PROPERTIES OF THE GENERALIZED EVALUATION SUBGROUP OF A TOPOLOGICAL PAIR

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Let $X$ be a topological space and $A$ be a subspace of $X$. A homotopy $H : X \times I \to X$ is called a cyclic homotopy [Go] if

$$H(x, 0) = H(x, 1) = x.$$ 

If $H$ is a cyclic homotopy and $x_0 \in A$ is a base point, the loop given by $h(s) = H(x_0, s)$ is called the trace of $H$.

The set of homotopy classes of those loops which are the trace of some cyclic homotopy form a subgroup $G(X, x_0)$ which is called the evaluation subgroup of the fundamental group $\pi_1(X, x_0)$ [Go]. $G(X, x_0)$ is denoted $J(X)$ by Jiang [J2]. If we consider the class of continuous functions $H : A \times I \to X$ such that $H(x, 0) = H(x, 1) = i(x)$ and $i : A \to X$ is the inclusion, then the trace $h(s) = H(x_0, s)$ of $H$ is a loop at $x_0$ in $X$. In this case, $H$ is called an affiliated homotopy to $[h]$ with respect to $A$. The trace subgroup $G(X, A, x_0)$ of $\pi_1(X, x_0)$ is defined by $G(X, A, x_0) = \{ \alpha \in \pi_1(X, x_0) :$ there exists an affiliated homotopy $H$ such that $[H(x_0, \cdot)] = \alpha \}$. In particular, we have that $G(X, X, x_0) = G(X, x_0)$ and $G(X, x_0, x_0) = \pi_1(X, x_0)$.

Let $A$ be locally compact and regular, and $X^A$ be the space of mappings from $A$ to $X$ with compact open topology. The map $\rho : X^A \to X$ given by $\rho(g) = g(x_0)$ is continuous. Thus $\rho$ induces a homomorphism $\rho_* : \pi_1(X^A, i) \to \pi_1(X, x_0)$. In this case, the image of $\rho_*$ is $G(X, A, x_0)$ [WK]. Thus $G(X, A, x_0)$ is called the generalized evaluation subgroup of the fundamental group. It is clear that $G(X, x_0)$ is a subgroup of $G(X, A, x_0)$.

In [J2], Jiang showed that $J(X) = G(X, x_0)$ is a subgroup of $Z(\pi_1(X, x_0))$. We generalize this result as follows:

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Theorem 1. \( G(X, A, x_0) \) is contained in \( Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0)) \),
where \( Z(H, K) \) denotes the centralizer of a subgroup \( H \) of \( K \).

Proof. Let \( \alpha \in G(X, A, x_0) \). Then there exists an affiliated homotopy \( H : A \times I \to X \) such that \( H(x, 0) = H(x, 1) = i(x) \) and \( [H(x_0)] = \alpha \). Let \( \beta = [f] \) be any element of \( \pi_1(A, x_0) \). We must show that \( a_i*(\beta) = i_*(\beta) \alpha \). Let \( K = H(f \times 1) : I \times I \to X \). Define a homotopy \( G : I \times I \to X \) by

\[
G(s, t) = \begin{cases} 
K(2s(1-t), 2st), & 0 \leq s \leq 1/2 \\
K(1 - (2-2s)t, (2-2s)t + 2s - 1), & 1/2 \leq s \leq 1.
\end{cases}
\]

Then \( [G(, 0)] = i_*(\beta) \alpha \) and \( [G(, 1)] = a_i*(\beta) \). Since \( G(0, t) = x_0 = G(1, t) \), we have \( a_i*(\beta) = i_*(\beta) \alpha \).

Corollary 2. \( J(X) = G(X, X, x_0) \) is a subgroup of \( Z(\pi_1(X, x_0), \pi_1(X, x_0)) = Z(\pi_1(X, x_0)) \).

If \( A \) is a connected aspherical polyhedron, then the reverse is also true.

Theorem 3. If \( A \) is a connected aspherical polyhedron and \( A \subset X \), then \( G(X, A, x_0) = Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0)) \).

Proof. By the previous theorem, it was proved that \( G(X, A, x_0) \) is contained in \( Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0)) \).

The proof of the reverse is quite analogous to Theorem 10 [7A, Br].

