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On the Total Curvature of Manifolds Immersed in a Euclidean Space

Yong-Soo Pyo and Sun-Sick Jang

1. Introduction

S.S.Chern and R.K.Lashof([3]) studied the total absolute curvature of immersed manifolds in a higher Euclidean space firstly through the Lipschitz-Killing curvature, and N.H.Kuiper([4]) who studied this area was contemporary with them.

Later, many mathematicians studied for the total absolute curvature (or total mean curature) of immersed manifolds ([1], [2], [6], [7], [8] and [9] etc.).

For an n-dimensional compact manifold M^n immersed in a Euclidean m-space E^m and the total absolute curvature $T(M^n)$ (that is, the intergral of the absolute value of the Lipschitz-Killing curvature over the unit normal bundle of M^n if it exists) of M^n, one of results Chern-Lashof and Kuiper proved in their papers [3, II] and [4].

(1.1) $$T(M^n) \geq C_{m-1}\,\beta(M^n),$$

where C_{m-1} is the volume of the unit $(m-1)$-sphere S^{m-1} and $\beta(M^n)$ is the sum of the betti numbers of M^n. The right-hand side of (1.1) depends on the coefficient field. And we know the Gauss-Bonnet theorem for a compact surface M in E^m.

(1.2) $\int_M G(p)\,dv = 2\pi\ \chi(M),$

where $G(p)$ is the Gauss curvature at p in M and $\chi(M)$ is the Euler characteristic of M. Besides, for any compact manifold M^n immersed in E^m, the inequality

(1.3) $\int_{M^n} \alpha^n(p)\,dv \geq C_n$

was proved in [1.I] and [7], where $\alpha(p)$ is the length of the mean curvature vector of M^n at p.

We have found that the idea in B.Y.Chern's([1, III]) was to choose the so-called Frenet frame e_1, e_2, e_3, e_4 in E^4 so that the Lipschitz-Killing curvature $K(p,e)$ at (p,e) is given by

(1.4) $K(p,e) = \lambda(p)\cos^2\theta + \mu(p)\sin^2\theta,\ \ \lambda(p) \geq \mu(p),$

where $e = \Sigma_{r=3}^{4} \cos\theta e_3 + \sin\theta e_4$ is a unit normal vecter at p.

In this paper, we have generalized this idea of choosing suitable local field of orthonormal frames $e_1, e_2, \cdots, e_m$ so that the partial Gauss curvatures $\lambda_1(p) \geq \lambda_2(p) \geq \cdots \geq \lambda_d(p)$ and $K(p,e) = (-1)^n \lambda_1(p) \cos^n n\theta_{n+1} + \cdots + \lambda_d(p) \cos^n \theta_{n+d}$ for a unit normal vector $e = \Sigma_{r=n+1}^{m} \cos\theta_r e_r$ at p if the N-index of M^n at p is d, and obtained some results for the total curvature of a manifold M^n immersed in E^m.

2. Preliminaries

Let E^n be an n-dimensional manifold immersed in a Euclidean space E^m of dimension $m (m > n)$. We choose a local field of orthonormal frames $e_1, \cdots e_m$ in E^m such that, restricted to M^n, the vectors $e_1 \cdots e_n$ are tangent to M^n(and consequently, $e_{n+1}, \cdots, e_m$ are normal to M^n).

We shall make use of the following convention on the ranges of indices :

(2.1) $$1 \leq i,\ j \leq n\ ;\quad n+1 \leq r \leq m\ ;$$
$$1 \leq A,\ B,\ C \leq m$$

unless otherwise stated. With respect to the frame field of E^m chosen above, let $\omega_1,\cdots,\omega^m$ be the field of dual frames. Then the structure equations of E^m are given by

(2.2) $$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{AB} = 0,$$
$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}.$$

We restrict these forms to M^n. Then $\omega_r = 0$. Since $0 = d\omega_r = \Sigma_i\, \omega_{ri} \wedge \omega_i$, by Cartan's lemma we may write

(2.3) $$\omega_{ir} = \sum_j h^r_{ij}\, \omega_j, \quad h^r_{ij} = h^r_{ji}.$$

From these formulas, we obtain

(2.4) $$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl}\, \omega_k \wedge \omega_l,$$

where R_{ijkl} denotes the curvature tensor on the manifold M^n. Thus we obtain

(2.5) $$R_{ijkl} = \sum_r (h^r_{il} h^r_{jk} - h^r_{ik} h^r_{jl}).$$

We call $h = \Sigma_{r,i,j}\, h^r_{ij}\, \omega_i\, \omega_j e_r$ the second fundamental form of M^n. The mean curvature vector H is given by $\frac{1}{n} \Sigma_r\, (\Sigma_i h^r{}_{ii}) e_r$.

