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ON THE CONTACT CONFORMAL CURVATURE TENSOR*

Dedicated to Professor Yong Bai Baik on his sixtieth birthday

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1. Introduction

In 1949 ([2]), S. Bochner has introduced "Bochner curvature tensor" on a Kaehlerian manifold analogous to the Weyl conformal curvature tensor on a Riemannian manifold. However, we have not known the exact meaning of his tensor yet.

In 1989, H. Kitahara, K. Matsuo and J. S. Pak ([7]) defined a new tensor field on a hermitian manifold which is conformally invariant and studied some properties of this new tensor field. They called this new tensor field conformal curvature tensor field". In particular, on a $2n$-dimensional Kaehlerian manifold the conformal curvature tensor field is given by

$$
\begin{aligned}
(1.1)\quad B_{0,dcba}=R_{dcba}&+\frac{1}{2n}(g_{da}R_{cb}-g_{ca}R_{db}+R_{da}g_{cb}-R_{ca}g_{db}\\
&-f_{da}S_{cb}+f_{ca}S_{db}-S_{da}f_{cb}+S_{ca}f_{db}+2f_{dc}S_{ba}-2S_{dc}f_{ba})\\
&+\frac{(n+2)s}{4n^2(n+1)}(f_{da}f_{cb}-f_{ca}f_{db}-2f_{dc}f_{ba})\\
&-\frac{(3n+2)s}{4n^2(n+1)}(g_{da}g_{cb}-g_{ca}g_{db}),
\end{aligned}
$$

where $(f_c{}^b, g_{ab})$ denotes the Kaehlerian structure, Ricci tensor and scalar curvature being respectively denoted by R_{ba} and s and $f_{ca}=f_c{}^b g_{ba}$ and $S_{cb}=f_c{}^e R_{eb}$.

In this paper, we define a new tensor field on a Sasakian manifold, which is constructed from the conformal curvature tensor field by using the Boothby-Wang's fibration ([3]), and study some properties of

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this new tensor field.

In Section 2, we recall definitions and fundamental properties of Sasakian manifold and ϕ-holomorphic sectional curvature.

In Section 3, we define contact conformal curvature tensor field on a Sasakian manifold and prove that it is invariant under D-homothetic deformation due to S. Tanno ([13]).

In Section 4, we study Sasakian manifolds with vanishing contact conformal curvature tensor field, and the last Section 5 is devoted to studying some properties of fibred Riemannian spaces with Sasakian structure of vanishing contact conformal curvature tensor field.

2. Preliminaries

We first of all recall definitions and fundamental properties of almost contact manifold for later use.

Let M be a $(2n+1)$-dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U ; x^h\}$ in which there are given a tensor field $\phi_i{}^h$ of type $(1, 1)$, a vector field ξ^h and 1-form η_i satisfying

(2.1) $$\phi_i{}^h\phi_h{}^i=-\delta_j{}^i+\eta_j\xi^i,\quad \phi_j{}^i\xi^j=0,\quad \eta_i\phi_j{}^i=0,\quad \eta_i\xi^i=1,$$

where here and in the sequel the indices h, i, j, k, l run over the range $\{1, 2, \dots, 2n+1\}$. Such a set (ϕ, ξ, η) of a tensor field ϕ, a vector field ξ and a 1-form η is called an *almost contact structure* and a manifold with an almost contact structure an *almost contact manifold.*

If the Nijenhuis tensor

$$N_{ji}{}^h=\phi_j{}^k\partial_k\phi_i{}^h-\phi_i{}^k\partial_k\phi_j{}^h-(\partial_j\phi_i{}^k-\partial_i\phi_j{}^k)\phi_k{}^h$$

formed with $\phi_i{}^h$ satisfies

$$N_{ji}{}^h+(\partial_j\eta_i-\partial_i\eta_j)\xi^h=0,$$

where $\partial_i=\partial/\partial x^i$, then the almost contact structure is said to be *normal* and the manifold is called a *normal almost contact manifold.*

Suppose that there is given, in an almost contact manifold, a Riemannian metric g_{ji} such that

(2.2) $$g_{kh}\phi_j{}^k\phi_i{}^h=g_{ji}-\eta_j\eta_i,\quad \eta_i=g_{ih}\xi^h,$$

then the almost contact structure is said to be *metric* and the manifold is called an *almost contact metric manifold.* In an almost contact metric manifold, the tensor field $\phi_{ji}=\phi_j{}^h g_{hi}$ is skew-symmetric.

