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HAUSDORFF φ-STRONG UNIQUENESS

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Let M be a closed subspace of a Banach space X. An element m_0 in M is called a best approximation to an element x in X if

$$\|x-m_0\|=\inf_{m\in M}\|x-m\|.$$

Let φ be astrictly increasing function from $\boldsymbol{R}^+$ into $\boldsymbol{R}^+$ with $\varphi(0)=0$. $P_M(x)$ is Hausdorff φ-strongly unique if there exists a positive number $\lambda=\lambda(x, M)$ such that, for all m in M,

$$\varphi(\|x-m\|)\geqq\varphi(d(x, M))+\lambda\varphi(d(m, P_M(x))).$$

P_M is uniform Hausdorff φ-strongly unique if there exists a positive number $\lambda=\lambda(M)$ such that, for all x in X and all m in M,

$$\varphi(\|x-m\|)\geqq\varphi(d(x, M))+\lambda\varphi(d(m, P_M(x))).$$

When M is Chebyshev, (uniform) Hausdorff φ-strong uniqueness is (uniform) φ-strong uniqueness.

It is known [3] that if M is a Haar subspace of $C(T)$, the space of continuous real-valued functions of a compact Hausdorff space T with the supremum norm, then for every x in $C(T)$ there exists a strongly unique best approximation in M. Also it is well known that in a Hilbert space, every best approximation is an φ-strongly unique best approximation with $\varphi(s)=s^2$ and $\gamma=1$. On the other hand, it is known that a strongly unique best approximation need not exists for every x in X when X is smooth.

Let M be a proximinal subspace of a Banach space X such that dim $X\geqq 2$.

REMARKS. 1) R. Smarzewski [7] defined φ-strong uniqueness on a linear closed subspace M of the real Banach space X when $P_M(x)$ is a singleton. But we defined Hausdorff φ-strong uniqueness when $P_M(x)$ is a set.

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2) When $\varphi(s)=s$, (Hausdorff, uniform Hausdorff) φ-strong uniqueness is equivalent to (Hausdorff, uniform Hausdorff) strong uniqueness.

Now we want to prove that (uniform) Hausdorff φ-strong uniqueness is preserved under the formation of quotient spaces.

LEMMA 1. *Let $P_M(x)$ be Hausdorff φ-strongly unique and N a closed subspace of M. Let $Q : X \longrightarrow X/N$ denote the quotient mapping. Then $P_{Q(M)}(Q(X))$ is the closure of $Q(P_M(x))$ in X/N.*

Proof. $\|Q(x)+Q(M)\|=\|x+M\|$, so $Q(P_M(x)) \subset P_{Q(M)}(Q(x))$. Since $P_{Q(M)}(Q(x))$ is closed, it only remains to prove that every element of this set is in the closure of $Q(P_M(x))$. Atter translation we can suppose hat the given element is zero. So we may assume that $0 \in P_{Q(M)}(Q(x))$, that is, $\|x+M\|=\|x+N\|$ and we must find elements in N arbitrary close to $P_M(x)$. Let $\{n_k\}$ be a sequence in N such that $\|x-n_k\| \longrightarrow d(x, M)=\|x+M\|$.
Then

$$\varphi^{-1}(\varphi(d(x, M))+\lambda\varphi(d(0, P_M(x-n_k))) \leqq \|x-n_k\| \longrightarrow d(x, M).$$

Choose $m \in P_M(x)$ and note that

$$\varphi^{-1}(\varphi(d(x, M))+\lambda\varphi(d(0, P_M(x-m)))=\|x+M\|.$$

Since φ^{-1} is also strictly increasing,

$$d(0, P_M(x-n_k)) \longrightarrow 0, \text{ that is, } d(n_k, P_M(x)) \longrightarrow 0, \text{ as}$$

required.

Remark. [3] The assertion of the above Lemma is no longer true if weassume M to be only a proximnal subspace of X, as the folloiwing example shows. In $X=l_1$,

$$M=\{x \in X : x_1+\sum_{n=2}^{\infty}\left(1-\frac{1}{n}\right)x_n=0\}$$

$$N=\{x \in M : \sum_{n=2}^{\infty} x_n=0\}$$

and let $x \in l_1$ be given by $2x_1=x_2=1$ and $x_n=0$ for $n \geqq 3$. It is not difficult to verify that $\|x+M\|=\|x+N\|=1$, and so $0 \in P_{Q(M)}(Q(x))$, where as $d(0, Q(P_M(x))=1$.

By the above lemma we obtain the following proposition.

