Bull. Korean Math. Soc. 27(1990), No. 2, pp. 151-155

NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF SOLUTIONS TO OPERATOR EQUATIONS

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Recently, H. Z. Ming [7] obtained a necessary and sufficient condition for the existence of a solution to a general operator equation. In the present paper, we obtain such conditions in general forms and give some examples.

We begin with the well-known Fan-Browder fixed point theorem, from which we deduce two general theorems on such necessary and sufficient conditions. We give some examples of such conditions, which are improved versions of fixed point theorems of Halpern-Bergman [5], Ky Fan [3], [4], Kaczynski [6], Reich [9], Schauder [10], Tychonoff [11], and Ming [7]. In fact, we restate Ming's result in its correct form.

The following is known as the Fan-Browder fixed point theorem [1], [2].

LEMMA. *Let X be a nonempty compact convex subset of a topological vector space E and $T : X \to 2^X$ a multifunction satisfying*

(i) *for each $x \in X$, Tx is nonempty and convex, and*

(ii) *for each $y \in X$, $T^{-1}y = \{x \in X : y \in Tx\}$ is open.*

Then T has a fixed point $x_0 \in X$, that is, $x_0 \in Tx_0$.

From Lemma, we obtain

THEOREM 1. *Let D be a nonempty subset of a topological vector space E. Then a function $f : D \to E$ has a fixed point if and only if there exist a nonempty compact convex subset X of D and a multifunction $T : X \to 2^X$ satisfying*

(1) *if $Tx = \emptyset$ for some $x \in X$, then $fx = x$;*

Received December 9, 1989.

Supported in part by the Basic Science Research Institute Program, Ministry of Education, 1989.

(2) *Tx is convex for each $x \in X$;*
(3) *$T^{-1}y$ is open for each $y \in X$; and*
(4) *$x \notin Tx$ for each $x \in X$.*

Proof. If $fx_0 = x_0$ for some $x_0 \in D$, then let us put $X = \{x_0\}$ and $Tx_0 = \emptyset$, whence the conditions (1)-(4) hold trivially.

Conversely, suppose that $Tx \neq \emptyset$ for each $x \in X$. Then by Lemma, (2) and (3) imply the existence of an $x_0 \in X$ such that $x_0 \in Tx_0$, which contradicts (4). Therefore, there exists an $x_1 \in X$ such that $Tx_1 = \emptyset$. From (1), we have $fx_1 = x_1$. This completes our proof.

Using Theorem 1, a number of well-known fixed point theorems in nonlinear analysis can be stated in the "if and only if" form.

EXAMPLE 1. Let E be a normed vector space and $f : E \to E$ a function. Then f has a fixed point if and only if there exists a nonempty compact convex subset X of E such that $f|_X$ is weakly inward and continuous.

Here, $f : X \to E$ is said to be *weakly inward* if for each $x \in X$, fx belongs to the closure of the inward set

$$I_X(x) = \{x + r(u - x) : r > 0, \ u \in X\}$$

of X at $x \in X$.

In fact, define $T : X \to 2^X$ by

$$Tx = \{y \in X : \|x - fx\| > \|y - fx\|\}$$

for $x \in X$. Then the conditions (2)-(4) hold trivially. For (1), suppose that $Tx_0 = \emptyset$. Then

$$\|x_0 - fx_0\| \leq \|y - fx_0\|$$

for all $y \in X$, and hence for all $y \in \bar{I}_X(x_0)$. Since $fx_0 \in \bar{I}_X(x_0)$, by putting $y = fx_0$, we have $x_0 = fx_0$. Therefore, Example 1 follows from Theorem 1.

The sufficiency part of Example 1 is due to Halpern-Bergman [5] and includes well-known fixed point theorems of Schauder [10] and Brouwer.

A slightly modified argument in the proof of Theorrm 1 is useful to obtain a generalized form of Example 1 as follows.

EXAMPLE 2. Let E be a topological vector space on which its topological dual E^* separates points. Then a function $f: E \to E$ has a fixed point if and only if there exists a nonempty compact convex subset X of E such that $f|_X$ is weakly inward and weakly continuous.

Note that $f: X \to E$ is *weakly continuous* if $pf: X \to C$ is continuous for all $p \in E^*$.