If an almost contact metric structure satisfies

$$\phi_{ji}=\frac{1}{2}(\partial_j\eta_i-\partial_i\eta_j),$$

then the almost contact metric structure is called a *contact structure.* A manifold with a normal contact structure is called a *Sasakian manifold.*

It is well known that in a Sasakian manifold we have

(2.3) $$\nabla_j\xi^i=\phi_j{}^i,$$

and

(2.4) $$\nabla_j\phi_i{}^h=\eta_i\delta_j^h-\xi^h g_{ji},$$

where ∇_j denotes the operator of covariant differentiation with respect to g_{ji}. If we denote by $\mathcal{L}$ the operator of Lie derivation with respect to the vector field ξ^h, we have

$$\mathcal{L}g_{ji}=\nabla_j\eta_i+\nabla_i\eta_j=\phi_{ji}+\phi_{ij}$$

and consequently

(2.5) $$\mathcal{L}g_{ji}=0$$

which shows that the vector field ξ^h is a Killing vector field.

Now, from (2.3), (2.4) and the Ricci identity

$$\nabla_k\nabla_j\xi^h-\nabla_j\nabla_k\xi^h=R_{kji}{}^h\xi^i,$$

we find

(2.6) $$R_{kji}\xi^i=\delta_k^h\eta_j-\delta_j^h\eta_k$$

or

(2.7) $$R_{kji}{}^h\eta_h=\eta_k g_{ji}-\eta_j g_{ki},$$

from which, bycontraction,

(2.8) $$R_{ji}\xi^i=2n\eta_j.$$

From (2.3), (2.4) and the Ricci identity

$$\nabla_k\nabla_j\phi_i{}^h-\nabla_j\nabla_k\phi_i{}^h=R_{kjl}{}^h\phi_i{}^l-R_{kji}{}^l\phi_l{}^h$$

we find

(2.9) $$R_{kjl}{}^h\phi_i{}^l-R_{kji}{}^l\phi_l{}^h=-\phi_k{}^h g_{ji}+\phi_j{}^h g_{ki}-\delta_k^h\phi_{ji}+\delta_j^h\phi_{ki},$$

from which, by contraction,

(2.10) $$R_{jh}\phi_i{}^h+R_{kjih}\phi^{kh}=-(2n-1)\phi_{ji},$$

where $\phi^{kh}=\phi_i{}^h g^{ki}$, g^{ki} being contravariant components of the metric tensor g_{ji}. Since

$$R_{kjih}\phi^{kh}=R_{hijk}\phi^{kh}=-R_{kijh}\phi^{kh},$$

we have from (2.10)

(2.11) $$S_{ji}+S_{ij}=0,$$

where $S_{ji}=\phi_j{}^h R_{hi}$.

A plane section in $T_x(M)$ is called a *ϕ-section* if there exists a unit vector X in $T_x(M)$ orthogonal to ξ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature $g(R(X, \phi X)\phi X, X)$ is called a *ϕ-sectional curvature.*

If the ϕ-sectional curvature at any point of a Sasakian manifold of dimension $\geqq 5$ is independent of the choice of ϕ-section at the point, then it is constant on the manifold and the curvature tensor is given by

$$R_{kjih}=\frac{1}{4}(k+3)(g_{kh}g_{ji}-g_{jh}g_{ki})+\frac{1}{4}(k-1)(\eta_k\eta_i g_{jh}-\eta_j\eta_i g_{kh} + g_{ki}\eta_j\eta_h-g_{ji}\eta_k\eta_h+\phi_{kh}\phi_{ji}-\phi_{jh}\phi_{ki}-2\phi_{kj}\phi_{ih}),$$

where k is the constant ϕ-sectional curvature (cf. [14]).

