Bull. Korean Math. Soc. **27**(1990), No. 2, pp. 121-126

PROPERTIES OF THE GENERALIZED EVALUATION SUBGROUP OF A TOPOLOGICAL PAIR

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Let X be a topological space and A be a subspace of X. A homotopy $H : X\times I\to X$ is called a *cyclic homotopy* [Go] if

$$H(x, 0)=H(x, 1)=x.$$

If H is a cyclic homotopy and $x_0\in A$ is a base point, the loop given by $h(s)=H(x_0, s)$ is called the *trace* of H.

The set of homotopy classes of those loops which are the trace of some cyclic homotopy form a subgroup $G(X, x_0)$ which is called the *evaluation subgroup* of the fundamental group $\pi_1(X, x_0)$ [Go]. $G(X, x_0)$ is denoted $J(X)$ by Jiang [J_2]. If we consider the class of continuous functions $H : A\times I\to X$ such that

$H(x, 0)=H(x, 1)=i(x)$ and $i : A\to X$ is the inclusion, then the trace $h(s)=H(x_0, s)$ of H is a loop at x_0 in X. In this case, H is called an *affiliated homotopy* to $[h]$ with respect to A. The trace subgroup $G(X, A, x_0)$ *of* $\pi_1(X, x_0)$ is defined by $G(X, A, x_0)=\{\alpha\in\pi_1(X, x_0):$ there exists an affiliated homotopy H such that $[H(x_0,)]=\alpha\}$. In particular, we have that $G(X, X, x_0)=G(X, x_0)$ and $G(X, x_0, x_0)=\pi_1(X, x_0)$.

Let A be locally compact and regular, and X^A be the space of mappings from A to X with compact open topology. The map $p : X^A\to X$ given by $p(g)=g(x_0)$ is continuous. Thus p induces a homomorphism $p_* : \pi_1(X^A, i)\to\pi_1(X, x_0)$. In this case, the image of p_* is $G(X, A, x_0)$ [WK]. Thus $G(X, A, x_0)$ is called the *generalized evaluation subgroup* of the fundmental group. It is clear that $G(X, x_0)$ is a subgroup of $G(X, A, x_0)$.

In [J_2], Jiang showed that $J(X)=G(X, x_0)$ is a subgroup of $Z(\pi_1(X, x_0))$. We generalize this result as follows:

Received October 4, 1989.
Revised January 30, 1990.
Research supported by Korea Science and Engineering Foundation.

THEOREM 1. *$G(X, A, x_0)$ is contained in $Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0))$, where $Z(H, K)$ denotes the centralizer of a subgroup H of K.*

Proof. Let $\alpha \in G(X, A, x_0)$. Then there exists an affiliated homotopy $H : A \times I \to X$ such that $H(x, 0) = H(x, 1) = i(x)$ and $[H(x_0,)] = \alpha$. Let $\beta = [f]$ be any element of $\pi_1(A, x_0)$. We must show that $\alpha i_*(\beta) = i_*(\beta)\alpha$. Let $K = H(f \times 1_I) : I \times I \to X$. Define a homotopy $G : I \times I \to X$ by

$$G(s, t) = \begin{cases} K(2s(1-t),\ 2st), & 0 \leq s \leq 1/2 \\ K(1-(2-2s)t,\ (2-2s)t+2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

Then $[G(\ ,0)] = i_*(\beta)\alpha$ and $[G(\ ,1)] = \alpha i_*(\beta)$. Since $G(0, t) = x_0 = G(1, t)$, we have $\alpha i_*(\beta) = i_*(\beta)\alpha$.

COROLLARY 2. *$J(X) = G(X, X, x_0)$ is a subgroup of $Z(\pi_1(X, x_0), \pi_1(X, x_0)) = Z(\pi_1(X, x_0))$.*

If A is a connected aspherical polyhedron, then the reverse is also true.

THEOREM 3. *If A is a connected aspherical polyhedron and $A \subset X$, then $G(X, A, x_0) = Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0))$.*

Proof. By the previous theorem, it was proved that $G(X, A, x_0)$ is contained in $Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0))$.

