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AN EXAMPLE OF A PARTIALLY ORDERED SHARKOVSKY SPACE

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1. Introduction

Let $f: R \longrightarrow R$ be a continuous function on the real line R, and denote the n-th iterate of f by f^n : $f^1=f$ and $f^n=f\circ f^{n-1}$ for $n>1$. A point $x\in R$ is a periodic point of f of peroid $k>0$ if $f^k(x)=x$ but $f^i(x)\neq x$ for all $0<i<k$. In the recent year the following question has aroused interest: If f has a point of period k, must f also have points of other periods $m\neq k$? The obvious answer would seem to be "no": why should there be any connection between points of period k and points of period m? Yet a little thought will show that there should be at least some results along these lines. For instance, if a continuous function f has a periodic point of period $k>1$, then it must also have a fixed point, by the Intermediate Theorem. Also the question has an intriguing answer which was found by ths Russian mathematician Sharkovky [6] in 1964.

THEOREM 1. (Sharkovsky's theorem). *Order the positive integers in a sequence as follows (we will call this sequence Sharkovsky's sequence):*

$$3, 5, 7, \dots, 2\cdot 3, 2\cdot 5, 2\cdot 7, \dots, 2^2\cdot 3, 2^2\cdot 5, 2^2\cdot 7, \dots, 2^3, 2^2, 2, 1.$$

If a continuous function $f: R \longrightarrow R$ has a point of period k, then f has points of all periods which follow k in the sequence.

Sharkovsky's original proof of Theorem 1 is quite complicated, even in the some what improved English version given by Stefan [8] in 1977. Clearly the periodic behavior of a function with points of all periods is extremely complex. In fact, considering the periodic behavior of physical and biological systems which can be modeled using such functions, Lie and Yorke [3] in 1975 called such behavior "chaos", and

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titled their paper "Period three implies chaos". But the efficient method of proof was found by Straffin, Jr., [9] in 1978, who used directed graphs to present information about the periodic point of f. There also exist some proofs of partial versions of Theorem 1 which were obtained by western mathematicians not yet aware of Sharkovsky's result (see, e.g., [1] and [3]).

Consider a (linearly) ordered set L with more than one point. We say that L is a linear continuum if

(1) L has the least upper bound property (or, equivalently, the greatest lower bound property),

(2) L is order dense, i.e., if $x<y$, then there exists z so that $x<z<y$,

and give L the order topology [4, page 84]. (Note that L is not a "continuum", which the topologist usually defines as a compact connected spaces, as L need not be compact). The real line and intervals of the real line are examples of linear continua in the order topology. But there are many others, among them the long line and the unit square in the dictionary order.

In 1985, Schirmer [5] extended Sharkovsky's theorem to all linear continua. Before introducing Schirmer's theorem, let us define a Sharkovsky space. A topological space X is a Sharkovsky space if Theorem 1 (with R replaced by X) is true, that is, if any continuous function $f: X \longrightarrow X$ has a point of period k, then f has points of all periods which follow k in Sharkovsky's sequence. Schirmer [5] proved the following theorem which is an extension of Sharkovsky's theorem.

THEOREM 2 (Schirmer's theorem). *An ordered set in the order topology is a Sharkovsky space if and only if it is a linear continuum.*

The proof of Schirmer's theorem is modeled on the proof of the Sharkovsky's theorem in [9] and [2], and apart from an inspection of the arguments in [9] and [2] nothing is needed but a more careful proof of three lemmas (Lemma 2.2, 2.3 and 2.4 in [5]).

We now look at partially ordered spaces in the context of Sharkovsky spaces. A partially ordered topological space X consists of a set with a partial order $\leqq$ and a topology which has a subbasis for its closed sets consisting of the sets

$$L(a)=\{x\in X \mid x\leqq a\},$$
$$M(a)=\{x\in X \mid a\leqq x\}$$

for all $a\in X$ ([10], page 148). Note that this topology equals to the order topology if the partial order is in fact a linear one.