3. Contact Conformal Curvature Tensor Field

In a $(2n+1)$-dimensional Sasakian manifold M^{2n+1}, we define contact conformal curvature tensor field $C_{0,kji}{}^h$ by

$$\begin{aligned}(3.1)\quad C_{0,kji}{}^h=&R_{kji}{}^h+\frac{1}{2n}(\delta_k^h R_{ji}-\delta_j^h R_{ki}+R_k{}^h g_{ji}-R_j{}^h g_{ki}-R_k{}^h\eta_j\eta_i \\ &+R_j{}^h\eta_k\eta_i-\eta_k\xi^h R_{ji}+\eta_j\xi^h R_{ki}-\phi_k{}^h S_{ji}+\phi_j{}^h S_{ki} \\ &-S_k{}^h\phi_{ji}+S_j{}^h\phi_{ki}+2\phi_{kj}S_i{}^h+2S_{kj}\phi_i{}^h) \\ &+\frac{1}{2n(n+1)}\left[2n^2-n-2+\frac{(n+2)s}{2n}\right](\phi_k{}^h\phi_{ji}-\phi_j{}^h\phi_{ki} \\ &-2\phi_{kj}\phi_i{}^h)+\frac{1}{2n(n+1)}\left[n+2-\frac{(3n+2)s}{2n}(\delta_k^h g_{ji}-\delta_j^h g_{ki})\right. \\ &\left.+\frac{1}{2n(n+1)}\right]-(4n^2+5n+2)+\frac{(3n+2)s}{2n}(\delta_k^h\eta_j\eta_i \\ &-\delta_j^h\eta_k\eta_i+\eta_k\xi^h g_{ji}-\eta_j\xi^h g_{ki}),\end{aligned}$$

which is constructed from the conformal curvature tensor field (1.1) in a Kaehlerian manifold by using the Boothby-Wang's fibration ([3]), where $s=R_{ji}g^{ji}$ denotes the scalar curvature of M^{2n+1}, $R_j{}^h=R_{ji}g^{ih}$ and $S_j{}^h=S_{ji}g^{ih}$.

We next recall definition and fundamental properties of D-homothetic deformation due to S. Tanno ([13]), where D denotes the distribution defined by a contact form η.

D-homothetic deformation $g \longrightarrow {}^*g$ is defined by

$${}^*g_{ji}=\alpha g_{ji}+\alpha(\alpha-1)\eta_j\eta_i$$

for a positive constant α. From $*g_{ji}$ we have ([13])

$$
\begin{aligned}
&*g^{kj}=\alpha^{-1}g^{kj}-\alpha^{-2}(\alpha-1)\xi^k\xi^j,\\
&*R_{kji}{}^h=R_{kji}{}^h+(\alpha-1)(\phi_{ki}\phi_j{}^h-\phi_{ji}\phi_k{}^h+2\phi_{kj}\phi_i{}^h)\\
(3.2)\qquad &\quad+(\alpha-1)^2(\eta_j\delta_k^h-\eta_k\delta_j^h)\eta_i+(\alpha-1)[\eta_k(g_{ji}\eta^h-\eta_i\delta_j^h)\\
&\qquad\qquad -{}_j(g_{ki}\eta^h-\eta_i\delta_k^h)+\eta_i(\eta_j\delta_k^h-\eta_k\delta_j^h)],\\
&*R_{kj}=R_{kj}-2(\alpha-1)g_{kj}+2(\alpha-1)(n\alpha+n+1)\eta_k\eta_j,\\
&*s=\alpha^{-1}s-2n\alpha^{-1}(\alpha-1).
\end{aligned}
$$

If (ϕ,ξ,η,g) is a Sasakian structure, then $(*\phi, *\xi, *\eta, *g)$ is also a Sasakian structure, where we put

$$(3.3)\qquad *\phi=\phi,\quad *\xi=\alpha^{-1}\xi,\quad *\eta=\alpha\eta,\quad *g=\alpha g+\alpha(\alpha-1)\eta\phi\eta$$

for a positive constant α ([13]). In this case it is said that *$M(\phi,\xi,\eta,g)$ is D-homothetic to $M(*\phi, *\xi, *\eta, *g)$*.