The proof of the reverse is quite analogous to Theorem 10 [7A, Br]. Take a triangulation (K, τ) of A and choose $x_0 \in A^0$ (0–skeleton of A). Define a map $h : (A \times \{0\}) \cup (A^0 \times I) \to A$

by $$h(x, t) = \begin{cases} x, & \text{if } t=0 \\ C_x(t), & \text{if } x \in A^0 \end{cases}$$

where C_x is a path from x to x_0 and C_{x_0} is the trivial path. Then there exists an extension $H : A \times I \to A$ of h. Define $d : A \to A$ by $d(x) = H(x, 1)$. Then the map d is homotopic to 1_A and $d(A^0) = x_0$. Define $\bar{i} = i \circ d$, then $\bar{i}(A^0) = x_0$. Let α be an element of $Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0))$ and $\alpha = [c]$. Define $h_1 : Q^1 = A \times \partial I \cup A^0 \times I \to X$ by

$$h_1(x, u) = \begin{cases} \bar{i}(x), & \text{if } u=0 \text{ or } u=1 \\ c(u), & \text{if } x \in A^0 \end{cases}$$

For 1–simplex s_j of K, let $\sigma_j = \tau(cl|s_j|) \times \{0\} \subset Q^1$, then there is a homeomorphism $\phi_j : I \to \sigma_j$. Define $c_j = h_1 \circ \phi_j = \bar{i} \circ \phi_j = i \circ d \circ \phi_j$ but $d \circ \phi_j$ is a loop in A based at x_0. Therefore $[c_j] = [i \circ d \circ \phi_j] = i_*([d \circ \phi_j]) \in i_*(\pi_1(A, x_0))$. Since $\alpha = [c]$ is contained in $Z(i_*(\pi_1(A, x)), \pi_1(X, x_0))$, $[c]$

$[c_j]=[c_j][c]$. Therefore, there exists a map $L_j : I\times I\to X$ such that $L_j(t, 0)=L_j(t, 1)=c_j(t)$ and $L_j(0, u)=L_j(1, u)=c(u)$ for all $t, u\in I$. Define $H_j : \sigma_j\times I\to X$ by $H_j(x, u)=L_j(\phi_j^{-1}(x), u)$. If $(x, u)\in\partial(\sigma_j\times I)\subset Q^1$, then $H_j(x, u)=h_1(x, u)$. Write the 1-simplices of K as $s_1, s_2, \cdots, s_{r(1)}$; then $Q^2=(A\times\partial I)\cup\bigcup_{j=1}^{r(1)}(\sigma_j\times I)$. Extend h_1 to a map $h_2 : Q^2\to X$ by

$$h_2(x, u)=\begin{cases}\bar{i}(u) & \text{, if } u=0 \text{ or } u=1\\ H_j(x, u), & \text{if } x\in\sigma_j \text{ for some } j=1, \ldots, r(1).\end{cases}$$

Assume that h_2 has been extended to a map $h_p : Q^p\to X$, $(p\geq 2)$. Take some p-simplex s_j of K and again define

$$\sigma_j=\tau(cl|s_j|)\times\{0\}\subset A\times\{0\}.$$

Since $\partial(\sigma_j\times I)\subset Q^p$, we have the restriction $h_{p,j} : \partial(\sigma_j\times I)\to X$ of h_p. We assumed that A was aspherical, so $\pi_p(A, x_0)$ is trivial. Since $\sigma_j\times I$ is homeomorpic to I^{p+1}, we can extend $h_{p,j}$ to a map $h_{p+1,j} : \sigma_j\times I\to X$. Write the p-simplices of K as $s_1, \ldots, s_{r(p)}$ and define $h_{p+1} : Q^{p+1}\to X$ by

$$h_{p+1}(x, u)=\begin{cases}\bar{i}(x), & \text{if } u=0 \text{ or } u=1\\ h_{p+1,j}(x, u), & \text{if } x\in\sigma_j \text{ for some } j=1, \ldots, r(p).\end{cases}$$

