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Nonlinear semigroups on locally convex spaces

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Abstract

Let E be a locally convex Hausdorff space and let Γ be a calibration for E. In this note we proved that if E is sequentially complete and a multi-valued operator A in E is Γ-accretive such that $D(A) \subset R(I+\lambda A)$ for all sufficiently small positive λ, then A generates a nonlinear Γ-contraction semiproup $\{T(t) ; t>0\}$. We also proved that if E is complete, Γ is a dually uniformly convex calibration, and an operator A is m-Γ-accretive, then the initial value problem

$$\begin{cases} \frac{d}{dt}u(t)+Au(t) \ni 0, \ t>0, \\ u(0)=x \end{cases}$$

has a solution $u : [0,\infty) \to E$ given by $u(t)=T(t)x=\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x$ for each $x\varepsilon D(A)$.

1. Γ-completions

Let E be a locally convex space and let Γ be a calibration for E, *i.e.*, Γ is a direct set of semi-norms on E which induces the topology of E. For $p\varepsilon\Gamma$, a sequence $\{x_i\}$ in E is called a p-Cauchy sequence if $p(x_i - x_j) \to 0$ as $i,j \to \infty$. Two p-Cauchy sequences $\{x_i\}$ and $\{y_i\}$ are said to be equivalent if $p(x_i - y_i) \to 0$ as $i \to \infty$. Let $\{x_i\}$ be a p-Cauchy sequence and $\underline{x}$ be the set of all p-Cauchy sequences in E which are equivalent to $\{x_i\}$. Such a set $\underline{x}$ is called a p-class on E. The set of all p-classes on E will be denoted by $E[p]$ and it will be called the p-completion of E. For $\underline{x}$, $\underline{y}$ $\varepsilon E[p]$ and real numbers α, β, $\alpha\underline{x} + \beta\underline{y}$ is defined to be the p-class which contains a p-Cauchy sequence $\{\alpha x_i + \beta y_i\}$ for some $\{x_i\} \varepsilon \underline{x}$ and $\{y_i\} \varepsilon \underline{y}$.Then $E[p]$ is a real vector space.

For $\underline{x} \varepsilon E[p]$, we define

$$p(\underline{x}) = \lim_{i \to \infty} p(x_i) \quad for \quad \{x_i\} \varepsilon \underline{x}.$$

Then the value $p(\underline{x})$ does not depend on the choice of $\{x_i\}$ from $\underline{x}$

In is obvious that p is a norm on $E[p]$ and, with this norm, $E[p]$ is a Banach space. The family of Banach spaces $\{E[p]\,;\, p \varepsilon \Gamma\}$ defined in this way will be called the Γ-completion of E. We denote by $S_p(x)$ the p-class which contains the p-Cauchy sequence whose terms are all identical to x. Then the zero element of the Banach space $E[p]$ is $S_p(0)$ and we have

$$p(S_p(x)) = p(x) \quad \text{for every } x \varepsilon E.$$

Let $\{E[p]\,;\, p \varepsilon \Gamma\}$ be the Γ-completion of E. First we have a linear and continuous map

$$S_p : E \to E[p] : x \to S_p(x),$$

which satisfies the equality $p(S_p(x))=p(x)$ for every $x\varepsilon E$. Next, when $p\geq q$ in Γ, that is, $q(x)\geq p(x)$ for every $x\varepsilon E$, we have the natural embedding

$$T_{q,p}: E[q]\rightarrow E[p],$$

which maps every $\underline{x}\ \varepsilon E[q]$ to be the p-class which contains elements of $\underline{x}$. Obviously, this map is linear,

$$p(T_{q,p}(\underline{x}))\leq q(\underline{x}) \text{ for every } \underline{x}\varepsilon E[q]$$

and

$$T_{q,p}\cdot S_q=S_p.$$

2. Γ-contractions and Γ-accretive operators

Let E and F be locally convex spaces and let Γ be a calibration for (E,F). In other words, each $p\varepsilon\Gamma$ has the E-component p_E and the F-component p_F and $\Gamma_E=\{p_E ; p\varepsilon\Gamma\}$ and $\Gamma_F=\{p_F ; p\varepsilon\Gamma\}$ are calibrations for E and F, respectively. We shall denote the embeddings S_{p_E} and S_{p_F} by the same S_p.