If $M(\phi,\xi,\eta,g)$ is D-homothetic to $M(*\phi, *\xi, *\eta, *g)$, then by using (3.2) and (3.3) we have

$$
\begin{aligned}
&*\phi_{kj}=\alpha\phi_{kj},\\
&*R_k{}^i=\alpha^{-1}R_k{}^i-2\alpha^{-1}(\alpha-1)\delta_k^i+2(n+1)\alpha^{-1}(\alpha-1)\eta_k\xi^i,\\
&*S_{kj}=S_{kj}-2(\alpha-1)\phi_{kj},\\
&*S_k{}^i=\alpha^{-1}S_k{}^i-2\alpha^{-1}(\alpha-1)\phi_k{}^i.
\end{aligned}
$$

Taking account of (3.2), (3.3) and the above equations, we can easily verify that

$$*C_{0,kji}{}^h=C_{0,kji}{}^h,$$

where $*C_{0,kji}{}^h$ denotes the contact conformal curvature tensor field with respect to $(*\phi, *\xi, *\eta, *g)$.

Thus we have

THEOREM 3.1. *The contact conformal curvature tensor $C_{0,kji}{}^h$ with respect to (ϕ,ξ,η,g) coincides with the one with respect to $(*\phi, *\xi, *\eta, *g)$.*

COROLLARY 3.2. *A Sasakian manifold with vanishing contact conformal curvature tensor field is D-homothetic to a Sasakian manifold with vanishing contact conformal curvature tensor field.*

4. Sasakian Manifolds with Vanishing Contact Conformal Curvature Tensor Field

In this section we assume that M^{2n+1} is a Sasakian manifold with vanishing contact conformal curvature tensor field. If $C_{0,kji}{}^h=0$, then we have from (3.1)

$$0=\frac{2(n-2)}{n}R_{ji}+\frac{1}{n}\left[2(n-2)-\frac{n-2}{n}s\right]g_{ji} +\frac{1}{n}\left[-2(2n+1)(n-2)+\frac{n-2}{n}s\right]\eta_j\eta_i$$

and consequently

$$R_{ji}=\left(\frac{s}{2n}-1\right)g_{ji}+\left(2n+1-\frac{s}{2n}\right)\eta_j\eta_i, \tag{4.1}$$

that is, M^{2n+1} is η-Einsteinian, provided $2n+1>2$. Substituting (4.1) into (3.1) with $C_{0,kji}{}^h=0$, we have

$$R_{kji}{}^h=\frac{k+3}{4}(\delta_k{}^h g_{ji}-\delta_j{}^h g_{ki})+\frac{k-1}{4}(\phi_k{}^h\phi_{ji}-\phi_j{}^h\phi_{ki} -2\phi_{kj}\phi_i{}^h-\delta_k{}^h\eta_j\eta_i+\delta_j{}^h\eta_k\eta_i-\eta_k\xi^h g_{ji}+\eta_j\xi^h g_{ki}) \tag{4.2}$$

where $k=\frac{1}{n(n+1)}[s-n(3n+1)]$.

Thus we have

THEOREM 4.1. *A Sasakian manifold M^{2n+1} $(n>2)$ with vanishing contact conformal curvature tensor field is of constant ϕ-holomorphic sectional curvature $[s-n(3n+1)]/n(n+1)$.*

Combining Theorem 4.1 with Proposition 4.1 ([14], p. 504) and Corollary ([12], p. 282), we have

COROLLARY 4.2. *Let $M^{2n+1}(n>2)$ be a complete and simply connected Sasakian manifold with vanishing contact conformal curvature tensor field.*

(i) *If $s>-2n$, M^{2n+1} is D-homothetic to the unit sphere S^{2n+1};*

(ii) *If $s=-2n$, M^{2n+1} is isometric to $E^{2n+1}(-3)$;*

(iii) *If $s<-2n$, M^{2n+1} is D-homothetic to the universal pseudo-Riemannian covering manifold of S^{2n+1}_{2n}, which is diffeomorphic to $E^{2n}\times S^1$.*