COROLLARY 2. *Let P_M be uniform Hausdorff φ-strongly unique and*

N a closed subspace of M. Let $Q: X \longrightarrow X/N$ denote the quotient mapping. Then, for any x in X, $P_{Q(M)}(Q(x))$ is the closure (in X/N) of $Q(P_M(x))$.

From Lemma 1, we get the following proposition.

PROPOSITION 3. *Let $P_M(x)$ be Hausdorff φ-strongly unique and N a closed subspace of M. Then $P_{M/N}(x+N)$ is Hausdorff φ-strongly unique.*

Proof. Since $P_M(x)$ is Hausdorff φ-strongly unique, there exists $\lambda>0$ such that

$$\varphi(\|x-m\|) \geqq \varphi(d(x, M)) + \lambda\varphi(d(m, P_M(x)))$$

for all m in M. Then

$$\begin{aligned}\|Q(x)\| = \|x+N\| &= \inf_{n\in N} \|x+n\| \\ &\geqq \inf_{n\in N} \varphi^{-1}(\lambda\varphi(d(0, P_M(x+n))) + \varphi(d(x, M))) \\ &= \varphi^{-1}(\inf_{n\in N}(\lambda\varphi(d(0, P_M(x+n))) + \varphi(d(x, M))) \\ &= \varphi^{-1}(\lambda\varphi(\inf\{\|m+n\| : m\in P_M(x), n\in N\}) + \varphi(\|x+M\|)) \\ &= \varphi^{-1}(\lambda\varphi(d(0, Q(P_M(x)))) + \varphi(\|x+M\|))\end{aligned}$$

Note that $\|x+M\| = \|Q(x)+Q(M)\|$. By Lemma 1,

$$d(0, Q(P_M(x))) = d(0, P_{Q(M)}(Q(x))).$$

Thus we have, for any x in X,

$$\varphi(\|Q(x)\|) \geqq \lambda\varphi(d(0, P_{Q(M)}(Q(x))) + \varphi(d(Q(x), Q(M))).$$

Hence $P_{M/N}(x+N)$ is Hausdorff φ-strongly unique.

COROLLARY 4. *Let P_M be uniform Hausdorff φ-strongly unique and N a closed subspace of M. Then $P_{M/N}$ is uniform Hausdorff φ-strongly unique.*

We recall that a Banach space X with dim $X \geqq 2$ is said to be uniformly convex if the modulus of convexity $\delta_X = \delta_X(\varepsilon)$, $0<\varepsilon\leqq 2$, of X defined by

$$\delta_X(\varepsilon) = \inf\left\{1-\frac{\|x-y\|}{2} : x, y\in X, \|x\|=\|y\|=1, \ \|x-y\|=\varepsilon\right\}$$

satisfies the inequality $\delta_X(\varepsilon)>0$ for every $\varepsilon\in(0, 2]$. Assume that $\varphi: \boldsymbol{R}^+ \longrightarrow \boldsymbol{R}^+$ be an increasing convex continuous function such that $\varphi(0)=0$ and $\varphi(1)=1$. We shall say that a uniformly convex space X

has modulus of convexity of the type φ if there is a constant K, $0<K<\infty$, such that

(1) $$\delta_X(\varepsilon) \geqq K\varphi(\varepsilon), \quad 0 \leqq \varepsilon \leqq 2.$$

The function φ is said to be submultiplicative if there is a constant L, $0<L<\infty$, such that the inequality

(2) $$\varphi(ts) \leqq L\varphi(t)\varphi(s)$$

holds for all positive t and s.

THEOREM 5 [8]. *Let M be a Chebyshev subspace of a uniformly convex space X having modulus of convexity δ_X of the type φ. Assume that φ is a submultiplicative function. Then the inequality*

$$\varphi(\|x-P_M(x)\|) \leqq \varphi(\|x-P_M(x)\|) - KL^{-1}\varphi(\|P_M(x)-m\|)$$

holds for all $m \in M$, where K and L are in (1)-(2).

COROLLARY 6[8]. *Let M be a Chebyshev subspace of L_p, $1<p<\infty$. Then*

$$\|x-P_M(x)\|^q \leqq \|x-m\|^q - C_p\|P_M(x)-m\|^q$$

for all m in M, where $q=\max\{2, p\}$ and

(3) $$C_p = \begin{cases} \dfrac{p-1}{8} & , \text{ if } 1<p\leqq 2 \\[2ex] \dfrac{1}{p2^p} & , \text{ if } 2\leqq p<\infty, \end{cases}$$

that is, P_M is uniform φ-strongly unique with $\varphi(s)=s^q$, where $q=\max\{2, p\}$.