Then h_{p+1} is an extension of h_p. Since $Q^n=A\times I$ for some n, we have proved the existence of a map $H=h_n : A\times I\to X$ whose restriction on Q^1 is h_1. Thus $H(x, 0)=H(x, 1)=\bar{i}(x)$ and $H(x_0, u)=h_1(u)=c(u)$. Since d is homotopic to 1_A, there is a homotopy G from $i\circ d$ to i (rel x_0). Define $K : A\times I\to X$

by
$$K(a, s)=\begin{cases}G(a, 1-3s), & 0\leq s\leq 1/3\\ H(a, 3s-1), & 1/3\leq s\leq 2/3\\ G(a, 3s-2), & 2/3\leq s\leq 1.\end{cases}$$

Then $K(a, 0)=i(a)=K(a, 1)$ and $K(x_0, u)=c(u)$. Therefore $[c]$ is an element of $G(X, A, x_0)$.

COROLLARY 4. *If the inclusion $i : A\to X$ has a left homotopy inverse, then $G(X, A, x_0)\cap i_*(\pi_1(A, x_0))$ is contained in $i_*(Z(\pi_1(A, x_0)))$.*

Proof. Let 1 be a left homotopy inverse of i. Then $1\circ i$ is homotopic to 1_A and hence i_* is a monomorphism. Let α be an element of $G(X, A, x_0)\cap i_*(\pi_1(A, x_0))$. Then $\alpha=i_*(\beta)$ for some $\beta\in\pi_1(A, x_0)$. Let γ be any element of $\pi_1(A, x_0)$. Since $G(X, A, x_0)\subset Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0))$, $\alpha=i_*(\beta)\in G(X, A, x_0)$ and $i_*(\gamma)\in i_*(\pi_1(A, x_0))$, we have $i_*(\gamma)$

$i_*(\beta)=i_*(\beta)i_*(\gamma)$. Therefore $\gamma\beta=\beta\gamma$ This implies $\beta\in Z(\pi_1(A, x_0))$. Hence $\alpha=i_*(\beta)$ belongs to $i_*(Z(\pi_1(A, x_0)))$.

THEOREM 5. *Let A be a connected aspherical polyhedron. Then the inclusion* $i: A\to X$ *satisfies* $i_*(\pi_1(A, x_0))\subset Z(\pi_1(X, x_0))$ *if and only if* $G(X, A, x_0)=\pi_1(X, x_0)$.

Proof. By Theorem 3, we have $G(X, A, x_0)=Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0))$. Since $i_*(\pi_1(A, x_0)\subset Z(\pi_1(X, x_0))$, we have $G(X, A, x_0)=\pi_1(X, x_0)$.

Conversely, if $G(X, A, x_0)=\pi_1(X, x_0)$, then $i_*(\pi_1(A, x_0))$ is contained in $Z(\pi_1(X, x_0))$.

Jiang [J_2] showed that $J(X, f(x_0))\subset J(f, x_0)=\{\S\in\pi_1(X, f(x_0)) :$ there exists a cyclic homotopy $H: f\simeq f$ such that $[H(x_0,)]=\S\}$ In the following theorem, we show that $J(X, f(x_0))\subset G(X, f(X), f(x_0)))\subset J(f, x_0)$.

THEOREM 6. *Let* $f: X\to X$ *be a self-map and* $y_0=f(x_0)$. *Then* $G(X, f(X), y_0)\subset J(f, x_0)$, *where* $J(f, x_0)$ *denotes the Jiang subgroup of* $\pi_1(X, y_0)$ [J_2, *Br*]. *In particular, if* $f^2=f$, *then* $G(X, f(X), y_0)=J(f, y_0)$, *where* $y_0\in f(X)$.