We suppose that the contact conformal curvature tensor coincides with the C-Bochner curvature tensor $C_{kji}{}^h$ (for the definition of $C_{kji}{}^h$, see [9]). Then it follows that

$$R_{ji}=\left(\frac{s}{2n}-1\right)g_{ji}+\left(2n+1-\frac{s}{2n}\right)\eta_j\eta_i.$$

Conversely, if M^{2n+1} is η-Einsteinian,

$$C_{0,kji}{}^{h}=C_{kji}{}^{h}=R_{kji}{}^{h}-\frac{k+3}{4}(\delta_k^h g_{ji}-\delta_j^h g_{ki})-\frac{k-1}{4}(\phi_k{}^h\phi_{ji}-\phi_j{}^h\phi_{ki}-2\phi_{kj}\phi_i{}^h-\delta_k^h\eta_j\eta_i+\delta_j^h\eta_k\eta_i-\eta_k\xi^h g_{ji}+\eta_j\xi^h g_{ki}).$$

Thus we have

THEOREM 4.2. *A necessary and sufficient condition in order that $C_{0,kji}{}^{h}$ coincides with $C_{kji}{}^{h}$ is that M^{2n+1} is η-Einsteinian.*

5. Sasakian manifold as fibred space with invariant Riemannian metric

It is well known that in a Sasakian manifold we have

(5.1) $$\mathcal{L}g_{ji}=0,\quad \mathcal{L}\phi_j{}^i=0,\quad \mathcal{L}\eta_i=0,\quad \mathcal{L}\xi^i=0.$$

Thus, assuming that ξ^h is regular, we can regard the Sasakian manifold M^{2n+1} as a fibred manifold with invariant Riemannian metric (cf. [16]). Such a manifold is called a *fibred Riemannian space with Sasakian structure.*

Denoting by $u^a(x)$ $2n$ functionally independent solutions of $\xi^i\partial_i u=0$, we see that u^a are local coordinates of the base space M^{2n}. We put

(5.2) $$E_i{}^a=\partial_i u^a,\quad E_i=\eta_i,\quad E^h=\xi^h,$$

where here and in the sequel the indices a, b, c, d, e run over the range $\{1, 2, \dots, 2n\}$. Then we have

$$\xi^i E_i{}^a=0,\quad E^i E_i=1.$$

Since $E_i{}^a$ and E_i are linearly independent, we set

$$\begin{bmatrix} E_i{}^a \\ E_i \end{bmatrix}^{-1}=[E^i{}_c, E^i].$$

Then we have

(5.3) $$E_i{}^a E^i{}_b=\delta_b^a,\quad E_i{}^a E^i=0,\quad E_i E^i{}_b=0,\quad E^i E_i=1$$

and

(5.4) $$E_i{}^e E^h{}_e+E_i E^h=\delta_i^h.$$

For the Lie derivatives of E's we have

(5.5) $$\mathcal{L}E_i{}^a=0,\quad \mathcal{L}E_i=0,\quad \mathcal{L}E^h{}_b=0,\quad \mathcal{L}E^h=0.$$

Using $\mathcal{L}g_{ji}=0$ and (5.5), it turns out that

(5.6) $$g_{ba}=g_{ji}E^j{}_b E^i{}_a$$

is a metric tensor of the base space M^{2n}. From (5.6) it is clear that

(5.7) $$g_{ji}=g_{ba}E_j{}^bE_i{}^a+E_jE_i.$$

It will be easily verified that

(5.8) $$E^i{}_a=E_j{}^bg^{ji}g_{ba},\quad E^i=E_jg^{ji},\quad E_i{}^a=E^j{}_bg_{ji}g^{ba},\quad E_i=E^jg_{ji},$$

where g^{ba} are contravariant components of the metric tensor g_{ba} of M^{2n}. Also using $\mathcal{L}\phi_i{}^h=0$ and (5.5), we see that

(5.9) $$f_b{}^a=\phi_i{}^hE^i{}_bE_h{}^a$$

is a tensor field of type (1, 1) of M^{2n} and defines an almost complex structure of M^{2n} From (5.6) and (5.9) it follows that

(5.10) $$g_{dc}f_b{}^df_a{}^c=g_{ba},$$

which means that g_{ba} is a hermitian metric with respect to this almost complex structure. Thus the base space M^{2n} is an almost hermitian manifold. From (5.9), we have

(5.11) $$\phi_i{}^hE^i{}_b=f_b{}^eE^h{}_e,\quad \phi_i{}^hE_h{}^a=f_e{}^aE_i{}^e,\quad \phi_i{}^h=f_b{}^aE_i{}^bE^h{}_a.$$

For a function $f(u(x))$ on the base space M^{2n}, we have

(5.12) $$\partial_if=E_i{}^a\partial_af,\quad \partial_af=E^i{}_a\partial_if,$$

where$\partial_a=\partial/\partial u^a$.