Take a triangulation \( (K, \tau) \) of \( A \) and choose \( x_0 \in A^0 \) (0-skeleton of \( A \)). Define a map \( h : (A \times \{0\}) \cup (A^0 \times I) \to A \)

by

\[
h(x, t) = \begin{cases} 
x, & \text{if } t = 0 \\
C_x(t), & \text{if } x \in A^0
\end{cases}
\]

where \( C_x \) is a path from \( x \) to \( x_0 \) and \( C_x(t) \) is the trivial path. Then there exists an extension \( H : A \times I \to A \) of \( h \). Define \( d : A \to A \) by \( d(x) = H(x, 1) \). Then the map \( d \) is homotopic to \( 1_A \) and \( d(A^0) = x_0 \). Define \( i = i \circ d \), then \( i(A^0) = x_0 \). Let \( \alpha \) be an element of \( Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0)) \) and \( \alpha = [c] \). Define \( h_1 : Q^1 = A \times \partial I \cup A^0 \times I \to X \) by

\[
h_1(x, u) = \begin{cases} 
i(x), & \text{if } u = 0 \text{ or } u = 1 \\
c(u), & \text{if } x \in A^0
\end{cases}
\]

For 1-simplex \( s_j \) of \( K \), let \( \sigma_j = \tau(\{c|s_j\}) \times \{0\} \subset Q^1 \). Then there is a homeomorphism \( \phi_j : I \to \sigma_j \). Define \( c_j = h_1 \circ \phi_j = i \circ \phi_j = i \circ d \circ \phi_j \) but \( d \circ \phi_j \) is a loop in \( A \) based at \( x_0 \). Therefore \( [c_j] = [i \circ d \circ \phi_j] = i_*(d \circ \phi_j) = i_*(\pi_1(A, x_0)) \). Since \( \alpha = [c] \) is contained in \( Z(i_*(\pi_1(A, x)), \pi_1(X, x_0)) \), [c]
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\([c_j] = [c_j][c]\). Therefore, there exists a map \(L_j : I \times I \to X\) such that \(L_j(t, 0) = L_j(t, 1) = c_j(t)\) and \(L_j(0, u) = L_j(1, u) = c(u)\) for all \(t, u \in I\). Define \(H_j : \sigma_j \times I \to X\) by \(H_j(x, u) = L_j(\phi_j^{-1}(x), u)\). If \((x, u) \in \partial_0(\sigma_j \times I) \subset Q^1\), then \(H_j(x, u) = h_1(x, u)\). Write the 1-simplices of \(K\) as \(s_1, s_2, \ldots, s_{r(1)}\); then \(Q^2 = (A \times \partial I) \cup \bigcup_{j=1}^{r(1)} (\sigma_j \times I)\). Extend \(h_1\) to a map \(h_2 : Q^2 \to X\) by

\[
h_2(x, u) =
\begin{cases}
  i(u), & \text{if } u = 0 \text{ or } u = 1 \\
  H_j(x, u), & \text{if } x \in \sigma_j \text{ for some } j = 1, \ldots, r(1).
\end{cases}
\]

Assume that \(h_2\) has been extended to a map \(h_p : Q^p \to X\), \((p \geq 2)\). Take some \(p\)-simplex \(s_j\) of \(K\) and again define

\[
\sigma_j = \tau(d|s_j|) \times \{0\} \subset A \times \{0\}.
\]

Since \(\partial(\sigma_j \times I) \subset Q^p\), we have the restriction \(h_{p,j} : \partial(\sigma_j \times I) \to X\) of \(h_p\). We assumed that \(A\) was aspherical, so \(\pi_p(A, x_0)\) is trivial. Since \(\sigma_j \times I\) is homeomorphic to \(I^{p+1}\), we can extend \(h_{p,j}\) to a map \(h_{p+1,j} : \sigma_j \times I \to X\). Write the \(p\)-simplices of \(K\) as \(s_1, \ldots, s_{r(p)}\), and define \(h_{p+1} : Q^{p+1} \to X\) by

\[
h_{p+1}(x, u) =
\begin{cases}
  i(x), & \text{if } u = 0 \text{ or } u = 1 \\
  h_{p+1,j}(x, u), & \text{if } x \in \sigma_j \text{ for some } j = 1, \ldots, r(p).
\end{cases}
\]

Then \(h_{p+1}\) is an extension of \(h_p\). Since \(Q^n = A \times I\) for some \(n\), we have proved the existence of a map \(H = h_n : A \times I \to X\) whose restriction on \(Q^1\) is \(h_1\). Thus \(H(x, 0) = H(x, 1) = i(x)\) and \(H(x_0, u) = h_1(u) = c(u)\).