Proof. Let α be an element of $G(X, f(X), y_0)$. Then there exists an affiliated homotopy $H: f(X)\times I\to X$ such that $H(y, 0)=i(y)=H(y, 1)$ and $[H(y_0,)]=\alpha$. Define $K=H(f_0\times 1_I): X\times I\to X$, where $f_0: X\to f(X)$ is a map such that $f_0(x)=f(x)$. Then $K(x, 0)=H(f_0(x), 0)=i(f_0(x))=f(x)=K(x, 1)$. Since $K(x_0, s)=H(f_0(x_0), s)=H(y_0, s)$, we have $\alpha=[H(y_0,)]=[K(x_0,)]$. This implies $\alpha\in J(f, x_0)$.

Suppose $f^2=f$. Let α be an element of $J(f, y_0)$. Then there exists a cyclic homotopy $H: X\times I\to X$ such that $H(x, 0)=f(x)=H(x, 1)$ and $[H(y_0,)]=\alpha$. Define $K=H(i\times 1_I): f(X)\times I\to X$. Then $K(y, 0)=H(y, 0)=f(y)=f(f(x))=f(x)=y=K(y, 1)$ and $K(y_0, t)=H(y_0, t)$. Thus $\alpha=[H(y_0,)]\in G(X, f(X), y_0)$.

COROLLARY 7. *Let f and g be self-maps of X such that* $f^2=f$, $g^2=g$ *and* $f(X)=g(X)$. *Then* $J(f, y_0)=J(g, y_0)$, *where* $y_0\in f(X)$.

In [Br], we know that if f is a self-map of X such that $J(f, x_0)$

$=\pi_1(X, x_0)$, then all the fixed point classes of f have the same index. If we use Theorem 6, we have the following;

Corollary 8. *Let f be a self-map of X such that $f^2=f$, $x_0\in Fix(f)$ and $G(X, f(X), x_0)=\pi_1(X, x_0)$. Then all the fixed point classes of f have the same index.*

Theorem 9. *Let f_i $(i=1, 2)$ be self-maps of X and f_1 is homotopic to f_2 by a homotopy K such that $K(f_i^{-1}f_i\times 1_I)$ is single valued. Then $G(X, f_1(X), f_1(x_0))$ is isomorphic to $G(X, f_2(X), f_2(x_0))$.*

Proof. Let K be the homotopy from f_1 to f_2 such that $K(f_i^{-1}f_i\times 1_I)$ is single valued. Let $P(t)=K(x_0, t)$, Then P is a path from $f_1(x_0)$ to $f_2(x_0)$. Since $P_*: \pi_1(X, f_1(x_0))\to\pi_1(X, f_2(x_0))$ is an isomorphism, it is sufficent to show $P_*(G(X, f_1(X), f_1(x_0)))\subset G(X, f_2(X), f_2(x_0))$. Let α be any element of $G(X, f_1(X), f_1(x_0))$. Then there exists an affiliated homotopy $H: f_1(X)\times I\to X$ such that $H(x, 0)=H(x, 1)=i(x)$ and $\alpha=[H(f_1(x_0),)]$. Define $G: f_2(X)\times I\to X$ by

$$G(f_2(x), t)=\begin{cases}K(x, 1-3t), & 0\leq t\leq 1/3\\ H(f_1(x), 3t-1), & 1/3\leq t\leq 2/3\\ K(x, 3t-2), & 2/3\leq t\leq 1\end{cases}$$

Then G is well defined and continuous. Since $G(y, 0)=y=G(y, 1)$ and

$$\begin{aligned}G(f_2(x_0), t)&=\begin{cases}K(x_0, 1-3t), & 0\leq t\leq 1/3\\ H(f_1(x_0), 3t-1), & 1/3\leq t\leq 2/3\\ K(x_0, 3t-2), & 2/3\leq t\leq 1\end{cases}\\ &=\begin{cases}P(1-3t), & 0\leq t\leq 1/3\\ h(3t-1), & 1/3\leq t\leq 2/3\\ P(3t-2), & 2/3\leq t\leq 1\end{cases}\\ &=(P*h*P)(t),\end{aligned}$$

thus $P_*(\alpha)$ belongs to $G(X, f_2(X), f_2(X), f_1(x_0))$, where $h(t)=H(f_1(x_0), t)$.

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