Recall that H_p, $1<p<\infty$, is the Banach space of all functions x analytic in the unit disc $|z|<1$ of the complex plane and such that

$$\|x\| = \|x\|_p = \lim_{r\to 1^-}\left[\frac{1}{2\pi}\int_0^{2\pi} |x(re^{i\theta})|^p d\theta\right]^{1/p} < \infty.$$

COROLLARY 7[8]. *Let M be a subspace of H^p, $1<p<\infty$. If $m_0 \in M$ is a best approximation in M to an element x, then*

$$\|x-m_0\|^q \leqq \|x-m\|^q - C_p\|m_0-m\|^q$$

for all y in M, where $q=\max\{2, p\}$ and C_p is as in (3).

DDFINITION 8. If, for any x, y in a subset E of X,

$$H(P_M(x), P_M(y)) \leqq \varphi(\lambda\|x-y\|),$$

then P_M is said to be φ-Lipschitz continuous relative to E.

LEMMA 9. *Let φ be a strictly increasing C^1 function on $\boldsymbol{R}^+$. Then φ is Lipschitz continuous relative to $[0, r]$ for any $r>0$.*

Proof. By Mean Value Theorem, for u_i with $0\leqq u_i\leqq r$ and $u_1<u_2$,

$$|\varphi(u_1)-\varphi(u_2)|=|\varphi'(\xi)|\,|u_1-u_2|$$

for some $\xi\in(u_1, u_2)$. Since $\varphi\in C^1$, there exists $u_0\in[0, r]$ such that

$$C_r=\sup_{0\leqq u\leqq r}|\varphi'(u)|=|\varphi'(u_0)|.$$

Thus for any u_i with $0\leqq u_1<u_2\leqq r$,

$$|\varphi(u_1)-\varphi(u_2)|\leqq C_r|u_1-u_2|.$$

THEOREM 10. *Let φ be as in Lemma 9 and $\varphi(0)=0$. If P_M is uniform Hausdorff φ-strongly unique, then P_M is φ^{-1}-Lipschitz continuous relative to $B_r(0)$ for any $r>0$, where $B_r(0)=\{x:\|x\|\leqq r\}$.*

Proof. Suppose that there is $\lambda=\lambda(M)>0$ such that

$$\varphi(d(x, P_M(x)))\leqq\varphi(\|x-m\|)-\lambda\varphi(d(m, P_M(x)))$$

for any $x\in X$ and $m\in M$. Choose any $r>0$.
Let $x, y\in B_r(0)$ be fixed, $a\in P_M(x)$, and $b\in P_M(y)$.
Then

$$\lambda\varphi(d(b, P_M(x))\leqq\varphi(\|x-b\|)-\varphi(\|x-a\|)$$

and

$$\lambda\varphi(d(a, P_M(y))\leqq\varphi(\|y-a\|)-\varphi(\|y-b\|),$$

or,

$$\varphi(d(b, P_M(x)))\leqq\lambda^{-1}\{\varphi(\|x-b\|-\varphi(\|x-a\|)\}$$

and

$$\varphi(d(a, P_M(y)))\leqq\lambda^{-1}\{\varphi(\|y-a\|-\varphi(\|y-b\|)\}.$$

Thus

$$\begin{aligned}\varphi(H(P_M(x), P_M(y)))&=\max\{\sup_{a\in P_M(x)}\sup_{a\in P_M(x)}\varphi((d(a, P_M(y))), \sup_{b\in P_M(y)}\varphi(d(b, P_M(x)))\}\\ &\leqq\sup_{\substack{a\in P_M(x)\\ b\in P_M(y)}}\{\varphi(d(a, P_M(y)))+\varphi(d(b, P_M(x)))\}\\ &\leqq\sup_{\substack{a\in P_M(x)\\ b\in P_M(y)}}\lambda^{-1}\{\varphi(\|y-a\|)-\varphi(\|y-b\|)+\varphi(\|x-b\|)-\varphi(\|x-a\|)\}\\ &\leqq 2\lambda^{-1}C_r\|x-y\|,\end{aligned}$$

whrer $C_r=\sup_{0\leqq u\leqq r}|\varphi'(u)|$. Since $x, y\in B_r(0)$ were arbitrary, for any $x, y\in B_r(0)$,

$$H(P_M(x), P_M(y))\leqq\varphi^{-1}(2\lambda^{-1}C_r\|x-y\|).$$

REMARK In Theorem 10, the convexity of φ didn't require, but the convexity of φ required in Theorem 5.

By Theorem 5, Corollary 6, Corollary 7 and Theorem 10, we obtained the following corollaries.