We shall deal with multi-valued operators. By a multi-valued operator A in E we mean that A assigns to each $x\varepsilon D(A)$ a subset $Ax\neq\phi$ of E, where $D(A)=\{x\varepsilon E ; Ax\neq\phi\}$. And $D(A)$ is called the domain of A, and the range of A is defined by $R(A)=\bigcup_{x\varepsilon D(A)} Ax$.

Let A be a multi-valued operator from E info F, that is, A is a subset of $E\times F$. For $p\varepsilon\Gamma$ and $[x,y]\varepsilon A$, we set

$$S_p([x,y])=[S_p(x),\ S_p(y)].$$

Then $S_p(A) \subset E[p] \times F[p]$ and we set

$$A_p=\overline{S_p(A)}$$

where the closure is taken in the product $E[p] \times F[p]$ of Banach spaces $E[p]$ and $F[p]$. Hence A_p is always closed and $A_p=(A)_p$.

Lemma 2.1[6]. (i) $\bar{A}=\bigcap_{p\varepsilon\Gamma} S_p^{-1}(A_p)$,

(ii) $\overline{D(A)}=\bigcap_{p\varepsilon\Gamma} S_p^{-1}(\overline{D(A)_p})$,

(iii) $\overline{D(A_p)}=\overline{S_p(D(A))}$.

Lemma 2.2.[6]. Assume that $q \geq$ in Γ. Then for every $x_q \varepsilon D(A_q)$,

(i) $T_{q,p} x_q \,\varepsilon\, D(A_p)$,

(ii) $T_{q,p} A_q x_q = A_q T_{q,p} x_q$.

Recall that a multi-valued operator A in a Banach space X with its norm $\| \cdot \|$ is said to be accretive if for each x_1, $x_2 \varepsilon D(A)$, $y_1 \varepsilon A x_1$, $y_2 \varepsilon A x_2$, and for every $\lambda>0$, the following inequality holds

$$\| (x_1+y_1)-(x_2+y_2) \| \geq \| x_1-x_2 \| .$$

Moreover, if $R(I+\lambda A)=X$ then A is said to be m-accretive.

Let Γ be a calibration for a locally convex space E.

Definition 2.3. An operator f from a subset $D(f)$ of E into E is said to be a Γ-contraction if

$$p(f(x)) - f(y)) \leq p(x-y)$$

for all $p \varepsilon \Gamma$ and $x, y \varepsilon D(f)$.

When f is a Γ-contraction and $p \varepsilon \Gamma$, $\{f(x_i)\}$ is p-Cauchy sequence whenever $\{x_i\}$ is a p-Cauchy sequence. hence for every $\underline{x} \varepsilon \overline{S_p(D(f))}$ we can set

$$f_p(\underline{x}) = \lim_{i \to \infty} S_p(f(x_i)).$$

Then f_p is a contraction of $\overline{S_p(D(f))}$ into $E[p]$ and

$$f_p \cdot S_p = S_p \cdot f.$$

Definition 2.4. An operator $A \subset E \times E$ is said to be Γ-accretive if, for every $\lambda > 0$, $(I + \lambda A)^{-1}$ is a single-valued Γ-contraction. If, furthermore, $R(I + \lambda A) = E$, then A is said to be m-Γ-accretive. Where I is an identity oprator on E.

Lemma 2.5[6]. For any operator $A \subset E \times E$ and $\lambda > 0$,

(i) $(I + \lambda A)_p = I + \lambda A_p$ for all $p \varepsilon \Gamma$,

(ii) $((I + \lambda A)^{-1})_p = (I + \lambda A_p)^{-1}$.

Lemma 2.6[6]. (i) If A is m-Γ-accretive, every A_p is m-accretive,

(ii) If E is complete, A is closed and every A_p is m-accretive, then A is m-Γ-accretive,

(iii) A is Γ-accretive if and only if every A_p is accretive,

(iv) A m-Γ-accretive operator $A \subset E \times E$ is closed in $E \times E$,

(v) If A is m-Γ-accretive and $x \varepsilon D(A)$, then Ax is closed.

3. Theorems

Definition 3.1. Let E be a locally convex space with a calibration Γ and let $\{T(t) : t \geq 0\}$ be a family of nonlinear operators from a closed subset C of E into itself satisfying the following conditions

(i) $T(0)=I$(identity), $T(t+s)=T(t)T(s)$ for $t,s \geq 0$,

(ii) Forevery $x \varepsilon C$, $T(t)x$ is continuous in $t \geq 0$,

(iii) For all $p \varepsilon \Gamma$, $t \geq 0$, and, $x,y \varepsilon C$,

$$p(T(t)x-T(t)y) \leq p(x-y).$$

Then we shall call this family $\{T(t) : t \geq 0\}$ a nonlinear Γ-contraction semigroup.