Since \(d\) is homotopic to \(1_A\), there is a homotopy \(G\) from \(id\) to \(i\) (rel \(x_0\)). Define \(K : A \times I \to X\) by

\[
K(a, s) =
\begin{cases}
  G(a, 1-3s), & 0 \leq s \leq 1/3 \\
  H(a, 3s-1), & 1/3 \leq s \leq 2/3 \\
  G(a, 3s-2), & 2/3 \leq s \leq 1.
\end{cases}
\]

Then \(K(a, 0) = i(a) = K(a, 1)\) and \(K(x_0, u) = c(u)\). Therefore \([c]\) is an element of \(G(X, A, x_0)\).

Corollary 4. If the inclusion \(i : A \to X\) has a left homotopy inverse, then \(G(X, A, x_0) \cap i_*(\pi_1(A, x_0))\) is contained in \(i_*(Z(\pi_1(A, x_0)))\).

Proof. Let \(1\) be a left homotopy inverse of \(i\). Then \(1 \circ i\) is homotopic to \(1_A\) and hence \(i_*\) is a monomorphism. Let \(\alpha\) be an element of \(G(X, A, x_0) \cap i_*(\pi_1(A, x_0))\). Then \(\alpha = i_*(\beta)\) for some \(\beta \in \pi_1(A, x_0)\). Let \(\gamma\) be any element of \(\pi_1(A, x_0)\). Since \(G(X, A, x_0) \subset Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0))\), \(\alpha = i_*(\beta) \in G(X, A, x_0)\) and \(i_*(\gamma) \in i_*(\pi_1(A, x_0))\), we have \(i_*(\gamma)\)
\( i_*(\beta) = i_*(\beta) i_*(\gamma) \). Therefore \( \gamma \beta = \beta \gamma \) This implies \( \beta \in Z(\pi_1(A, x_0)) \). Hence \( \alpha = i_*(\beta) \) belongs to \( i_*(Z(\pi_1(A, x_0))) \).

**Theorem 5.** Let \( A \) be a connected aspherical polyhedron. Then the inclusion \( i : A \to X \) satisfies \( i_*(\pi_1(A, x_0)) \subset Z(\pi_1(X, x_0)) \) if and only if \( G(X, A, x_0) = \pi_1(X, x_0) \).

**Proof.** By Theorem 3, we have \( G(X, A, x_0) = Z(i_*(\pi_1(A, x_0)), \pi_1(X, x_0)) \). Since \( i_*(\pi_1(A, x_0)) \subset Z(\pi_1(X, x_0)) \), we have \( G(X, A, x_0) = \pi_1(X, x_0) \).

Conversely, if \( G(X, A, x_0) = \pi_1(X, x_0) \), then \( i_*(\pi_1(A, x_0)) \) is contained in \( Z(\pi_1(X, x_0)) \).

Jiang [\( J_2 \)] showed that \( J(X, f(x_0)) \subset J(f, x_0) = \{ g \in \pi_1(X, f(x_0)) : \text{there exists a cyclic homotopy } H : f \simeq f \text{ such that } [H(x_0, \cdot)] = g \} \) In the following theorem, we show that \( J(X, f(x_0)) \subset G(X, f(X), f(x_0)) \subset J(f, x_0) \).

**Theorem 6.** Let \( f : X \to X \) be a self-map and \( y_0 = f(x_0) \). Then \( G(X, f(X), y_0) \subset J(f, x_0) \), where \( J(f, x_0) \) denotes the Jiang subgroup of \( \pi_1(X, y_0) \) [\( J_2 \), Br]. In particular, if \( f^2 = f \), then \( G(X, f(X), y_0) = J(f, y_0) \), where \( y_0 \in f(X) \).

**Proof.** Let \( \alpha \) be an element of \( G(X, f(X), y_0) \). Then there exists an affiliated homotopy \( H : f(X) \times I \to X \) such that \( H(y, 0) = i(y) = H(y, 1) \) and \([H(y_0, \cdot)] = \alpha\). Define \( K = H(f_0 \times 1_I) : X \times I \to X \), where \( f_0 : X \to f(X) \) is a map such that \( f_0(x) = f(x) \). Then \( K(x, 0) = H(f_0(x), 0) = i(f_0(x)) = f(x) = K(x, 1) \). Since \( K(x_0, s) = H(f_0(x_0), s) = H(y_0, s) \), we have \( \alpha = [H(y_0, \cdot)] = [K(x_0, \cdot)] \). This implies \( \alpha \in J(f, x_0) \).