COROLLARY 11 [8]. *Let M be a Chebyshev subspace of a uniformly convex X having modulus of convexity δ_X of the type φ. Assume that φ is a submultiplicative function. Then P_M is φ^{-1}–Lipschitz continuous relative to $B_r(0)$ for all $r>0$.*

COROLLARY 12 [8]. *Let M be a Chebyshev subspace of L_p, $1<p<\infty$. Then P_M is q^{-1}–Lipschitz continuous relative to $B_r(0)$ for all $r>0$, where $q=\max\{2, p\}$.*

COROLLARY 13 [8]. *Let M be a proximinal subspace of H^p, $1<p<\infty$. Then P_M is q^{-1}–Lipschitz continuous relative to $B_r(0)$ for all $r>0$, where $q=\max\ \{2, p\}$.*

Now we get the property that if P_M is uniform φ–strongly unique, then P_M is φ^{-1}–Lipschitz continuous relative to $B_r(0)$ for all $r>0$. But we are interesting in the following question: If P_M is uniform hausdorff φ–strongly unique, is P_M Lipschitz continuous?

THEOREM 14 [1, 6] *The following statements are equivalent:*
(i) *P_M is Lipschitz continuous;*
(ii) *P_M is uniformly continuous.*

By the previous Theorem, P_M is φ–Lipschitz continuous relative to X if and only if P_M is Lipschitz continuous.

THEOREM 15. *Suppose φ is a strictly increasing continuous function on $\boldsymbol{R}^+$, $\varphi(0)=0$ and there is $\gamma>0$ such that*

(4) $$\varphi(a+b)\leqq\varphi(a)+\gamma\varphi_1(b),\quad a\geqq 0,\quad b\geqq 0$$

where φ_1 is a function, $\varphi_1(0)=0$ and $\varphi_1(s)\leqq\varphi(s)$ for all s in $\boldsymbol{R}^+$. If P_M is uniform Hausdorff φ–strongly unique, then P_M is Lipschitz continuous.

Proof. Since P_M is uniform Hausdorff φ–strongly unique, there is

$\lambda>0$ such that for each $x\in X$,

$$\varphi(\|x-m\|)\geqq\varphi(d(x,M))+\lambda\varphi(d(m,P_M(x))),\quad m\in M.$$

Let $x_1, x_2\in X$. For any $m_2\in P_M(x_2)$, we have

$$\begin{aligned}\lambda\varphi(d(m_2,P_M(\dot{x}_1))&\leqq\varphi(\|x_1-m_2\|)-\varphi(d(x_1,M))\\&\leqq\varphi(\|x_1-x_2\|+\|x_2-m_2\|)-\varphi(dx_1,M))\\&=\varphi(\|x_1-x_2\|+d(x_2,M)-d(x_1,M)+d(x_1,M))-\varphi(d(x_1,M)\\&\leqq\varphi(2\|x_1-x_2\|+d(x_1,M))-\varphi(d(x_1,M)).\end{aligned}$$

If $d(x_1,M)\leqq 2\|x_1-x_2\|$, then

$$\lambda\varphi(d(m_2,P_M(x_1)))\leqq\varphi(4\|x_1-x_2\|). \tag{5}$$

If $d(x_1,M)>2\|x_1-x_2\|$, by (4), we get

$$\begin{aligned}\lambda\varphi(d(m_2,P_M(x_1))&\leqq\varphi(d(x_1,M))+\gamma\varphi_1(2\|x_1-x_2\|)-\varphi(d(x_1,M))\\&\leqq\gamma\varphi_1(2\|x_1-x_2\|).\end{aligned}$$

Thus

$$\begin{aligned}\lambda\varphi(d(m_2,P_M(x_1))&\leqq\max\{\varphi(4\|x_1-x_2\|),\gamma\varphi(2\|x_1-x_2\|)\}\\&\leqq\max\{\varphi(2\|x_1-x_2\|+\gamma\varphi_1(2\|x_1-x_2\|),\ \gamma\varphi_1(2\|x_1-x_2\|)\}\\&\leqq(1+\gamma)\varphi(2\|x_1-x_2\|).\end{aligned}$$

Similarly, for any $m_1\in P_M(x_1)$,

$$\lambda\varphi(d(m_1,P_M(x_2))\leqq(1+\gamma)\varphi(2\|x_1-x_2\|).$$

Hence $H(P_M(x_1),P_M(x_2))\leqq\varphi^{-1}(\lambda^{-1}(1+\gamma)\varphi(2\|x_1-x_2\|)$. Therefore P_M is uniformly continuous. By Theorem 14, P_M is Lipschitz continuous.

COROLLARY 16 [5]. *If P_M is uniform Hausdorff strongly unique, i.e., $\varphi(s)=s$, then P_M is Lipschitz continuous.*

Proof. Note that $\varphi(a+b)=\varphi(a)+\varphi(b)$. It follows from Theorem 15.

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