Theorem 3.2. Let E be a sequentially complete, locally convex Hausdorff space with a calibration Γ and A be a Γ-accretive operator in E such that $\overline{D(A)} \subset R(I+\lambda A)$ for all sufficiently small positive λ. Then

$$(3.1) \qquad T(t)x=\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x$$

exists for $x\varepsilon\overline{D(A)}$, uniformly in t on every compact interval of [0, ∞). Moreover, $T(t)$ defined by the formula (3.1) is a Γ-contraction semigroup on $D(A)$.

Proof. If A is Γ-accretive and $\overline{D(A)}\subset R(I+\lambda A)$, then, for every $p\varepsilon\Gamma$, A_p is accretive and $\overline{D(A_p)}\subset R(I+\lambda A_p)$. Thus, for $p\varepsilon\Gamma$ and $x\varepsilon D(A_p)$, $T^p(t)\underline{x}=\lim_{n\to\infty}(I+\frac{t}{n}A_p)^{-n}\underline{x}$ exists and $\{T^p(t) : t\geq 0\}$ is a contraction semigroup on $\overline{D(A_p)}$ ([2]). Let $x\varepsilon\overline{D(A)}$ and let n and m be positive integers such that n≥m. Then, for any $p\varepsilon\Gamma$,

$$p((I+\frac{t}{n}A)^{-n}x-(I+\frac{t}{m}A)^{-m}x)=p(S_p(I+\frac{t}{n}A)^{-n}x)-S_p((I+\frac{t}{m}A)^{-m}x))$$

$$=p((I+\frac{t}{n}A_p)^{-n}S_p(x)-(I+\frac{t}{m}A_p)^{-m}S_p(x))$$

$$\leq 2t(\frac{1}{n}-\frac{1}{m})^{\frac{1}{2}}\cdot inf\{p(\underline{y}) : \underline{y}\ \varepsilon A_pS_p(x)\} \quad ([2])$$

and hence $p((I+\frac{t}{n}A)^{-n}x-(I+\frac{t}{m}A^{-m}x))\to 0$ as $n,m\to\infty$.

Therefore $\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x=T(t)x$ exists uniformly in t on every compact subset of [0,∞). Then, for every $p\varepsilon\Gamma$ and $x\varepsilon\overline{D(A)}$,

$$S_p(T(t)x)=S_p(\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x)$$

$$=\lim_{n\to\infty}S_p(I+\frac{t}{n}A)^{-n}x)$$

$$=\lim_{n\to\infty}(I+\frac{t}{n}A_p)^{-n}S_p(x)$$

$$=T^p(t)S_p(x)$$

and hence $T(t)x\varepsilon\overline{D(A)}$. Since $(I+\frac{t}{n}A)^{-n}$ is Γ-contraction, we find that $p(T(t)x-T(t)y)\leq p(x-y)$ for every $t\geq 0$, $x,y\varepsilon\overline{D(A)}$, and for all $p\varepsilon\Gamma$. Therefore $T(t)$ is Γ-contraction on $\overline{D(A)}$. Moreover, for all $p\varepsilon\Gamma$ and $x\varepsilon\overline{D(A)}$, we obtain

$$\begin{aligned} p(T(t)x-T(s)x) &= p(S_pT(t)x)-S_p(T(s)x)) \\ &= p(T^p(t)S_p(x)-T^p(s)S_p(x)) \\ &\leq 2\,|\,t-s\,|\cdot inf\{p(\underline{y}):\underline{y}\varepsilon A_pS_p(x)\} \quad ([2]). \end{aligned}$$

In particular, this shows that $T(t)x$ is continuous in for every $x\varepsilon\overline{D(A)}$. In order to complete the proof, we shall verify the semigroup property $T(t+s)=T(t)T(s)$. For all $p\varepsilon\Gamma$ and $t,s\geq 0$, we have

$$\begin{aligned} S_p(T(t+s)x) &= T^p(t+s)S_p(x) \\ &= T^p(t)T^p(s)S_p(x) \\ &= T^p(t)\,(S_p(T(s)x)) \\ &= S_p(T(t)\,T(s)x), \qquad \text{for } x\varepsilon\overline{D(A)}. \end{aligned}$$

Since E is Hausdorff, $T(t+s)=T(t)T(s)$ for $t,s\geq 0$. This completes the proof.

