \geq 0$.

(ii) For every $x \in C$, $T(t)x$ is continuous in $t \geq 0$.

(iii) For all $p \in \Gamma$, $t \geq 0$, and, $x, y \in C$,

\[ p(T(t)x - T(t)y) \leq p(x - y). \]

Then we shall call this family $\{T(t) : t \geq 0\}$ a nonlinear $\Gamma$-contraction semigroup.

**Theorem 3.2.** Let $E$ be a sequentially complete, locally convex Hausdorff space with a calibration $\Gamma$ and $A$ be a $\Gamma$-accretive operator in $E$ such that $\overline{D(A)} \subseteq R(I + \lambda A)$ for all sufficiently small positive $\lambda$. Then
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(3.1) \[ T(t)x = \lim_{n \to \infty} (I + \frac{t}{n} A)^n x \]
exists for \( x \in \overline{D(A)} \), uniformly in \( t \) on every compact interval of \([0, \infty)\). Moreover, \( T(t) \) defined by the formula (3.1) is a \( \Gamma \)-contraction semigroup on \( D(A) \).

**Proof.** If \( A \) is \( \Gamma \)-accretive and \( \overline{D(A)} \subset R(I + \lambda A) \), then, for every \( p \in \Gamma \), \( A_p \) is accretive and \( \overline{D(A_p)} \subset R(I + \lambda A_p) \). Thus, for \( p \in \Gamma \) and \( x \in D(A_p) \),

\[ p((I + \frac{t}{n} A_p)^n x) = p((I + \frac{t}{m} A_p)^m x) \]

exists and \( \{ p((I + \lambda A_p)^n x) : t \geq 0 \} \) is a contraction semigroup on \( \overline{D(A_p)} \) ([2]). Let \( x \in \overline{D(A)} \) and let \( n \) and \( m \) be positive integers such that \( n \geq m \). Then, for any \( p \in \Gamma \),

\[ p((I + \frac{t}{n} A)^n x - (I + \frac{t}{m} A)^m x) = p((S_p(I + \frac{t}{n} A)^n x - S_p(I + \frac{t}{m} A)^m x)) \]

\[ = p((S_p(I + \frac{t}{n} A)^n - (I + \frac{t}{m} A)^m S_p(x)) \]

\[ \leq 2n \frac{1}{n} \frac{1}{m} \inf \{ p(x) : x \in A_p S_p(x) \} \]

and hence \( p((I + \frac{t}{n} A)^n x - (I + \frac{t}{m} A)^m x) \to 0 \) as \( n, m \to \infty \).

Therefore \( \lim_{n \to \infty} (I + \frac{t}{n} A)^n x = T(t)x \) exists uniformly in \( t \) on every compact subset of \([0, \infty)\). Then, for every \( p \in \Gamma \) and \( x \in \overline{D(A)} \),

\[ S_p(T(t)x) = S_p(\lim_{n \to \infty} (I + \frac{t}{n} A)^n x) \]

\[ = \lim_{n \to \infty} S_p(I + \frac{t}{n} A)^n x \]

\[ = \lim_{n \to \infty} (I + \frac{t}{n} A)^n S_p(x) \]
and hence $T(t)x \in D(A)$. Since $(I + \frac{t}{n}A)^n$ is $\Gamma$-contraction, we find that $p(T(t)x - T(t)y) \leq p(x - y)$ for every $t \geq 0, x, y \in D(A)$, and for all $p \in \Gamma$. Therefore $T(t)$ is $\Gamma$-contraction on $\overline{D(A)}$. Moreover, for all $p \in \Gamma$ and $x \in D(A)$, we obtain

$$p(T(t)x - T(t)y) = p(S_p(T(t)x) - S_p(T(t)y))$$

$$= p(T^p(t)x) - T^p(t)y)$$

$$\leq 2 |t-s| \cdot \inf \{p(x) : x \in A, S_p(x)\} \quad (2).$$

In particular, this shows that $T(t)x$ is continuous in $x$ for every $x \in D(A)$. In order to complete the proof, we shall verify the semigroup property $T(t+s) = T(t)T(s)$. For all $p \in \Gamma$ and $t, s \geq 0$, we have

$$S_p(T(t+s)x) = T^p(t+s)x)$$

$$= T^p(t)T^p(s)x)$$

$$= T^p(t) (S_p(T(s)x))$$

$$= S_p(T(t) T(s)x), \quad \text{for } x \in D(A).$$

Since $E$ is Hausdorff, $T(t+s) = T(t)T(s)$ for $t, s \geq 0$. This completes the proof.
We shall call a calibration $\Gamma$ dually uniformly convex if, for every $p \in \Gamma$, $E[p]$ and its dual are uniformly convex.