Also in his paper [5], Schirmer proved that a partially ordered space which is dentrite is a Sharkovsky space if and only if it is a linear continuum. However he asked in his paper whether or not any partially ordered topological space is a Sharkovsky space if and only if it is a linear continuum. But the following simple example shows that the answer is negative. It is trivial to construct examples of partially ordered Sharkovsky spaces which are not linear continua. Let $(L, <)$ be a linear continuum and let $z\in L$ be any point other than the smallest or largest points of L. Define a partial order $\leqq$ on L as follows: $x\leqq y$ if and only if $x\leq y\leq z$ or $z\leq y\leq x$. Then $(L, \leqq)$ is a partially ordered topological space which is not a linear continuum. On the other hand, $<$ and $\leqq$ induce the same topology on L, so $(L, \leqq)$ is a Sharkovsky space.

The preceeding example shows that every linear continuum is homeomorphic to a partially ordered topological space which is not a linear continuum. Therefore Schirmer's question may be changed as follows; Is any partially ordered topological space a Sharkovsky space if and only if it is homeomorphic to a linear continuum?

2. Example

Let Ω be the first uncountable ordinal. Let X be a linearly ordered set from $[0, \Omega]$ by replacing between each ordinal α and its successor $\alpha+1$ a copy of the unit interval $I=(0, 1)$ and let Y be a closed interval $[-1, 1]$ of the real line. Then X and Y are linear continua in the order topology. It is well-known [7] that X is connected and compact, but it is not path connected, for no path can join from Ω to any other point. But $[0, \Omega)=X\setminus\{\Omega\}$ is path connected. Now we construct a partially ordered set. Let Z be the union of X and Y attached Ω to 0, that is, $Z=X\dot{\cup} Y/\Omega\sim 0$. For simplicity we will denote Ω or 0 instead of the class $\{\Omega\}=\{0\}$ by thinking that Ω and 0 are the same point. Give a partial order on Z as follows: For $x, y\in Z, x\leqq y$ provided that

$x \leq y$ holds when $x, y \in X$ or $x, y \notin Y$, and $y \geq 0$ whenever $x \in X$ and $y \in Y$. Note that X and Y have subspace topologies which are same as original order topologies of themselves, that is, as subspaces of Z, X and Y are linear continua. At this point, we want to prove that the space Z is a Sharkovsky space which is not homeomorphic to any linear continuum.

THEOREM. *The space Z defined as above is a Sharkovsky space, which is not homeorphic any linear continuum.*

Proof. Since $Z-\{0\}$ has three connected components, it is clear that Z is not homeomorphic to any linear continuum. In order to prove that Z is a Sharkovsky space, suppose that a continuous function $f: Z \longrightarrow Z$ has a periodic point of period $k>0$. Let us consider the following cases.

Case 1. If $f(\Omega) \in X \setminus \{\Omega\}$ then $f(Z) \subset X$, since X and Y are path connected but any point in X other than Ω can not be joined to Ω by a path. Hence all periodic points must lie in X. Since X is a linear continuum as a subspace of Z and $f(X) \subset X$, f has points of all periods which follow k in Sharkovsky's sequence by Schirmer's theorem.

Case 2. If $f(\Omega) \in Y \setminus \{\Omega\}$, then $f(Z) \subset Y$ by the same reason as in Case 1. By the same way as in Case 1, we can show that f has points of all periods which follow k in Sharkovsky's sequence.

Case 3. Let $f(\Omega)=\Omega$. Then $f(Y) \subset Y$ since also Y is path connected but any point in $X \setminus \{\Omega\}$ can not be joined to Ω by a path. If $f(X) \subset Y$, then $f(Z) \subset Y$, so that we have the same conclusion of Case 2. Suppose that $f(X) \not\subset Y$. Then we have $f(X) \subset X$. Let $a \in Z$ be any periodic point of f of order k. Then the orbit $\{a, f(a), \ldots, f^{k-1}(a)\}$ of a must contained in either X or Y. Since $f(X) \subset X$ and $f(Y) \subset Y$, and since X and Y are linear continua (as subspaces of Z) f has points of all periods which follow k in Sharkovsky's sequence.

Hence the proof of theorem is completed.

REMARK. In [5], Schirmer also proved that a linear continuum L in the order topology which contains an arc is a Sharkovsky space for which Sharkovsky's sequence is sharp in the sense that no implications from right to left are possible. Note that since Z contains an arc,

Z is also a Sharkovsky space for which Sharkovsky's sequence is sharp. Also Schirmer asked whether or not the assumption that L contains an arc can be omited.

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