We shall call a calibration Γ dually uniformly convex if, for every $p\varepsilon\Gamma$, $E[p]$ and its dual are uniformly convex.

Lemma 3.3.[6]. Assume that B is a closed subset of E and

$$S_p(x) \ \varepsilon \ S_p(B) \text{ for all } p\varepsilon\Gamma.$$

Then $x\varepsilon B$.

Theorem 3.4. If E is complete, locally convex Hausdorff space with a dually uniformly convex calibration Γ and A is a m-Γ-accretive operator in E. Then for each $x\varepsilon D(A)$ the initial value problem

$$(E) \begin{cases} \frac{du}{dt}(t)+Au(t) \ \ni \ 0, & t\geq 0, \\ u(0)=x \end{cases}$$

has a solution $u:[0,\infty)\rightarrow E$ given by $u(t)=T(t)x=\lim_{n\rightarrow\infty}(I+\frac{t}{n}A)^{-n}x$, $t\geq 0$.

Proof. By theorem 3.2, for each $x\varepsilon D(A)$, $T(t)x=\lim_{n\rightarrow\infty}(I+\frac{t}{n}A)^{-n}x$ exists. Since A is m-Γ-acretive and Γ is a dually uniformly convex calibration, A_p is m-accretive and $E[p]$ is uniformly convex space for every $p\varepsilon\Gamma$. Hence the initial value problem

$$\begin{cases} \frac{d}{dt}u(t)+A_p\underline{u}(t) \ \ni \ S_p(0), \\ \underline{u}(0)=S_p(x) \end{cases}$$

has a unique solution $\underline{u}: [0,\infty)\to E[p]$ given by

$$\underline{u}_p(t)=T^p(t)S_p(x)=\lim_{n\to\infty}(I+\frac{t}{n}A_p)^{-n}S_p(x) \quad ([1]).$$

Then, if $q\geq p$ in Γ, $T_{q,p}\underline{u}_p(0)=T_{q,p}S_q(x)=S_p(x)$ and

$$T_{q,p}(\frac{d}{dt}\underline{u}_q(t))+T_{q,p}(A_q\underline{u}_q(t) \quad \ni \quad T_{q,p}S_q(0),$$

which implies

$$\frac{d}{dt}T_{q,p}\underline{u}_q(t)+A_p T_{q,p}\ \underline{u}_q(t) \quad \ni \quad S_p(0),t\geq 0.$$

Hence, by the uniqueness of solution, $T_{q,p}\underline{u}_q(t)=\underline{u}_p(t)$ for $t\geq 0$. Since E is complete, there exists $u(t)\varepsilon E$ such that $\underline{u}_p(t)=S_p(u(t))$ for all $p\varepsilon\Gamma$ and $t\geq 0$([4]). Then, for all $p\varepsilon\Gamma$ and $t\geq 0$,

$$\begin{aligned}S_p(u(t))=\underline{u}_p(t)&=\lim_{n\to\infty}(I+\frac{t}{n}A_p)^{-n}S_p(x)\\&=\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n})_pS_p(x)\\&=S_p(\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x)\\&=S_p(T(t)x).\end{aligned}$$

Hence $u(t)=T(t)x==\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x$ for $x\varepsilon D(A)$ and $t\geq 0$. Furthermore, for all $p\varepsilon\Gamma$ and $t\geq 0$,

$$S_p(\frac{d}{dt}u(t)+Au(t))\ni S_p(0).$$

Then, by lemma 3.3, we have

$$\frac{d}{dt}u(t)+Au(t)) \ni 0.$$

Therefore $u(t)=T(t)x=\lim_{n\to\infty}(I+\frac{t}{n}A)^{-n}x$ is a solution of the initial value problem *(E)*.

References

1. V. Barbu, Nonlinear semi-groups and Differential equations in Banach spaces. Noordhoff International Publishing, 1976.
2. M.G Crandall and T.M. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces. Amer. J. Math 93(1971), 265-298.
3. J. Horvath, Topological vector spaces and Distributions. Addision-Wesley publishing, 1966
4. G. Kothe, Topological vector space I. Die Grundlehrem der mathemaschen Wissenchaften, 159. springer-Verlag, Berlin, Heidelberg, New York, 1967.
5. S. Yamamuro, Notes on the inverse mapping theorem in locally convex space. Bull. Austral. Math. Soc. Vol 21(1980), 419-461.
6. ______________, On a functional differential equation in locally convex spaces. Bull. Austral. Math. Soc Vol. 27(1983), 269-283.

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