For a normal vector $e = \Sigma_r\, \alpha_r\, e_r$ at p in M^n, the second fundamental form $A(p,e)$ at (p,e) is given by $(\Sigma_r \alpha_r h^r_{ij})$ as $n \times n$ matrix. The Lipschitz-killing curvature $K(p,e)$ is defined by

(2.6) $$K(p,e) = (-1)^n det(A(p,e)).$$

3. Some Results

For each $p \in M^n$, we denote by T_p the normal space of M^n at p. We define a linear mapping γ from T_p into the space of all symmetric matrices of order n by

(3.1) $$\gamma(\sum_r \alpha_r e_r) = \sum_r \alpha_r A(p,e_r).$$

Let O_p denote the kernel of γ. Then we have $A(p,e)=0$ for any $e \in O_p$ and dim $O_p \rangle m - \frac{n}{2}(n+3)$. We define the N-index of M^n at p by

(3.2) $$N-index_p = m - n - dim O_p.$$

In fact, the N-index of any surface M is $\langle$ 3 everywhere.

Suppose that the N-index of M^n at p is d. Then we choose $e_3 \cdots, e_m$ at p in such a way that $e_{n+d+1}, \cdots, e_m \in O_p$. For any unit normal vector $e = \Sigma_r \cos \theta_r e_r$ at p, the Lipschitz-Killing curvature $K(p,e)$ at (p,e) is a form of degree n on $\cos\theta_r$. Hence, by choosing a suitable unit orthogonal vectors $e_{n+1} \cdots, e_{n+d}$ at p, we may write

(3.3) $$K(p,e) = (-1)^n(\lambda_1(p)\cos^n\theta_{n+1} + \cdots + \lambda_d(p)\cos^n\theta_{n+d}),$$
$$\lambda_1(p) \geq \cdots \geq \lambda_d(p).$$

Theorem 1. Let M^n be an n-dimensional compact manifold immersed in a Euclidean m-space E^m. If the N-index of $M^n \leq d$ everywhere and $\lambda_d \geq 0$, then the total absolute curvature $T(M^n)$ of M^n is given by

(3.4) $$T(M^n) \leq \frac{2C_{m-1}}{C_n} \int_{M^n} \sum_{i=1}^{d} \lambda_i(p)\, dv,$$

where C_m is the volume of the unit m-sphere S^m. The equality sign holds when and only when $\lambda_1 \neq 0$, and n is even or $\lambda_2 = 0$.

Proof. From (3.3), the total absolute curvature $K^*(p)$ at p is given by

(3.5) $$\begin{aligned} K^*(p) &= \int_{S^{m-n-1}} |K(p,e)|\, d\sigma \leq \sum_{i=1}^{d} \lambda_i(p) \int_{S^{m-n-1}} |\cos^n \theta_{n+i}|\, d\sigma \\ &= \sum_{i=1}^{d} \lambda_i(p) \frac{C_{m-1}}{C_{n+1}} \int_0^{2\pi} |\cos^n \theta|\, d\theta \\ &= \frac{2C_{m-1}}{C_n} \sum_{i=1}^{d} \lambda_i(p) \end{aligned}$$

by spherical integration [5], where S^{m-n-1} is the unit hypersphere of $T_p^\perp$, $d\sigma$ is the volume element of S^{m-n-1} and Γ is the Gamma function. Therefore the total absolute curvature $T(M^n)$ of M^n is given by

(3.6) $$T(M^n) = \int_{M^n} K^*(p)\, dv \leq \frac{2C_{m-1}}{C_n} \int_{M^n} \sum_{i=1}^{d} \lambda_i(p)\, dv.$$

If the equality sign of (3.6) holds, the inequality in (3.5) is actually equality. Since $\lambda_1 \geq \cdots \geq \lambda_d \geq 0$, n is even or $\lambda_2 = 0$. If $\lambda_1 = 0$, then this is impossible because $T(M^n) \geq 2C_{m-1}$ (see [3, 1]). Hence $\lambda_1 \neq 0$. The converse of this is trivial.

And also, we can prove the following theorem.

Theorem 2. Let M^n be an n-dimensional compact manifold immersed in Euclidean m-space E^m. If the N-index of $M^n \leq d$ everywhere and $\lambda_1 \leq 0$, then we have

(3.7) $$T(M^n) \leq -\frac{2C_{m-1}}{C_n} \int_{M^n} \sum_{i=1}^{d} \lambda_i(p)\, dv.$$

If the equality sign holds when and only when $\lambda_d \neq 0$, and n is even or $\lambda_{d-1}=0$.