Suppose \( f^2 = f \). Let \( \alpha \) be an element of \( J(f, y_0) \). Then there exists a cyclic homotopy \( H : X \times I \to X \) such that \( H(x, 0) = f(x) = H(x, 1) \) and \([H(y_0, \cdot)] = \alpha\). Define \( K = H(i \times 1_I) : f(X) \times I \to X \). Then \( K(y, 0) = H(y, 0) = f(y) = f(f(x)) = f(x) = y = K(y, 1) \) and \( K(y_0, t) = H(y_0, t) \). Thus \( \alpha = [H(y_0, \cdot)] \in G(X, f(X), y_0) \).

**Corollary 7.** Let \( f \) and \( g \) be self-maps of \( X \) such that \( f^2 = f \), \( g^2 = g \) and \( f(X) = g(X) \). Then \( J(f, y_0) = J(g, y_0) \), where \( y_0 \in f(X) \).

In [\( Br \)], we know that if \( f \) is a self-map of \( X \) such that \( J(f, x_0) \)
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\[ = \pi_1(X, x_0), \] then all the fixed point classes of \( f \) have the same index. If we use Theorem 6, we have the following:

**Corollary 8.** Let \( f \) be a self-map of \( X \) such that \( f^2 = f, x_0 \in \text{Fix}(f) \) and \( G(X,f(X),x_0) = \pi_1(X,x_0) \). Then all the fixed point classes of \( f \) have the same index.

**Theorem 9.** Let \( f_i \) \((i = 1, 2)\) be self-maps of \( X \) and \( f_1 \) is homotopic to \( f_2 \) by a homotopy \( K \) such that \( K(f_i^{-1}f_i \times 1) \) is single valued. Then \( G(X,f_1(X),f_1(x_0)) \) is isomorphic to \( G(X,f_2(X),f_2(x_0)) \).

**Proof.** Let \( K \) be the homotopy from \( f_1 \) to \( f_2 \) such that \( K(f_i^{-1}f_i \times 1_i) \) is single valued. Let \( P(t) = K(x_0, t) \), Then \( P \) is a path from \( f_1(x_0) \) to \( f_2(x_0) \). Since \( P_* : \pi_1(X,f_1(x_0)) \rightarrow \pi_1(X,f_2(x_0)) \) is an isomorphism, it is sufficient to show \( P_*(G(X,f_1(X),f_1(x_0))) \subset G(X,f_2(X),f_2(x_0)) \). Let \( \alpha \) be any element of \( G(X,f_1(X),f_1(x_0)) \). Then there exists an affiliated homotopy \( H : f_1(X) \times I \rightarrow X \) such that \( H(x,0) = H(x,1) = i(x) \) and \( \alpha = [H(f_1(x_0),0)] \). Define \( G : f_2(X) \times I \rightarrow X \) by

\[
G(f_2(x), t) = \begin{cases}
K(x, 1 - 3t), & 0 \leq t \leq 1/3 \\
H(f_1(x), 3t - 1), & 1/3 \leq t \leq 2/3 \\
K(x, 3t - 2), & 2/3 \leq t \leq 1
\end{cases}
\]

Then \( G \) is well defined and continuous. Since \( G(y,0) = y = G(y,1) \)

\[
G(f_2(x_0), t) = \begin{cases}
K(x_0, 1 - 3t), & 0 \leq t \leq 1/3 \\
H(f_1(x_0), 3t - 1), & 1/3 \leq t \leq 2/3 \\
K(x_0, 3t - 2), & 2/3 \leq t \leq 1
\end{cases}
\]

\[
= \begin{cases}
P(1 - 3t), & 0 \leq t \leq 1/3 \\
h(3t - 1), & 1/3 \leq t \leq 2/3 \\
P(3t - 2), & 2/3 \leq t \leq 1
\end{cases}
\]

\[
= (P \ast h \ast P)(t),
\]

thus \( P_*(\alpha) \) belongs to \( G(X,f_2(X),f_2(X),f_1(x_0)) \), where \( h(t) = H(f_1(x_0),t) \).

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