**Lemma 3.3.** Assume that $B$ is a closed subset of $E$ and

$$S_p(x) \in S_p(B) \text{ for all } p \in \Gamma.$$  

Then $x \in B$.

**Theorem 3.4.** If $E$ is complete, locally convex Hausdorff space with a dually uniformly convex calibration $\Gamma$ and $A$ is an $m$-$\Gamma$-accretive operator in $E$. Then for each $x \in \mathcal{D}(A)$ the initial value problem

$$(E) \left\{ \begin{array}{l}
\frac{du}{dt}(t) + Au(t) \ni 0, \\
u(0) = x
\end{array} \right.$$  

has a solution $u : [0, \infty) \rightarrow E$ given by $u(t) = T(t)x = \lim_{n \to \infty} (I + \frac{t}{n} A)^n x$, $t \geq 0$.

**Proof.** By theorem 3.2, for each $x \in \mathcal{D}(A)$, $T(t)x = \lim_{n \to \infty} (I + \frac{t}{n} A)^n x$ exists. Since $A$ is $m$-$\Gamma$-accretive and $\Gamma$ is a dually uniformly convex calibration, $A_p$ is $m$-accretive and $E[p]$ is uniformly convex space for every $p \in \Gamma$. Hence the initial value problem

$$\left\{ \begin{array}{l}
\frac{du}{dt}(t) + A_p u(t) \ni S_p(0), \\
u(0) = S_p(x)
\end{array} \right.$$
has a unique solution \( u : [0, \infty) \to E[\rho] \) given by

\[
u_p(t) = T^f(t)S_p(x) = \lim_{n \to \infty} (I + \frac{t}{n} A_p)^n S_p(x) \quad (11).
\]

Then, if \( q \geq p \) in \( \Gamma \), \( T_{q, q}^{-\rho}(0) = T_{q, q} S_p(x) = S_p(x) \) and

\[
T_{q, q} \left( \frac{d}{dt} u_q(t) + T_{q, q} A u_q(t) \right) \supseteq T_{q, q} S_q(0),
\]

which implies

\[
\frac{d}{dt} T_{q, q} u_q(t) + A \rho T_{q, q} u_q(t) \supseteq S_p(0), t \geq 0.
\]

Hence, by the uniqueness of solution, \( T_{q, q}^{-\rho} u_q(t) = u_q(t) \) for \( t \geq 0 \). Since \( E \) is complete, there exists \( u(t) \in E \) such that \( u_p(t) = S_p(u(t)) \) for all \( p \in \Gamma \) and \( t \geq 0([A]) \). Then, for all \( p \in \Gamma \) and \( t \geq 0 \),

\[
S_p(u(t)) = u_p(t) = \lim_{n \to \infty} (I + \frac{t}{n} A_p)^n S_p(x)
\]

\[
= \lim_{n \to \infty} (I + \frac{t}{n} A)^n S_p(x)
\]

\[
= S_p(T(t)x).
\]

Hence \( u(t) = T(t)x = \lim_{n \to \infty} (I + \frac{t}{n} A)^n x \) for \( x \in D(A) \) and \( t \geq 0 \). Furthermore, for all \( p \in \Gamma \) and \( t \geq 0 \),

\[
S_p \left( \frac{d}{dt} u(t) + A u(t) \right) \supseteq S_p(0).
\]
Then, by lemma 3.3, we have
\[
\frac{d}{dt}u(t) + Au(t) \geq 0.
\]
Therefore \( u(t) = T(t)x = \lim_{n \to \infty} (I + \frac{t}{n}A)^nx \) is a solution of the initial value problem \((E)\).

References


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