Corollary 3. Let M^n be a compact manifold immersed in E^m with N-index of $M^n \leq d$ everywhere. If $\int_{M^n} \sum_{i=1}^{d} |\lambda_i(p)| \, dv \geq \frac{3}{2} C_n$, then M^n is homeomorphic to a sphere S^n of n-dimensions.

Proof. By Theorem 1 and Theorem 2, we have

$$(3.8) \qquad T(M^n) \leq \frac{2C_{m-1}}{C_n} \int_{M^n} \sum_{i=1}^{d} \lambda_i(p) dv.$$

Hence we abtain $T(M^n) \leq 3C_{m-1}$ by the assumption. Therefore M^n is homeomorphic to S^n (see [3, 1]).

We have the following corollary by (1.1) and (3.8).

Corollary 4. Let M^n be a compact manifold immersed in E^m with the N-index of $M^n \leq d$ everywhere. Then we have

$$(3.9) \qquad C^n \beta(M^n) \leq 2 \int_{M^n} \sum_{i=1}^{d} |\lambda_i(p)| \, dv,$$

where $\beta(M^n)$ is the sum of the betti numbers of M^n.

And also, we abtain the following corollary.

Corollary 5. Let M^n be a compact manifold immersed in E^m with even dimension n. If the N-index of $M^n \leq d$ everywhere, then we have

$$(3.10) \qquad T(M^n) \int_{M^n} \alpha^n(p) \, dv \geq 2C_{m-1} \int_{M^n} \sum_{i=1}^{d} \lambda_i(p) \, dv,$$

where $\alpha(p)$ is the lengh of the mean curvature vector H at p in M^n.

Proof. By spherical integration, the total absolute curvature $K^*(p)$

at p is given by

$$K^*(p) \geq \sum_{i=1}^{d} \lambda_i(p) \int_{S^{m-n-1}} \cos^n \theta_{n+1} \, d\sigma = \frac{2C_{m-1}}{C_n} \sum_{i=1}^{d} \lambda_i(p). \tag{3.11}$$

Hence we have

$$T(M^n) \geq \frac{2C_{m-1}}{C_n} \int_{M^n} \sum_{i=1}^{d} \lambda_i(p) \, dv. \tag{3.12}$$

Therefore we complete a proof of the corollary by (1.3).

Theorem 6. Let M be a compact surface in E^m. If the N-index of M is 2 and $\lambda_2=0$. Then M is homeomorphic to a 2-sphere.

Proof. From (3.3), since $G(p)=\lambda_1(p)$, we have

$$\begin{aligned} K^*(p) &= \lambda_1(p) \int_{S^{m-3}} \cos^2 \theta_3 \, d\sigma \\ &= \lambda_1(p) \frac{C_{m-1}}{C_3} \int_0^{2\pi} \cos^2 \theta \, d\theta \\ &= \frac{C_{m-1}}{2\pi} G(p) \end{aligned} \tag{3.13}$$

by spherical integration, where $G(p)$ is the Gauss curvature at p in M. Hence the total absolute curvature $T(M)$ of M is given by

$$T(M) = \int_M K^*(p) \, dv = C_{m-1} \chi(M) \tag{3.14}$$

by (1.2), where $\chi(M)$ is the Euler chracteristic of M. Since $T(M) \geq C_{m-1}\beta(M)$ (see (1.1)), $\chi(M) \geq \beta(M)$. Therefore $\chi(M)=\beta(M)=2$. Hence M is homeomorphic to a 2-sphere.

From Theorem 6, we can prove the following corollary because $G=\lambda_1+\lambda_2+\lambda_3$ on a surface M in E^m.

Corollary 7. Let M be a compact surface in E^m with $\lambda_3=0$.

Then M is homeomorphic to a 2-sphere.

Theorem 8. Let M^n be a compact manifold immersed in E^m. Then we have

(3.15) $$\int_{M^n} R(p)dv \leq n^2 \int_{M^n} \alpha^2(p)dv,$$

where $R(p)$ is the scalar curvature at p in M^n.

Proof. From (2.5), the scalar curvature R is given by

(3.16) $$\begin{aligned} R &= \sum_{i,j} R_{ijji} \\ &= \sum_{r}(\sum_{i} h^r_{ii})^2 - \sum_{r,i,j}(h^r_{ij})^2 \\ &= n^2\alpha^2 - S^2, \end{aligned}$$

where $S=(\Sigma_{r,i,j}(h^r_{ij})^2)^{1/2}$. Hence $R(p) \leq n^2\alpha^2(p)$ for at any point p in M^n, since $S \neq 0$. Therefore this completes a proof the theorem.

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National Fisheries University of Pusan
Pusan 608−737,
Korea.