In fact, for each $p \in E^*$, consider the closed set

$$X_p = \{y \in X : p(y - fy) = 0\}.$$

A point $y \in X$ is a fixed point of f if and only if $y \in \cap \{X_p : p \in E^*\}$. By compactness of X, it suffices to show that $y \in \bigcap_{i=1}^{n} X_{p_i}$ for every finite subset $\{p_1, p_2, \cdots, p_n\}$ of E^*. Given $\{p_1, p_2, \cdots, p_n\}$, define $T: X \to 2^X$ by

$$Tx = \{y \in X : \sum_{i=1}^{n} |p_i(x - fx)| > \sum_{i=1}^{n} |p_i(y - fx)|\}$$

for $x \in X$. Then the condition (2)-(4) of Theorem 1 hold trivially. For (1), suppose that $Tx_0 = \emptyset$ for some $x_0 \in X$. Then

$$\sum_{i=1}^{n} |p_i(x_0 - fx_0)| \le \sum_{i=1}^{n} |p_i(y - fx_0)|$$

for all $y \in X$, and hence for all $y \in \bar{I}_X(x_0)$. Since $fx_0 \in \bar{I}_X(x_0)$, by putting $y = fx_0$, we have $p_i(x_0 - fx_0) = 0$ for all i, that is, $x_0 \in \bigcap_{i=1}^{n} X_{p_i}$. Therefore, x_0 is a fixed point of f.

The sufficiency part of Example 2 includes results of Halpern–Bergman [5], Browder [1], Tychonoff [11], Ky Fan [3], [4], Reich [9], and Kaczynski [6]. For the literature, see Park [8]. Note that, in Examples 1 and 2, the weak inwardness can be replaced by the *weak outwardness* of f, that is, for each $x \in X$, $fx \in \bar{O}_X(x)$, where

$$O_X(x) = \{x + r(u - x) : r < 0, u \in X\}.$$

Now the argument of Ming [7] can be generalized as follows.

THEOREM 2. *Let D be a nonempty subset of a topological vector space E. Then a function $f: D \to E$ has a fixed point if and only if there exist a nonempty compact convex subset X of D and a multifunction $T: X \to 2^X$ satisfying*

(1) *if $Tx = \emptyset$ for some $x \in X$, then $fx = x$;*

(2) *for any $x \in X$, $x_1, x_2 \in Tx$, and $x_t \in (x_1, x_2) \equiv \{tx_1 + (1-t)x_2 : 0 < t < 1\}$, if $x_t \ne fx_t$, then $x_t \in Tx$;*

(3) *$T^{-1}y$ is open for each $y \in X$; and*

(4) *$x \notin Tx$ for each $x \in X$.*

Proof. If $fx_0 = x_0$ for some $x_0 \in D$, then by putting $X = \{x_0\}$ and $Tx_0 = \emptyset$, the conditions (1)-(4) hold trivially.

Conversely, if $Tx_0 = \emptyset$ for some $x_0 \in X$, then by (1), f has a fixed point x_0. Therefore, we may assume that $Tx \neq \emptyset$. for all $x \in X$. Suppose that Tx is convex for each $x \in X$. Then by Lemma, (3) implies that T has a fixed point, which contradicts (4). Therefore, there exists an $x \in X$ such that Tx is not convex, that is, there exist $x_1, x_2 \in Tx$ and $r \in (0, 1)$ such that $x_r = rx_1 + (1-r)x_2 \notin Tx$. By (2), this implies $x_r = fx_r$. This completes our proof.

Example 3 (Ming [7]). Let E be a real inner product space and $f : E \to E$ weakly continuous. Then f has a fixed point if and only if there exist two points $x_1^*, x_2^* \in E$ satisfying

(i) for any $x_0 \in B = [x_1^*, x_2^*] \equiv \{tx_1^* + (1-t)x_2^* : 0 \leq t \leq 1\}$ with $fx_0 \neq x_0$, there exists an $x \in B$ such that

$$\langle fx - x,\ fx_0 - x_0 \rangle < 0 ;$$

(ii) for every $x_0 \in B$ and for all $x_1, x_2 \in B$ with $\langle fx_i - x_i, fx_0 - x_0 \rangle < 0$, $i = 1, 2$, and for every $y \in (x_1, x_2)$, $fy \neq y$ implies

$$\langle fy - y,\ fx_0 - x_0 \rangle < 0.$$

In fact, note that B is a compact convex subset of E. For any $x \in B$, define

$$Tx = \{y \in B : \langle fy - y,\ fx - x \rangle < 0\}.$$

Note that Tx may be empty. We show that the conditions (1)-(4) of Theorem 2 follows.

(1) If $Tx_0 = \emptyset$ for some $x_0 \in B$, then $\langle fy - y,\ fx_0 - x_0 \rangle \geq 0$ holds for all $y \in B$. Therefore, by (i), we have $fx_0 = x_0$.

(2) follows from (ii).

(3) It suffices to show that $E \backslash T^{-1}y = \{x \in X : \langle fy - y,\ fx - x \rangle \geq 0\}$ is closed. This follows from the weak continuity of $f - 1_X$.

(4) is clear from the definition of T.

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