Now using (5.7), we compute the Christoffel symbols $\left\{ {h \atop j\ i} \right\}$ formed with g_{ji} and find

(5.13) $$\left\{ {h \atop j\ i} \right\}=\left\{ {a \atop c\ b} \right\}E_j{}^cE_i{}^bE^h{}_a+(\partial_jE_i{}^a)E^h{}_a+\frac{1}{2}(\partial_jE_i+\partial_iE_j)E^h$$
$$+E_j\phi_i{}^h+E_i\phi_j{}^h,$$

where $\left\{ {a \atop c\ b} \right\}$ are Christoffel symbols formed with g_{ba}. From (5.13) we have

(5.14) $$\partial_jE_i{}^a-\left\{ {h \atop j\ i} \right\}E_h{}^a+\left\{ {a \atop c\ b} \right\}E_j{}^cE_i{}^b=-(E_jE_i{}^e+E_iE_j{}^e)f_e{}^a$$

by the help with (5.11). Setting

$$\nabla_jE_i{}^a=\partial_jE_i{}^a-\left\{ {h \atop j\ i} \right\}E_h{}^a+\left\{ {a \atop c\ b} \right\}E_j{}^cE_i{}^b,$$

it follows from (5.14) that

(5.15) $$\nabla_jE_i{}^a=-(E_jE_j{}^e+E_iE_j{}^e)f_e{}^a.$$

Thus putting $\nabla_a=E^i{}_a\nabla_i$, we have

(5.16) $$\nabla_bE_i{}^a=-f_b{}^aE_i$$

and

(5.17) $$\nabla_bE^h{}_a=-f_{ba}E^h,$$

where $f_{ba}=f_b{}^eg_{ea}$. Hence we have from (5.9), (5.16) and (5.17)

$$\nabla_c f_b{}^a = 0,$$

which shows that the base space M^{2n} is Kaehlerian.

From (5.15) and the Ricci identity

$$\nabla_k \nabla_j E_i{}^a - \nabla_j \nabla_k E_i{}^a = -R_{kji}{}^h E_h{}^a + R_{dcb}{}^a E_k{}^d E_j{}^c E_i{}^b,$$

we find

$$R_{dcb}{}^a E_k{}^d E_j{}^c E_i{}^b = R_{kji}{}^h E_h{}^a - (E_k E_j{}^a - E_j E_k{}^a) E_i + (E_k{}^e \phi_{ji} - E_j{}^e \phi_{ki} - 2\phi_{kj} E_i{}^e) f_e{}^a,$$

and consequently

$$(5.18) \qquad R_{dcb}{}^a = R_{kji}{}^h E^k{}_d E^j{}_c E^i{}_b E_h{}^a + f_d{}^a f_{cb} - f_c{}^a f_{db} - 2 f_{dc} f_b{}^a.$$

We now assume that the contact conformal curvature tensor field of the Sasakian manifold M^{2n+1} vanishes identically. Then we have from (4.2) and (5.18)

$$R_{dcb}{}^a = \frac{k+3}{4} (\delta_d{}^a g_{cb} - \delta_c{}^a g_{db} + f_d{}^a f_{cb} - f_c{}^a f_{db} - 2 f_{dc} f_b{}^a).$$

Thus we have

THEOREM 5.1. *If the contact conformal curvature tensorfield of a $(2n+1)$-dimensional fibred Riemannian space M^{2n+1} $(n>2)$ with Sasakian structure vanishes identically, then the base space is a Kaehlerian manifold of constant holomorphic sectional curvature* $\frac{1}{n(n+1)}(s+2n)$.

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